ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 3, 2013

NON-COMMUTING PAIRS OF SYMMETRIES OF RIEMANN SURFACES

EWA KOZŁOWSKA-WALANIA

ABSTRACT. In this paper we study Riemann surfaces with the group of automorphisms being mainly a nonabelian dihedral 2-group. Our aim is to give the upper and lower bounds for the maximal possible power of 2 which can be realized as the order of the product of a pair of non-conjugate $(M-q)$- and $(M-q')$-symmetries on a Riemann surface of genus $g \geq 2$. The results we obtain refine earlier results of Bujalance, Costa, Singerman and Natanzon, and depend strongly on the parity structure of g and the total number of ovals of the symmetries.

1. Introduction. Let X be a compact Riemann surface of genus $g > 1$. By a *symmetry* of X we mean an antiholomorphic involution σ of X which has fixed points. As is known, by the classical result of Harnack, the set of fixed points of σ consists of at most $g+1$ disjoint simple closed curves, which are called *ovals*. If σ has $g+1-q$ ovals, then we shall call it an $(M-q)$*-symmetry*, following the terminology of Natanzon, the pioneer in the modern study of symmetries of Riemann surfaces.

Studying Riemann surfaces and their symmetries is important due to the categorical equivalence under which a compact, connected Riemann surface X corresponds to a smooth, complex, projective and irreducible algebraic curve $\mathcal{C}_X$. Moreover, under this equivalence, a Riemann surface X admits a symmetry σ if and only if the corresponding curve $\mathcal{C}_X$ has a real form $\mathcal{C}_X(\sigma)$ and two such symmetries give rise to real forms non-isomorphic over the reals $\mathbf{R}$ if and only if they are not conjugate in the group $\mathrm{Aut}^{\pm}(X)$ of all, including antiholomorphic,

2010 AMS *Mathematics subject classification.* Primary 30F50, Secondary 14H37.
Keywords and phrases. Riemann surface, symmetry of Riemann surface, oval of a symmetry of a Riemann surface.
This paper was supported by the Research Grant N N201 366436 of the Polish Ministry of Sciences and Higher Education, Foundation for Polish Science and written partially during the author's one year postdoctoral stay at the Mathematical Institute of the Polish Academy of Sciences.
Received by the editors on January 10, 2010, and in revised form on October 18, 2010.
DOI:10.1216/RMJ-2013-43-3-989

automorphisms of X. Finally, the set $\mathrm{Fix}(\sigma)$ is homeomorphic to a smooth projective model of the corresponding real form $\mathcal{C}_X(\sigma)$.

In this paper we shall study pairs of non-conjugate symmetries of Riemann surfaces, mainly non-commuting ones. As is known by the Sylow theorem, all Sylow 2-groups are conjugate and so we shall restrict ourselves to the study of symmetries which generate a dihedral 2-group. The starting point for our studies is the main theorem of Bujalance, Costa and Singerman from [**5**] (see also [**13**]), by which we know that the total number of ovals of two symmetries on a Riemann surface of genus g, whose product has order 2^n, does not exceed $g/2^{n-2}+2$. Here we make the further study of this result, and we show that if integers $g \geq 2$ and $0 \leq q \leq q' \leq g$ hold $2g - g/2^{n-2} \leq q + q' < 2g - g/2^{n-1}$, then the product of a pair of $(M-q)$- and $(M-q')$-symmetries on a Riemann surface of genus g has order at most 2^n. In order to investigate the attainment of this bound, we introduce $\mu_g(q, q')$ to be the maximal exponent of the power of 2 to be realized as the order of the product of two non-conjugate $(M-q)$- and $(M-q')$-symmetries on a Riemann surface of genus g. The main goal of the paper is the study of μ, and the results we obtain depend strongly on the parity structure of g. Here we also remind the reader of the known fact that, for $q + q' < g$, two $(M-q)$- and $(M-q')$-symmetries on a Riemann surface of genus g always commute; hence, $\mu_g(q, q') = 1$ in such a case. In [**12**] we have shown that $\mu_g(q, q') \geq 2$ for $q + q' \geq g$, except for the case $\mu_g(1, g) = 1$ for $g > 2$, which allows us to restrict our studies to non-commuting symmetries with the order of the product being mainly at least 8. In this paper we also point out a gap from [**12**], concerning $q = q' = g - 1$ for $g = 3$ and $g = 2$. In the first case the result stays correct, but the construction has to be changed; in the second one $\mu_2(1, 1) = 1$, as the symmetries constructed in [**12**] are in fact conjugate, which follows from the proof of Theorem 3.7 in this paper. We also treat separately the case of two symmetries with 1 oval each, as the mentioned Bujalance, Costa and Singerman original result does not cover it. Such symmetries shall be called 1-*symmetries* throughout the paper.

On the other hand, our work also relates with the results of Izquierdo and Singerman from [**9**], where the authors study Riemann surfaces admitting pairs of symmetries with given numbers of ovals and they find some necessary and sufficient conditions for such pairs to exist. The examples constructed in [**9**] are pairs for which the order of the product of the symmetries is smallest possible; often these are commuting pairs of symmetries. Here, we bring up the problem of finding the maximal possible order of the product of two symmetries, but we also give the lower bound for $\mu_g(q, q')$.

Problems similar to the ones brought up here are also studied in the last part of paper [**11**], where we allow our symmetries to be fixed point free, which requires a different approach. The extensive study of symmetric Riemann surfaces can be found in [**2**].

Throughout the paper, the letter μ stands both for the area of the fundamental region of an NEC group and for the principal function considered here, which however does not lead to distinction problems.

2. Preliminaries. We shall prove our results using the theory of non-Euclidean crystallographic groups (*NEC groups* in short), by which we mean discrete and cocompact subgroups of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$. The algebraic structure of such a group Λ is determined by the signature:

(1)
$$s(\Lambda) = (h; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}),$$

where the brackets $(n_{i1}, \dots, n_{is_i})$ are called the *period cycles*, the integers n_{ij} are the *link periods*, m_i*-proper periods*, and, finally, h is the *orbit genus* of Λ.

A group Λ with signature (1) has the presentation with the following generators, called *canonical generators*:

$$x_1, \dots, x_r, e_i, c_{ij},\ 1 \leq i \leq k,\ 0 \leq j \leq s_i$$

and

$$a_1, b_1, \dots, a_h, b_h$$

if the sign is + or $d_1, \dots, d_h$ otherwise, and relators:

$$x_i^{m_i}, \quad i = 1, \dots, r,$$
$$c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i, \quad i = 1, \dots, k, j = 1, \dots, s_i$$

and

$$x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1}$$

or

$$x_1 \cdots x_r e_1 \cdots e_k d_1^2 \dots d_h^2,$$

according to whether the sign is $+$ or $-$. The elements x_i are elliptic transformations, a_i, b_i hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections. Reflections c_{ij-1} and c_{ij} are said to be *consecutive*. Every element of finite order in Λ is conjugate either to a canonical reflection, to a power of some canonical elliptic element x_i or to a power of the product of two consecutive canonical reflections.

Now an abstract group with such a presentation can be realized as an NEC group Λ if and only if the value below is positive

$$2\pi\left(\varepsilon h + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\varepsilon = 2$ or 1 according to the sign being $+$ or $-$. This value turns out to be the hyperbolic area $\mu(\Lambda)$ of an arbitrary fundamental region for such a group, and we have the following Hurwitz-Riemann formula

$$[\Lambda : \Lambda'] = \frac{\mu(\Lambda')}{\mu(\Lambda)}$$

for a subgroup Λ' of finite index in an NEC group Λ.

NEC groups having no orientation reversing elements are classical Fuchsian groups. They have signatures $(g; +; [m_1, \ldots, m_r]; \{-\})$, which are abbreviated as $(g; m_1, \ldots, m_r)$. Given an NEC group Λ, the subgroup Λ^+ of Λ consisting of the orientation preserving elements is called the *canonical Fuchsian subgroup of* Λ and, for a group with signature (1) it has, by [**16**], signature

$$(\varepsilon h + k - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{ks_k}). \tag{2}$$

Recall that an NEC group Λ is called *maximal* if there is no other NEC group containing it properly. The complete list of all signatures which lead to non-maximal NEC groups can be found in [**1, 7, 15**] by Bujalance, Estevez and Izquierdo, and Singerman, respectively.

A torsion free Fuchsian group Γ is called a *surface group*, and it has signature $(g; -)$. Any Riemann surface can be represented as $\mathcal{H}/\Gamma$, with Γ a surface Fuchsian group. Furthermore, given a Riemann surface so represented, a finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ. Observe that, if Λ is maximal, then G is the full group of automorphisms. From now on, we shall consider Riemann surfaces uniformized by surface Fuchsian groups.

The following result from [**8**] is crucial for the paper.

Proposition 2.1. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface, with Γ a Fuchsian surface group, and let G be the group of all automorphisms of X, where $G = \Lambda/\Gamma$ for some NEC group Λ with $\theta : \Lambda \to G$ being the canonical epimorphism. Then, the number of ovals of a symmetry σ of X equals*

$$\sum [C(G, \theta(c_i)) : \theta(C(\Lambda, c_i))],$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to σ.

For a symmetry σ we shall denote by $\|\sigma\|$ the number of its ovals. The index $w_i = [C(G, \theta(c_i)) \colon \theta(C(\Lambda, c_i))]$ will be called a *contribution* of c_i to $\|\sigma\|$.

Lemma 2.2 (see also [**5**, Theorem 2]). *Let $\mathrm{D}_{2^n} = \Lambda/\Gamma$ be the 2-group of automorphisms of a Riemann surface $X = \mathcal{H}/\Gamma$ generated by two non-central symmetries σ and τ, and let $C = (n_1, \dots, n_s)$ be a period cycle of Λ. Then reflections corresponding to C contribute to $\|\sigma\|$ and $\|\tau\|$ at most s ovals in total if $s \geq 1$ and at most 2 ovals if C is empty.*

Proof. The centralizer of any non-central involution in D_{2^n} has order 4. Since $c_i \in C(\Lambda, c_i)$, we have that $w_i \leq 2$, and since σ and τ are not conjugate, we can assume that $s \geq 2$ or $s = 1$ and n_1 is even. If c belongs to an even link period n' and cc' has order n', then $(cc')^{n'/2} \in C(\Lambda, c)$. Now $\theta((cc')^{n'/2}c) \neq 1$, since $\ker\theta$ is a Fuchsian group, and therefore we see that $\theta(C(\Lambda, c))$ has order 4. Observe also, that an empty period cycle contributes 2 ovals to the respective symmetry if and only if the image of the corresponding connecting generator is trivial. $\square$

3. The order of the product of two symmetries of a Riemann surface. The starting point for this paper is the result of Bujalance, Costa and Singerman from [**5**] (see also Natanzon in [**13**]), which we reproduce below. In this work we study some consequences of this result, concerning the bound for the order of the product of two symmetries of a Riemann surface.

Theorem 3.1 [**5, 13**]. *Let σ and τ be two symmetries of a Riemann surface X of genus g, whose product has order N. Then σ and τ have at most $2(g-1)/N+4$ and $4g/N+2$ ovals in total for N odd and even, respectively.*

The bounds given in the previous theorem were shown in [**5**] to be attained for arbitrary N and g for which N divides $g-1$ and $4g$, respectively. Theorem 3.1 gives, in particular, bounds $[2(g-1)/N]+4$ and $[4g/N]+2$ (where $[\cdot]$ denotes the integer part), which were studied in [**12**]. In particular, the first bound turned out to be attained only for N dividing $g-1$ and, in contrast, the second is attained also for N not dividing $4g$. The following corollary, concerning pairs of commuting symmetries, is a direct consequence of the above theorem.

Corollary 3.2. *Arbitrary $(M-q)$- and $(M-q')$-symmetries of a Riemann surface of genus g commute for $g \geq q+q'+1$.*

Proof. Observe that, for the total number t of ovals of both symmetries, $t = 2g+2-q-q' \geq g+3$. Let N denote the order of the product of our symmetries, and assume that $N \neq 2$. By Theorem 3.1, for even N we get $g+3 \leq 4g/N+2 \leq g+2$, a contradiction. For odd N, $g+3 \leq 2(g-1)/N+4 \leq 2(g-1)/3+4$ and so $g \leq 1$ which is not our case. $\square$

In this paper we focus our attention on the pairs of non-commuting and non-conjugate symmetries of a Riemann surface. In fact, by the Sylow theorem, we may assume that these symmetries generate a dihedral 2-group, as we know that all Sylow 2-groups are conjugate. Now we define a function

$$\mu_g\colon \{0,\dots,g\} \times \{0,\dots,g\} \longrightarrow \mathbf{N},$$

where $\mu_g(q,q') = n$ if and only if 2^n is the biggest power of 2 being realized as the order of the product of a pair of non-conjugate $(M-q)$-

and $(M-q')$-symmetries on a Riemann surface of genus g. Clearly, there might be some configurations of g, q, q' for which it is impossible to construct a pair of $(M-q)$- and $(M-q')$-symmetries on a Riemann surface of genus g, and for these values we understand $\mu_g(q,q')=0$. However, the next result from [**9**] shows that these exceptions occur if and only if $q=0$, which mainly is not our case in this paper as, by Theorem 3.1, if a Riemann surface of genus g admits two symmetries, one with $g+1$ ovals, then the product of these symmetries is at most 4.

Theorem 3.3 [**9, 13**]. *Let $g \geq 2, q, q'$ be integers such that $0 \leq q \leq q' \leq g$. Then there exists a Riemann surface of genus g admitting a pair of $(M-q)$- and $(M-q')$-symmetries if and only if $q \geq 1$ or $q=0$ and either g, q' are even, or else g is odd and $q'=g$ or q' is even.*

From now on, we shall assume that σ and τ are two $(M-q)$- and $(M-q')$-symmetries on a Riemann surface of genus $g \geq 2$ and, without loss of generality, $q \leq q'$ throughout the paper. In order to give some bounds for $\mu_g(q,q')$, one has to take the parameters q, q' into account. Throughout the paper assume, unless directly stated otherwise, that the parameters $g \geq 2, q, q'$ satisfy

$$2g - g/2^{n-2} \leq q + q' < 2g - g/2^{n-1} \tag{3}$$

for some positive integer n. Observe that the case $q=q'=g$ shall be treated separately, as there is no integer n such that (3) holds and so, for the first part of the paper, we assume that $q+q'<2g$. Now we shall give an upper bound for $\mu_g(q,q')$, depending on the parity structure of g, and show its attainment. Immediately, by Theorem 3.1, we obtain

Theorem 3.4. *The order of the product of two $(M-q)$- and $(M-q')$-symmetries on a Riemann surface of genus $g \geq 2$, for integers g, q, q' and n satisfying* (3)*, does not exceed 2^n, and so $\mu_g(q,q') \leq n$ in such a case.*

Proof. Observe, as above, that the total number t of ovals of both symmetries holds $t = 2g+2-q-q' > 2g+2-2g+g/2^{n-1} = 2+g/2^{n-1}$. Let d denote the order of the product of the symmetries and assume, to the contrary, that d is a power of 2 such that $d \geq 2^{n+1}$. Now, by Theorem 3.1, we have $t \leq 4g/d + 2 \leq g/2^{n-1} + 2$, which leads to a contradiction as at the same time $t > g/2^{n-1} + 2$. Clearly, for $q + q' < 2g - g/^{n-2}$, the order of the product cannot exceed $2^{n-1} < 2^n$, and so $\mu_g(q, q') \leq n - 1 < n$ in such a case. $\square$

Remark 3.5. It is also necessary to remind the reader here that in [**12**] we have shown that, for any $g \geq 2, q, q'$ such that $g \leq q + q'$ but $\{q, q'\} = \{1, g\}$, $g > 2$ there exists a Riemann surface having a pair of $(M - q)$- and $(M - q')$-symmetries, whose product has order 4 and that $(M - 1)$- and 1-symmetry on a Riemann surface of genus $g > 2$ always commute. However, there is an error, concerning $g = 2$ and $q = q' = 1$, in the proof of Theorem 4.1 in [**12**]. In fact, on a Riemann surface of genus 2, two non-conjugate symmetries with 2 ovals each, must commute. This, roughly speaking, follows from the fact that the signature of an NEC group, used for the construction in [**12**], is not maximal and does not lead to the full group of automorphisms of the surface in question. The second part of the proof of Theorem 3.7 in this paper gives an explanation for this fact.

Summing up this remark, the previous results and the definition of $\mu_g(q, q')$, we obtain

Corollary 3.6. *For $g \geq 2$ and $0 \leq q \leq q' \leq g$ such that there exists a Riemann surface having a pair of non-conjugate $(M - q)$- and $(M - q')$-symmetries, the following conditions hold*:

1. $\mu_g(1, g) = 1$ *for* $g > 2$ *and* $\mu_2(1, 1) = 1$;
2. $\mu_g(q, q') = 1$ *for* $g \geq q + q' + 1$;
3. $\mu_g(q, q') \geq 2$ *for* $g \leq q + q'$ *and* $\{q, q'\} \neq \{1, g\}$ *or* $\{1, 1\}$ *with* $g > 2$ *or* $g = 2$, *respectively.*

In Table 1 we give an example of $\mu_2(q, q')$ for all values $0 \leq q \leq q' \leq 2$. The computations for most of the cases can be found in [**5, 9, 12**].

It is much more difficult to give some conditions under which the previously given bound is attained. Let us first consider the pair of integers $g \geq 2$, $n \in \mathbf{N}$ such that 2^{n-2} divides g, which means that the Bujalance, Costa and Singerman bound for the total number of ovals of two symmetries on a Riemann surface of genus g, where the product

TABLE 1.

q	q'	$\mu_2(q,q')$
0	0	1
0	1	0
0	2	2
1	1	1
1	2	3
2	2	2

is of order 2^n, is an integer. In such a case $g = 2^u a$, where $n-2 \leq u$ and a is an odd integer. Observe that, for $n > 2$, this means that g is even, and we shall assume that $n > 2$ as the case of $n = 1$ was treated before in [**9, 10**] and the case of $n = 2$ in [**12**], where we have shown that, for $\{q, q'\} \neq \{1, g\}$ for $g > 2$ there always exists a Riemann surface having a pair of $(M-q)$- and $(M-q')$-symmetries with the product of order 4; in the second part of the proof we shall correct the error from [**12**], concerning the case $g = 2, q = q' = 1$.

Theorem 3.7. *Let g, q, q' and $n > 2$ be integers satisfying the conditions that 2^{n-2} divides g and* (3)*, given before. Then $\mu_g(q, q') = n$, that is, there exists a Riemann surface of genus g having a pair of non-conjugate $(M-q)$- and $(M-q')$-symmetries σ, τ with the product of order 2^n, except for the cases when $q = q' = g-1$ and $g = 2^{n-1}$ or $g = 3 \cdot 2^{n-2}$. For these two cases, $\mu_g(q, q') = n-1$.*

Proof. In the first part of the proof, we shall cover the case when $\mu_g(q, q') = n$. Assume that $q + q' = 2g - g/2^{n-2} + x$, where $x \geq 0$ is an integer. To simplify the proof, we employ the following convention: for the canonical epimorphism $\theta : \Lambda \to \mathrm{D}_{2^n}$, we take $\theta(x_i) = (\sigma\tau)^{2^{n-1}}$ on all the elliptic generators x_i and omit the generators which are mapped to 1.

First let $x = 2\alpha$. Consider an NEC group Λ with signature

$$(0; +; [2, \overset{\alpha}{\ldots}, 2]; \{(2, \overset{g-q}{\ldots}, 2, 2^n, 2, \overset{g-q'}{\ldots}, 2, 2^n)\})$$

and an epimorphism $\theta : \Lambda \to G = \mathrm{D}_{2^n} = \langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^{2^n} \rangle$ defined as follows for the consecutive canonical reflections corresponding to the nonempty period cycle:

$$\underbrace{\sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\ \sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\ldots,\sigma \text{ or } \sigma(\sigma\tau)^{2^{n-1}},}_{g+1-q}\ \ldots$$
$$\ldots\underbrace{\tau,\ \tau(\sigma\tau)^{2^{n-1}},\ \tau,\ \tau(\sigma\tau)^{2^{n-1}},\ldots,\tau \text{ or } \tau(\sigma\tau)^{2^{n-1}},}_{g+1-q'}\ \sigma.$$

By writing 'or' in the definition of the epimorphism for consecutive canonical reflections, as above, we shall understand that the image of this particular reflection, here the last in the sequence of reflections sent alternatively to σ and $\sigma(\sigma\tau)^{2^{n-1}}$, depends on the parity of the length of the sequence, i.e., if $g+1-q$ is even, then the last of the reflections in the bracket is sent to $\sigma(\sigma\tau)^{2^{n-1}}$ and, if $g+1-q$ is odd, then the image is σ. We shall use this convention throughout the paper. Furthermore, if α is odd, we take $\theta(e) = (\sigma\tau)^{2^{n-1}}$. Such a definition of θ gives rise to the configuration of two $(M-q)$- and $(M-q')$-symmetries on a Riemann surface $X = \mathcal{H}/\ker\theta$ of genus g, whose product has order 2^n. Observe that, for $x > 0$ or $q + q' < 2g-2$, which means that the symmetries have more than 4 ovals in total, the signature of Λ is maximal, by [**7**], and so the symmetries are non-conjugate and generate the full group of automorphisms for X. For $x = 0$ and $q = g-2$, $q' = g$ or $q = g-1$, $q' = g$, we have symmetries with different numbers of ovals and so the symmetries are indeed non-conjugate, as desired.

We shall deal with the case $x = 0$ and $q = q' = g-1$ in the second part of the proof as, for these parameters, we have $2g-2 = q+q' = 2g - g/2^{n-2}$ and so $g = 2^{n-1}$.

Now let $x = 2\alpha+1$. Assume first that $q' < g$, and consider an NEC group Λ with signature

$$(0; +; [2, \overset{\alpha}{\ldots}, 2]; \{(2, \overset{g-1-q}{\ldots}, 2, 4, 2^n, 2, \overset{g-1-q'}{\ldots}, 2, 4, 2^n)\})$$

and an epimorphism $\theta : \Lambda \to \mathrm{D}_{2^n} = \langle \sigma, \tau\rangle$ for which the consecutive canonical reflections corresponding to the nonempty period cycle are mapped to

$$\underbrace{\sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\ \sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\ldots,\sigma \text{ or } \sigma(\sigma\tau)^{2^{n-1}},\ \sigma(\sigma\tau)^{2^{n-2}},}_{g+1-q}\ \ldots$$
$$\ldots\underbrace{\tau,\ \tau(\sigma\tau)^{2^{n-1}},\ \tau,\ \tau(\sigma\tau)^{2^{n-1}},\ldots,\tau \text{ or } \tau(\sigma\tau)^{2^{n-1}},\ \tau(\sigma\tau)^{2^{n-2}},}_{g+1-q'}\ \sigma,$$

and for α odd, as before, we take $\theta(e) = (\sigma\tau)^{2^{n-1}}$. Again, θ defines the configuration of two $(M-q)$- and $(M-q')$-symmetries on a Riemann surface $\mathcal{H}/\ker\theta$ of genus g, whose product has order 2^n. Similarly, as in the previous case, if $x > 1$ or $q \leq g-2$, then the symmetries generate the full group of automorphisms of X, by [**7**].

The case $x = 1, q = q' = g-1$ gives $2g-2 = q+q' = 2g - g/2^{n-2}+1$ and so $g = 3 \cdot 2^{n-2}$, which shall be considered in the second part of the proof.

Now let $q' = g$ and $q < g-1$. In such a case one takes an NEC group Λ with signature

$$(0; +; [2, \overset{\alpha}{\ldots}, 2]; \{(2, \overset{g-2-q}{\ldots}, 2, 4, 4, 2^n, 2^n)\}),$$

and an epimorphism $\theta : \Lambda \to \mathrm{D}_{2^n} = \langle \sigma, \tau \rangle$ defined for the consecutive canonical reflections corresponding to the nonempty period cycle as

$$\underbrace{\sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\ \sigma,\ \sigma(\sigma\tau)^{2^{n-1}}, \ldots, \sigma \text{ or } \sigma(\sigma\tau)^{2^{n-1}},}_{g-1-q}\ \sigma(\sigma\tau)^{2^{n-2}},\ \sigma,\ \tau,\ \sigma.$$

Now, if α is odd, as before we take $\theta(e) = (\sigma\tau)^{2^{n-1}}$. Once again, θ defines a configuration of two $(M-q)$- and $(M-q')$-symmetries on a Riemann surface of genus g, with the product of order 2^n. Again, observe that, for $x > 1$ or $q < g-2$, the signature of Λ is maximal, by [**7**], and the symmetries σ, τ are non-conjugate, generating the full automorphism group for X. Now for $x = 1$ and $q = g-2, q' = g$ again we have symmetries with different numbers of ovals and so they cannot be conjugate.

Now we shall study the two exceptional cases for $q = q' = g-1$, that is, $g = 2^{n-1}$ and $g = 3 \cdot 2^{n-2}$. Let Λ denote an NEC group with signature $(0; +; [-]; \{(2^\epsilon, 2^n, 2^\epsilon, 2^n)\})$, where $\epsilon = 1$ or 2, and consider an epimorphism $\theta : \Lambda \to \mathrm{D}_{2^n}$ which sends the consecutive canonical reflections c_i, respectively, to $\sigma, \sigma(\sigma\tau)^{2^{n-\epsilon}}, \tau, \tau(\tau\sigma)^{2^{n-\epsilon}}, \sigma$. The signature of Λ is not maximal and Λ is an index 2 subgroup of an NEC group Λ' with maximal, by [**7**], signature $(0; +; [2]; \{(2^\epsilon, 2^n)\})$ and canonical generators x, c_0', c_1'. Let $G' = \langle \sigma, \tau, \rho \mid \sigma^2, \tau^2, \rho^2, (\sigma\tau)^{2^n}, \rho\sigma\rho\tau \rangle$, and consider an epimorphism $\theta' : \Lambda' \to G'$ for which $\theta'(x) = \rho, \theta'(c_0') = \sigma, \theta'(c_1') = \sigma(\sigma\tau)^{2^{n-\epsilon}}$. This extends the epimorphism θ after taking an embedding of Λ into Λ' defined as $c_0 = c_0'$, $c_1 = c_1'$, $c_2 = xc_0'x, c_3 = xc_1'x$. Now, for

$\epsilon = 1$, we have $g = 2^{n-1}$ and, for $\epsilon = 2$, we have $g = 3 \cdot 2^{n-2}$. In both cases we obtain a pair of symmetries with 2 ovals, but the symmetries are conjugate. Moreover, letting $n = 2$ for the case $\epsilon = 1$ corrects an error for the case of $g = 2$, $q = q' = 1$, discussed earlier in the context of paper [**12**]. Observe also that there is no epimorphism from Λ' to $G'' = \mathrm{D}_{2^n} \rtimes \mathrm{Z}_2 = \langle \sigma, \tau \rangle \rtimes \langle \rho \rangle$ such that σ and τ are non-conjugate. Indeed, if such an epimorphism would exist, then, without loss of generality, we may assume that the reflections c'_1, c'_2 are mapped to σ and τ. But c'_0 is conjugate to c'_2 and so c'_0 must be mapped to a symmetry η conjugate with τ. On the other hand, the order of $\eta\sigma$ equals 2^ϵ, with $\epsilon = 1, 2$ a contradiction. From [**7**] it follows that Λ can also be seen as an index 2 subgroup of an NEC group Λ'' with signature $(0; +; [-]; \{(2, 2, 2^\epsilon, 2^n)\})$ and canonical generators c''_i, $i = 0, 1, 2, 3$. We shall also show that, in this case, θ cannot be extended to $\theta'' : \Lambda'' \to G''$, where G'' is as above. Assume, to the contrary, that such an epimorphism exists. In such a case at least one of the canonical reflections is mapped to a symmetry $\eta \notin \langle \sigma, \tau \rangle$. As σ, τ have 2 ovals each, it follows that one of them, say σ, gets ovals from only one reflection. First let c''_3 and c''_0 contribute to σ and τ, respectively, and assume, without loss of generality, that the images of these reflections are σ and τ. The centralizer of c''_3 in Λ'' contains $c''_3, (c''_3 c''_0)^{2^{n-1}}$ and $(c''_2 c''_3)^{2^{\epsilon-1}}$. Now the centralizer of σ in G'' has order 8 and so, by Proposition 2.1, for the symmetry σ to have 2 ovals, it has to be that the image of $(c''_2 c''_3)^{2^{\epsilon-1}}$ equals $(\sigma\tau)^{2^{n-1}}$. It follows that $\theta''(c''_2)$ is conjugate to σ, a contradiction as c''_3 is the only reflection contributing to σ. Assume now that no two consecutive canonical reflections contribute to σ and τ, respectively (here we consider c''_3 and c''_0 consecutive). It follows that two among the canonical reflections are mapped to symmetries $\eta_1, \eta_2 \notin \langle \sigma, \tau \rangle$ and each of σ, τ gets ovals from only one reflection. Now, if $\epsilon = 1$, then, without loss of generality, we may assume that the canonical reflections are mapped respectively to $\eta_1, \tau, \eta_2, \sigma$. Observe that η_1, η_2 and τ are in the image of the centralizer of the reflection c''_1. So, for τ to have 2 ovals, it must be that $\eta_1 = \eta_2$, as the order of the centralizer of τ in G'' is equal to 8. This, however, leads to a contradiction, as $\eta_1\sigma$ is an involution and, on the other hand, has order 2^n. Assume now that $\epsilon = 2$. Here the consecutive canonical reflections can be mapped, without loss of generality, to $\eta_1, \tau, \eta_2, \sigma$ or $\sigma, \eta_1, \tau, \eta_2$. In the first case, as above, $\eta_1 = \eta_2$, and we have a contradiction. In the second case, observe that the image of the centralizer of c''_2 in Λ'' contains $\tau, \eta_1\tau$

and $(\eta_2\tau)^2$. Moreover, for τ to have 2 ovals, it must that $\eta_1 = \eta_2\tau\eta_2$ or $\tau = \eta_2\tau\eta_2$, a contradiction again. We have shown that the only epimorphism extending θ leads to conjugate symmetries σ and τ.

The only thing that we still need to do is to prove that, in fact, $\mu_g(q, q') = n - 1$ for $q = q' = g - 1$ and g is one of the exceptional cases. Here it is enough to take an NEC group Λ with maximal signature $(0; +; [2, \overset{\epsilon}{\ldots}, 2]; \{(2, 2^{n-1}, 2, 2^{n-1})\})$ and an epimorphism onto $\mathrm{D}_{2^{n-1}}$ which maps the consecutive canonical reflections to $\sigma, \sigma(\sigma\tau)^{2^{n-2}}, \tau, \tau(\sigma\tau)^{2^{n-2}}$ and, for $\epsilon = 1$, we take $\theta(e) = (\sigma\tau)^{2^{n-2}} = \theta(x_i)$. This completes the proof. $\square$

Remark 3.8. There is one more gap in the proof of Theorem 4.1 in [**12**], which this time, however, fortunately does not lead to an error. In fact, the NEC signature $(0; +; [-]; \{(4, 4, 4, 4)\})$ used in that paper for the construction of a pair of symmetries with 2 ovals each for $g = 3$ is also not maximal, and the corresponding NEC group Λ is an index 4 subgroup of an NEC group Λ' with maximal signature $(0; +; [-]; \{(2, 2, 2, 4)\})$ and canonical generators c_i', $i = 0, 1, 2, 3$. Consider a group $G' = G \times H$ where $G = \mathrm{D}_4 = \langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^4 \rangle$, $H = \mathrm{Z}_2 \times \mathrm{Z}_2 = \langle \rho, \eta \mid \rho^2, \eta^2, (\rho\eta)^2 \rangle$, and take an epimorphism $\theta' : \Lambda' \to G'$, mapping the consecutive canonical reflections, respectively, to σ, ρ, η, τ. Now consider an embedding given by $c_0 = c_0'$, $c_1 = c_3'$, $c_2 = c_2'c_0'c_2'$, $c_3 = c_1'c_3'c_1'$, where c_i denote the canonical generators for Λ. With such definitions, θ' leads to the non-conjugate symmetries σ, τ having 2 ovals each, by Proposition 2.1.

Recall that in this part we assume that $g = 2^u a$ for some odd integer a and an integer u such that $n-2 \le u$. The results above give us a natural lower bound for $\mu_g(q, q')$. Observe that in our constructions actually we have only used the inequality $q + q' \ge 2g - g/2^{n-2}$ to assure that the length of the nonempty period cycle is positive and, if we omit the condition $q + q' < 2g - g/2^{n-1}$, all the proofs remain correct; however, for $q + q' \ge 2g - g/2^{n-1}$, they only give us a lower bound for $\mu_g(q, q')$. In particular, for any g, q, q' not being the exceptional cases, we have the following result.

Proposition 3.9. *For $g = 2^u a$ and $q + q' \geq 2g - g/2^u$, but $q = q' = g - 1$ with $a = 2$ or 3, we have $\mu_g(q, q') \geq u + 2$, and the equality holds for $q + q' < 2g - g/2^{u+1}$. Moreover, for any $m \leq u + 2$ and $2g - g/2^{m-2} \leq q + q' < 2g - g/2^{m-1}$, we have $\mu_g(q, q') = m$.*

Observe, however, that with our assumption $n \geq 3$ all the values of g divisible by 2^{n-2} are even. Now we shall try to investigate the case when the bound from Theorem 3.1 is not an integer. By the results from [**12**], we know that in such a case the natural bound $[4g/N] + 2$ holds for the total number of ovals of a pair of symmetries. We shall consider the case when g is of the form $g = 2^u a + 1$ with some odd a. Let $n \geq 3$ be an integer such that $2 \leq n - 1 \leq u$. Here $g = 2^{n-1} b + 1$ for some (possibly even) integer b. Note that $g/2^{n-2} = 2b + 1/2^{n-2}$ and $[g/2^{n-2}] = 2b$. By Theorem 3.1, in such a case the largest possible total number of ovals of a pair of symmetries, whose product has order 2^n, equals $2b + 2$. Let q, q' be integers such that (3) holds. Observe that, in our case, it means that $2g - 2b \leq q + q' < 2g - b$.

Theorem 3.10. *Let g, q and q' be integers such that $g = 2^{n-1} b + 1$ and* (3) *holds. Then $n - 1 \leq \mu_g(q, q') \leq n$, and the upper bound is attained if and only if one of the following is true*:

1. $q + q' \geq 2g + 2 - 2b$;
2. $q + q' = 2g + 1 - 2b$ *and* $q' \geq g - 1$ *or* $n \leq 4$;
3. $q + q' = 2g - 2b$ *and* $q' = g - 1$ *or* $n = 3$.

Proof. Observe that in particular we shall also show that, for the sets of parameters not holding conditions 1–3 of the theorem, the lower bound for $\mu_g(q, q')$ is attained as $\mu_g(q, q') = n - 1$ in these cases. The upper bound is true by virtue of Theorem 3.4. Now we shall show that in fact for any of the cases 1–3 given in the theorem, the equality $\mu_g(q, q') = n$ holds. To simplify and shorten the proof, throughout the constructions of respective surfaces, we define all elliptic generators to be mapped to the only nontrivial central element in the corresponding dihedral group, all the glide reflections are mapped to σ and we omit in the definitions all the generators which are mapped to 1.

In case 1 our study will depend on the parity of $q + q'$. Assume first that $q + q' = 2g + 2 - 2b + 2\alpha$. Now it is enough to consider an NEC group Λ with maximal, by [**7**] signature

$$(0;+;[2,\overset{\alpha}{\ldots},2];\{(2,\overset{g+1-q}{\ldots},2),(2,\overset{g+1-q'}{\ldots},2)\})$$

and an epimorphism θ which maps the consecutive canonical reflections corresponding to the first period cycle alternatively to σ and $\sigma(\sigma\tau)^{2^{n-1}}$, the reflections corresponding to the second period cycle alternatively to τ and $\tau(\sigma\tau)^{2^{n-1}}$. In addition, if α is even, q,q' are odd, then we take $\theta(e_1)=\theta(e_2)^{-1}=(\sigma\tau)^{2^{n-2}}$. If α is odd and q,q' are even, we take $\theta(e_2)=(\sigma\tau)^{2^{n-1}}$ and finally, if α,q,q' are odd, we put $\theta(e_1)=\theta(e_2)=(\sigma\tau)^{2^{n-2}}$. In each case we obtain a pair of non-conjugate $(M-q)$- and $(M-q')$-symmetries σ,τ on the Riemann surface $\mathcal{H}/\ker\theta$ of genus g with the product of order 2^n.

Now let $q+q'=2g+2-2b+2\alpha+1$. As $q+q'$ is odd, q,q' have different parity. Assume first that q is even and q' is odd, and observe that, in such a case, $(M-q)$-symmetry σ has at least two ovals. Consider an NEC group Λ with maximal signature:

$$(0;+;[2,\overset{\alpha}{\ldots},2];\{(2,\overset{g-1-q}{\ldots},2,4,4),(2,\overset{g+1-q'}{\ldots},2)\})$$

and an epimorphism θ which maps the consecutive canonical reflections corresponding to the first period cycle, respectively, to

$$\underbrace{\sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\ \sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\ldots,\sigma,}_{g-q}\ \sigma(\sigma\tau)^{2^{n-2}},\ \sigma(\sigma\tau)^{2^{n-1}},$$

and the reflections corresponding to the second cycle alternatively to τ and $\tau(\sigma\tau)^{2^{n-1}}$, finishing with $\tau(\sigma\tau)^{2^{n-1}}$. If α is even, we take $\theta(e_1)=\theta(e_2)^{-1}=(\sigma\tau)^{2^{n-2}}$ and, for α odd, we take $\theta(e_1)=\theta(e_2)=(\sigma\tau)^{2^{n-2}}$. Now, if q is odd and q' is even, we use the analogous definitions, exchanging roles of σ and τ.

For case 2, first let $q'=g-1$ and observe that, in such a case, q is an odd integer and so the number $g+1-q$ of ovals of symmetry σ is also odd and, by assumption $q\le q'$, not smaller than 3. Now consider an NEC group Λ with maximal signature

$$(0;+;[-];\{(2,\overset{g-1-q}{\ldots},2,4,4),(-)\})$$

and an epimorphism which maps the consecutive canonical reflections corresponding to the nonempty period cycle to

$$\underbrace{\sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\ \sigma,\ \sigma(\sigma\tau)^{2^{n-1}},\dots,\sigma(\sigma\tau)^{2^{n-1}}}_{g-q},\ \sigma(\sigma\tau)^{2^{n-2}},\ \sigma$$

and $\theta(c_{20})=\tau$. With such a definition, by Lemma 2.2, symmetry σ has $g+1-q$ ovals and symmetry τ has 2 ovals, and the product of the symmetries is an element of order 2^n. By [**7**], these symmetries generate the full group of automorphisms for our surface.

For $q'=g$ we take an NEC group with signature

$$(0;+;[-];\{(2,\overset{g+1-q}{\dots},2),(-)\}) \tag{4}$$

and an epimorphism θ which sends the consecutive canonical reflections of the nonempty period cycle alternatively to σ and $\sigma(\sigma\tau)^{2^{n-1}}$ with $\theta(e_1)=\theta(e_2)=(\sigma\tau)^{2^{n-1}}$. Observe that, for $q<g-1$, the signature of Λ is maximal, by [**7**], and so we get a pair of non-conjugate symmetries with $g+1-q$ and 1 oval, by Lemma 2.2, and the product has order 2^n. Now, if $q=g-1$, then the symmetries in question are non-conjugate as they have different numbers of ovals.

Let us now deal $n\le 4, q'<g-1$ for case 2. First let $n=4$, and consider an NEC group with maximal, by [**7**], signature

$$(0;+;[-];\{(2,\overset{g-1-q}{\dots},2,4,16,2,\overset{g-1-q'}{\dots},2,8,16)\})$$

and an epimorphism sending the consecutive canonical reflections to

$$\underbrace{\sigma,\ \tau(\sigma\tau)^7,\ \sigma,\ \tau(\sigma\tau)^7,\dots,\sigma \text{ or } \tau(\sigma\tau)^7,\ \tau(\sigma\tau)^3}_{g+1-q}\ \dots$$
$$\dots\underbrace{\tau,\ \sigma(\tau\sigma)^7,\ \tau,\ \sigma(\tau\sigma)^7,\dots,\tau \text{ or } \sigma(\tau\sigma)^7,\ \sigma\tau\sigma,}_{g+1-q'}\ \sigma.$$

This definition gives rise to the desired configuration of symmetries.

For $n=3$, consider an NEC group with signature

$$(0;+;[-];\{(2,\overset{g-1-q}{\dots},2,8,2,\overset{g-1-q'}{\dots},2,8,8,8)\}).$$

This signature is maximal by [**7**], as both symmetries have at least three ovals due to $q' < g-1$. Take an epimorphism θ sending the first $g-q$ consecutive canonical reflections alternatively to σ and $\sigma(\sigma\tau)^4$, the next $g-q'$ reflections alternatively to τ and $\tau(\sigma\tau)^4$, and the last three reflections, respectively, to σ, τ and σ. This definition gives rise to the configuration we looked for.

Now we shall treat case 3 for $q' = g-1$. Observe, that q is an even integer, as g is odd. Therefore, the number $g+1-q$ of ovals of symmetry σ is also even. To obtain the configurations we are looking for, consider an NEC group with signature (4) and an epimorphism $\theta : \Lambda \to \mathrm{D}_{2^n}$ defined as in case 2 for $q' = g$ on all the consecutive canonical reflections with $\theta(e_1) = \theta(e_2) = 1$. Recall that the above signature is maximal if $g+1-q > 2$ and so, with such a definition, we obtain a Riemann surface having non-conjugate symmetries σ, τ with $g+1-q > 2$ and 2 ovals respectively, by Lemma 2.2. Now, for $q = g-1$, the signature for Λ is $(0; +; [-]; \{(2,2), (-)\})$, which is not maximal. Consider Λ' with signature $(0; +; [-]; \{(2,2,2,2,2)\})$, where $[\Lambda' : \Lambda] = 2$, and take an epimorphism $\theta' : \Lambda' \to \langle \sigma, \tau, \rho \mid \sigma^2, \tau^2, \rho^2, (\sigma\tau)^{2^n}, (\rho\sigma)^2, (\rho\tau)^2 \rangle$ defined on the canonical generators $c_0', c_1', c_2', c_3', c_4'$ of Λ' by sending them, respectively, to $\rho, \sigma, \sigma(\sigma\tau)^{2^{n-1}}, \rho, \tau$. Now take an embedding of Λ with canonical generators $c_{10}, c_{11}, e_1, c_{20}, e_2$ into Λ' by defining $c_{10} = c_1'$, $c_{11} = c_2'$, $c_{20} = c_4'$, $e_1 = c_3' c_0' = e_2^{-1}$. After restricting θ' to Λ, we obtain the original epimorphism θ. Moreover, here σ has 2 ovals and τ has 2 ovals, as desired, with the symmetries in question being non-conjugate (see also [**12**, Theorem 3.4]).

For $n = 3$, regardless of q', consider an NEC group with signature

$$(0; +; [-]; \{(2, \overset{g-1-q}{\ldots}, 2, 4, 8, 2, \overset{g-q'}{\ldots}, 2, 8)\}),$$

and observe that the symmetries have at least 4 ovals in total. Therefore, by [**7**], our signature is maximal. We define $\theta : \Lambda \to \mathrm{D}_{2^n} = \langle \sigma, \tau \rangle$ to map the consecutive canonical reflections to

$$\underbrace{\sigma,\ \tau(\sigma\tau)^3,\ \sigma,\ \tau(\sigma\tau)^3, \ldots, \sigma \text{ or } \tau(\sigma\tau)^3,}_{g-q}\ \sigma(\sigma\tau)^2 \ldots$$
$$\ldots \underbrace{\tau,\ \sigma(\tau\sigma)^3,\ \tau,\ \sigma(\tau\sigma)^3, \ldots, \tau \text{ or } \sigma(\tau\sigma)^3,}_{g+1-q'}\ \sigma.$$

This leads to the configuration we were looking for.

Observe now that, to prove the necessary condition, it is enough to show that, for the total number of ovals $t = 2g+2-q-q' = 2b+1$ with $q' < g-1, n > 4$ and $t = 2g+2-q-q' = 2b+2$ with $q' \neq g-1, n > 3$, there is no Riemann surface of genus $g = 2^u a + 1 = 2^{n-1}b + 1$ having a pair of non-conjugate $(M-q)$- and $(M-q')$-symmetries with the product of order 2^n. For, assume to the contrary that such a configuration indeed exists for one of these cases. Let $G = \langle \sigma, \tau \rangle = \mathrm{D}_{2^n}$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [m_1, \dots, m_r]; \{C_1, \dots, C_k, (-), \overset{l}{\dots}, (-)\}).$$

By the Hurwitz-Riemann formula, $\mu(\Lambda)/2\pi = (g-1)/2^n = a/2^{n-u} = b/2$ and, by Lemma 2.2, $t \leq 2l + s$, where t denotes the total number of ovals of σ and τ and s denotes the total number of link periods in the signature of Λ. Therefore, we have

$$\tag{5} \begin{aligned} 2\pi(g-1)/2^n &= \mu(\Lambda) \\ &\geq 2\pi(\varepsilon h - 2 + r/2 + k + l + s/4) \\ &\geq 2\pi(\varepsilon h - 2 + r/2 + k + l/2 + t/4), \end{aligned}$$

which gives

$$b/2 \geq \varepsilon h - 2 + r/2 + k + l/2 + t/4,$$

and in turn $\varepsilon h + r/2 + k + l/2 \leq 7/4$ or $\varepsilon h + r/2 + k + l/2 \leq 3/2$ for $t = 2b+1$ and $t = 2b+2$, respectively. Hence, the only configurations we have to consider are:

(a) $k = l = 1$, $r = h = 0$;

(b) $k = r = 1$, $l = h = 0$;

(c) $k = 1$, $l = r = h = 0$;

(d) $k = 0$, $l \geq 2$, $r \leq 1$.

In case (a), first let $t = 2b+1$. Observe that there must be at least two link periods equal 2^n in the nonempty period cycle, as otherwise, by Lemma 2.2, one of the symmetries has 1 or 2 ovals which is not our case. Thus, $b/2 \geq -1/2 + b/2 + 1/4 - 2/4 + 1 - 1/2^n$, and so $1/4 \leq 1/2^n$, which is impossible as $n > 2$.

Now let $t = 2b+2$. First of all, the reflection corresponding to the empty period cycle contributes two ovals as, otherwise, by Lemma 2.2,

$t-1 \le s$ and, by (5), $b/2 \ge -2+2+(t-1)/4$, which in turn gives $t \le 2b+1$, a contradiction. Therefore, as $r=0$ and there are two period cycles, the condition $\theta(e_1) = \theta(e_2) = 1$ must hold. Now, as above, if there is at least one link period greater than 2 in the nonempty period cycle, then $b/2 \ge -2+2+(2b-1)/4+1/2-1/8$, a contradiction again. Hence, we arrive at the signature of the form

$$(0; +; [-]; \{(2, \overset{2b}{\ldots}, 2), (-)\},$$

which is not our case, as $q' = g-1$ here.

In case (b) we have only one, nonempty, period cycle. As the order of two conjugates of the same symmetry is strictly smaller than 2^n, there are at least two link periods equal to 2^n in our signature. Now, again by (5), if there is a third link period greater than 2, then

$$b/2 \ge -1/2 + (t-3)/4 + 1/2 - 1/8 + 1 - 1/2^n \ge b/2 + 3/8 - 1/2^n$$

since $t = 2b+1$ or $2b+2$. This leads to a contradiction as $n > 3$. Therefore, we arrive at an NEC group Λ with a signature of the form

$$(0; +; [2^l]; \{(2, \ldots, 2, 2^n, 2, \ldots, 2, 2^n)\})$$

with $0 < l \le n$. However, here $b/2 = (g-1)/2^n = (t-2)/4 - 1/2^n + 1 - 1/2^l$, and so $3/4 \le 1/2^n + 1/2^l$, a contradiction.

In case (c) we also have exactly one, nonempty, period cycle, and we may assume that there are at least two link periods equal to 2^n. Recall that we always have an even number of these as the first and the last reflections are conjugate. If there are at least four, then by Lemma 2.2 both symmetries have at least two ovals and so $q' \le g-1$. Now, since $t \ge 2b+1$, we have $b/2 \ge -1 + (t-4)/4 + 2 - 1/2^{n-1} \ge (2b+1)/4 - 1/2^{n-1}$, and the equality holds only if $t = 2b+1$ and $n=3$, which is not our case here. So we may assume that there are exactly two maximal link periods. Now, if four of the remaining link periods are greater than 2, we get a contradiction since, in such a case, by (5), we have $b/2 \ge t/4 - 1/2^n \ge b/2 + 1/4 - 1/2^n$ and, on the other hand, $n > 2$. Hence, we may assume that there are at most three link periods $n' \le n'' \le n''' < 2^n$ greater than 2.

Now let $t = 2b+1$. If $n' > 2$ and $n''' \ge 8$, then $b/2 \ge -2 + 1 + (t-5)/4 + 1 - 1/2^n + 3/4 + 7/16 = b/2 + 3/16 - 1/2^n$ and

$3/16 - 1/2^n \leq 0$, a contradiction. Now, if $n' = n'' = n''' = 4$, we get $b/2 = b/2 + 1/2 - 1/2^n - 3/8$ and the equality holds only for $n = 3$ which is not our case again. Now let $n' = 2, n'' \geq 8$. Similarly to the previous case, we obtain $b/2 \geq b/2 + 1/2 - 1/2^n - 1/4 - 1/8$, and the equality holds for $n = 3$ which is not our case. For $n' = 2, n'' = 4, n''' \geq 8$, we have $b/2 \geq b/2 + 1/2 - 1/2^n - 1/4 - 1/8 - 1/16$, and the equality holds for $n = 4$, again not our case. Moreover, in the remaining cases by the Hurwitz-Riemann formula respectively we would have: if $n' = 2, n'' = n''' = 4$, then $b/2 = (2b-3)/4 - 1/2^n + 3/4 = b/2 - 1/2^n$, if $n' = n'' = 2, n''' = 4$, then $b/2 = b/2 - 1/2^n - 1/8$, and if $n' = n'' = n''' = 2$, then $b/2 = b/2 - 1/2^n - 1/4$, giving a contradiction in all the cases for $t = 2b + 1$. We omit the proof for $t = 2b + 2$, as it is also based on the Hurwitz-Riemann formula and assumption $n > 3$.

The only case left is (d), when $k = 0$, and the possibilities for the signature are $l = 2$, $r = 1$ and $l = 3$, $r = 0$. In the first case, observe that the only proper period must be equal to 2 and exactly one of the generators e_i has a nontrivial image under θ for the relation $\theta(x_1 e_1 e_2) = 1$ to hold. Hence, by Lemma 2.2, one of the symmetries has 1 and the other 2 ovals, so the total number of ovals is odd and equal $2b + 1$ with $q' = g$, a contradiction. Now, if $l = 3$, $r = 0$, then, by the Hurwitz-Riemann formula, $g = 2^n + 1$ and the maximal possible number of ovals is $2b + 2 = 6$. For the relation $\theta(e_1 e_2 e_3) = 1$ to hold, either none, or exactly two, of the generators e_i have nontrivial image under θ. If the first occurs, then one of the symmetries has exactly two ovals and so $q' = g - 1$, which is not our case. In the second case, the symmetries have only 4 ovals in total, and so the assumption $t \geq 2b+1$ does not hold.

The last thing we need to show is that $\mu_g(q, q') = n - 1$ for $2g + 2 - q - q' = 2b + 1$ with $q' < g - 1$, $n > 4$ and $2g + 2 - q - q' = 2b + 2$ with $q' \neq g - 1$, $n > 3$. Therefore, we shall construct, for all possible sets of parameters in question, a Riemann surface of genus g having a pair of non-conjugate $(M - q)$- and $(M - q')$-symmetries with the product of order 2^{n-1}. Consider an NEC group Λ with maximal signature

$$(0; +; [2, \overset{m}{\ldots}, 2]; \{(2, 2)^{\epsilon}, (-)^l\})$$

for some $m, l, \epsilon \geq 0$, and let θ be an epimorphism $\theta : \Lambda \to G = \mathrm{D}_{2^{n-1}} = \langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^{2^{n-1}} \rangle$. We shall divide our considerations into cases, depending on the parity of q and q'.

First let q and q' be even. Here we take $m = 2$, $\epsilon = 0$ and $l = b+1$ in the definition of Λ. We also define θ to map the canonical reflections corresponding to the first $(g+1-q)/2$ empty period cycles to σ and reflections corresponding to the remaining $(g+1-q')/2$ empty period cycles to τ.

Now let q and q' be odd. We take $m = \epsilon = 0$ and $l = b+2$ in the definition of Λ. Moreover, define θ as $\theta(c_{i0}) = \sigma$ for $i = 1, \ldots, (g+2-q)/2$, $\theta(c_{i0}) = \tau$ for all the remaining canonical reflections and $\theta(e_1) = \theta(e_{b+2}) = (\sigma\tau)^{2^{n-2}}$. Then, by Lemma 2.2, reflections corresponding to the first and the last period cycle contribute with only one oval to the symmetry σ and τ, respectively, while the remaining canonical reflections contribute with two ovals each.

Now let q be odd and q' even. Here we take $m = \epsilon = 1$, $l = b$ in the signature of Λ. The epimorphism θ is given by $\theta(e_2) = (\sigma\tau)^{2^{n-2}}$, $\theta(c_{10}) = \theta(c_{12}) = \sigma$, $\theta(c_{11}) = \sigma(\sigma\tau)^{2^{n-2}}$, $\theta(e_i) = 1$ for $i \neq 2$. Also, c_{i0} is mapped to σ for $2 \leq i \leq (g+2-q)/2$ and to τ for the remaining values of i. Observe also that, for q even and q' odd, it suffices to take the same definitions of Λ and θ changing only $\theta(c_{i0}) = \tau$ for $2 \leq i \leq (g+4-q')/2$ and $\theta(c_{i0}) = \sigma$ for the remaining empty period cycles.

Each of these definitions leads to a Riemann surface $X = \mathcal{H}/\ker\theta$, which, by the Hurwitz-Riemann formula, has genus g, having a pair of non-conjugate $(M-q)$- and $(M-q')$-symmetries σ, τ, generating the full group of automorphisms for X, with the product of order 2^{n-1}. This finishes the proof of the theorem. $\square$

Remark 3.11. The above theorem gives a lower bound on $\mu_g(q, q')$ for $g = 2^u a + 1$, $u \geq 2$ and $q + q' \geq 2g - g/2^{u-1}$. In such case we have $\mu_g(q, q') \geq u$.

4. Pairs of symmetries with one oval. Here we shall consider $(M-q)$-symmetries for $q = g$, i.e., symmetries with one oval. For convenience, we shall refer to them as 1-symmetries. In the beginning of the previous section we mentioned that this case has to be considered separately, as there is no n such that the inequalities on $q + q'$ given in (3) hold. The result below gives an upper bound for $\mu_g(g, g)$.

Theorem 4.1. *Let g and $n \geq 2$ be integers such that $2^{n-1} \leq g < 2^n$. Then $\mu_g(g,g) \leq n$, and this bound is attained only for $g = 2^n - 2^{n-l}$ for some $0 < l \leq n$.*

Proof. Let n, g be integers satisfying the condition $2^{n-1} \leq g < 2^n$. To show that the bound $\mu_g(g,g) \leq n$ holds, assume to the contrary that we have a Riemann surface of genus g, having a pair of 1-symmetries with the product of order 2^v for $v > n$. Also let $G = \langle \sigma, \tau \rangle = \mathrm{D}_{2^v} = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h; \pm; [m_1, \dots, m_r]; \{C_1, C_2, \dots, C_k\}). \tag{6}$$

Note that, by Lemma 2.2, there are at most two period cycles, as each of our symmetries has exactly 1 oval. Hence, $1 \leq k \leq 2$. By the Hurwitz-Riemann formula, we know that $(g-1)/2^v = \mu(\Lambda)/2\pi$. Now, as $g < 2^n$ and $v \geq n+1$, we see that

$$\mu(\Lambda)/2\pi = (g-1)/2^v < 1/2 - 1/2^v, \tag{7}$$

and so there are proper or link periods in the signature of Λ. First let $k = 2$. If both cycles are empty, then there must be at least two proper periods in the signature for the epimorphism $\theta : \Lambda \to G$ to exist. Indeed, both symmetries have 1 oval and so, by Lemma 2.2, the connecting generators must be mapped to $(\sigma\tau)^{2^{v-1}}$, which is the only nontrivial central element in G. Now, for $r \geq 2$, we see that $\mu(\Lambda)/2\pi \geq 1$ and (7) does not hold, a contradiction.

Now let just one of the cycles be nonempty. Similarly to the previous case, the connecting generator of the empty cycle must be mapped to $(\sigma\tau)^{2^{v-1}}$. By Lemma 2.2, there is only one link period, say 2^l, $l < v$, in the nonempty cycle, as each of the symmetries has exactly one oval. It follows that the connecting generator of the nonempty cycle is mapped to some non-central element in G as the images of the first and second reflections in the cycle are distinct, but conjugate. Recall that we have the relation $\theta(x_1 \dots x_r e_1 e_2) = 1$ in G. This means that there is at least one proper period $x_i \geq 4$ in Λ. But, then $\mu(\Lambda)/2\pi \geq 3/4 + 1/2 - 1/2^{l+1} \geq 1$ and (7) again does not hold.

If we have two nonempty cycles, then by Lemma 2.2, each of them has just one link period for the symmetries to have 1 oval each. Let these link periods be equal to $2^l, 2^u$ where $1 \leq l \leq u < v$. Now $\mu(\Lambda)/2\pi \geq 1 - 1/2^{l+1} - 1/2^{u+1} \geq 1/2$, which again contradicts (7).

So let us assume that there is only one, nonempty, period cycle in the signature of Λ. Observe that, as $q = q' = g$, this cycle must be of the form $(2^v, 2^v)$, by Lemma 2.2. Now, as $\mu(\Lambda) > 0$, it follows that $h + r > 0$. But then $\mu(\Lambda)/2\pi \geq 1/2 - 1/2^v$, again a contradiction.

Observe that, in the considerations above, we have only used the inequality $g < 2^n$. Similarly, one might prove that, for $g < 2^{n-1}$, the order is strictly smaller than 2^n.

Now we shall deal with the attainment of this bound. Let $2^{n-1} \leq g < 2^n$. To find the values of g for which the bound can be attained, one has to look for the possible NEC groups Λ for which $1/2 - 1/2^n \leq \mu(\Lambda)/2\pi < 1 - 1/2^n$, as the epimorphism $\theta : \Lambda \to \mathrm{D}_{2^n}$ must exist. Observe that, from the first part of the proof, it follows that the only such NEC signatures are $(0; +; [-]; \{(2^l), (2^u)\})$ for some $0 < l \leq u < n$ and $(0; +; [2^l]; \{(2^n, 2^n)\})$ for some $0 < l \leq n$. We shall show that, in fact, the only possible values of g for which the bound is attained are of the form given in the theorem.

First let Λ have signature $(0; +; [-]; \{(2^l), (2^u)\})$ for $0 < l \leq u < n$. For the epimorphism θ onto $G = \mathrm{D}_{2^n}$ to exist, it must be $l = u$. Define θ as $\theta(e_1) = \theta(e_2)^{-1} = (\tau\sigma)^{2^{n-l-1}}$, $\theta(c_{10}) = \sigma$, $\theta(c_{11}) = \sigma(\sigma\tau)^{2^{n-l}}$, $\theta(c_{21}) = \tau$, $\theta(c_{20}) = \tau(\sigma\tau)^{2^{n-l}}$. However, the signature of Λ is not maximal and, by [**7**], Λ is a subgroup of index 2 of an NEC group Λ' with maximal signature $(0; +; [2, 2]; \{(2^l)\})$ and canonical generators x_1, x_2, c'_0. Moreover, there is an epimorphism $\theta' : \Lambda' \to G' = G \rtimes \langle\rho\rangle$, where ρ is an involution acting on G by $\rho\sigma\rho = \tau$, defined by $\theta'(x_1) = \rho$, $\theta'(x_2) = \rho(\sigma\tau)^{2^{n-l-1}}$, $\theta'(c'_0) = \sigma$. This epimorphism extends θ after embedding Λ in Λ' as $c_{10} = c'_0$, $e_1 = x_2 x_1 = e_2^{-1}$, $c_{20} = x_2 c'_0 x_2$. Observe, however, that the symmetries σ, τ are conjugate in G', which is the full group of automorphisms of the surface in question. Moreover, as c'_0 and c'_1 are conjugate, there is no epimorphism extending θ such that σ, τ are non-conjugate. Therefore, the signature $(0; +; [-]; \{(2^l), (2^l)\})$ does not give the configuration we looked for.

Now let Λ have signature $(0; +; [2^l]; \{(2^n, 2^n)\})$ for some $0 < l \leq n$. We also have an epimorphism $\theta : \Lambda \to G = \mathrm{D}_{2^n}$ for which the consecutive canonical reflections c_0, c_1, c_2 are sent alternatively to σ, τ and $\sigma(\tau\sigma)^{2^{n-l+1}}$, $\theta(e) = \theta(x)^{-1} = (\sigma\tau)^{2^{n-l}}$. Here again, by [**7**], the signature of Λ is not maximal and Λ is an index 2 subgroup of Λ' with signature $(0;+;[-];\{(2,2^l,2,2^n)\})$ and canonical generators c_i', $i=0,1,2,3$. The last signature is maximal for $l<n$. For $l=n$, the group Λ' is an index 2 subgroup of an NEC group Λ'' with maximal signature $(0;+;[2];\{(2,2^n)\})$. Assume to the contrary that the symmetries are conjugate. Take $G' = \mathrm{D}_{2^n} \rtimes \langle \rho \mid \rho^2\rangle = \mathrm{D}_{2^{n+1}}$ and G'' such that $[G'':G]=4$. There is no epimorphism from Λ' to G', as there are only two conjugacy classes of symmetries in $\mathrm{D}_{2^{n+1}}$ and the product of non-conjugate symmetries has order 2^{n+1}. Therefore, for $l<n$, the proof is finished. Now for $l=n$ and $G''=G\rtimes H$ such that $H=\mathrm{Z}_2\times\mathrm{Z}_2=\langle\eta_1,\eta_2\rangle$ or $H=\mathrm{Z}_4=\langle\gamma\rangle$ with $\gamma^2\sigma\gamma^2$ conjugate to τ, we can take $G'=\langle\sigma,\tau\rangle\rtimes\langle\rho\rangle$, where $\rho\in H$ is an involution which makes σ conjugate to τ, and the proof is also finished. The last case is when γ is an element of order 4 such that $\gamma\in G''\setminus G$ with $\gamma\sigma\gamma^{-1}=\overline{\tau}$ conjugate to τ, $\gamma\tau\gamma^{-1}=\overline{\sigma}$ conjugate to σ but σ,τ not being conjugate in $G\rtimes\langle\gamma^2\rangle$. Observe that, in fact, $\overline{\sigma},\overline{\tau}\in G$. Indeed, if not, then $\overline{\tau}=\gamma^2\lambda$, where $\lambda\in G$, as otherwise we would have that $\gamma\in G$. But now $\gamma^2\lambda\notin G$ is a symmetry conjugate to τ and so it equals $\gamma\tau_1\gamma^{-1}$ for some τ_1 in G, conjugate to τ. We get a contradiction, as it follows that σ is conjugate to τ in G. Hence, $\overline{\sigma},\overline{\tau}\in G$. Observe now that, in G'', there are no involutions of the form $\gamma\delta$ or $\gamma^{-1}\delta$, for $\delta\in G$. Indeed, otherwise we would have $1=\gamma\delta\gamma\delta=(\gamma\delta\gamma^{-1})\gamma^2\delta$. In turn, we obtain that $\gamma^2\in G$, as $\gamma\delta\gamma^{-1}\in G$, a contradiction. Hence, the epimorphism from Λ'' to G'' cannot exist, as γ cannot be in the image. Therefore, in the construction of $\theta:\Lambda\to G$, the symmetries σ and τ are non-conjugate 1-symmetries on a Riemann surface of genus 2^n-2^{n-l}. $\square$

Now we shall give some results concerning a lower bound for $\mu_g(g,g)$. First let $g=2^u a$ for some $u\geq 1$ and a odd.

Proposition 4.2. *With g as above, $\mu_g(g,g)\geq u+1$.*

Proof. We shall construct a pair of 1-symmetries with the product of order 2^{u+1} on a Riemann surface of genus $g=2^u a$ as in the proposition. Consider an NEC group Λ with maximal signature

$$(0;+;[2,\overset{a}{\ldots},2];\{(2^{u+1},2^{u+1})\}),$$

where $a\geq 3$, and an epimorphism $\theta:\Lambda\to\mathrm{D}_{2^{u+1}}=\langle\sigma,\tau\rangle$ which sends all the canonical elliptic generators and the only connecting generator e_1 to $(\sigma\tau)^{2^u}$ and the consecutive canonical reflections alternatively to σ and τ. With such a definition, we obtain a Riemann surface $\mathcal{H}/\ker\theta$, which, by the Hurwitz-Riemann formula, has genus g, admitting a pair of 1-symmetries σ, τ, with $\sigma\tau$ of order 2^{u+1}. For $a = 1$, the proof follows from the previous theorem as $g = 2^u$ in such a case. $\square$

Now let $g = 2^u a + 1$ where $u \geq 1$ and $a \geq 3$ is odd.

Proposition 4.3. *For g being of the form above, $\mu_g(g, g) \geq u + 1$.*

Proof. Similarly to the previous result, we shall construct the configuration of symmetries in question. For, consider an NEC group Λ with maximal signature

$$(0; +; [2, \overset{a-1}{\ldots}, 2]; \{(2), (2)\})$$

and an epimorphism $\theta : \Lambda \to \mathrm{D}_{2^{u+1}} = \langle\sigma, \tau\rangle$ for which $\theta(x_i) = (\sigma\tau)^{2^u}$, $i = 1, \ldots, a-1$, $\theta(e_1) = \theta(e_2)^{-1} = (\tau\sigma)^{2^{u-1}}$, $\theta(c_{10}) = \sigma$, $\theta(c_{11}) = \sigma(\sigma\tau)^{2^u}$, $\theta(c_{20}) = \tau$, $\theta(c_{21}) = \tau(\sigma\tau)^{2^u}$. This definition leads to the configuration we looked for. $\square$

Remark 4.4. Observe that the bound above is attained in a sense that there exist infinitely many values of g of the form $g = 2^u$ for which the maximal possible order of the product of two 1-symmetries is 2^{u+1}. Indeed, as we know, for $g < 2^{u+1}$, the inequality $\mu_g(g, g) \leq u + 1$ holds. On the other hand, for g, as above $\mu_g(g, g) \geq u + 1$ by the previous results. Hence, the equality holds. $\square$

Acknowledgments. The author is grateful to the referee for all the valuable comments and corrections, especially for pointing out gaps in the proofs, leading in particular to discovery of some errors in the earlier work [**12**], which we mention and correct in this paper.

REFERENCES

1. E. Bujalance, *Normal* NEC *signatures*, Illinois J. Math. **26** (1982), 519–530.

2. E. Bujalance, J.F. Cirre, J.M. Gamboa and G. Gromadzki, *Symmetries of compact Riemann surfaces*, Lect. Notes Math., Springer-Verlag, to appear.

3. E. Bujalance and A.F. Costa, *A combinatorial approach to the symmetries of* M *and* $M-1$ *Riemann surfaces*, Discrete Groups and Geometry, Lond. Math. Soc. Lect. Note **173** (1993), 16–25.

4. ———, *On symmetries of p-hyperelliptic Riemann surfaces*, Math. Ann. **308** (1997), 31–45.

5. E. Bujalance, A.F. Costa and D. Singerman, *Application of Hoare's theorem to symmetries of Riemann surfaces*, Ann. Acad. Sci. Fenn. **18** (1993), 307–322.

6. E. Bujalance and D. Singerman, *The symmetry type of a Riemann surface*, Proc. Lond. Math. Soc. **51** (1985), 501–519.

7. J.L. Estevez and M. Izquierdo, *Non-normal pairs of non-euclidean crystallographic groups*, Bull. Lond. Math. Soc. **38** (2006), 113–123.

8. G. Gromadzki, *On a Harnack-Natanzon theorem for the family of real forms of Riemann surfaces*, J. Pure Appl. Alg. **121** (1997), 253–269.

9. M. Izquierdo and D. Singerman, *Pairs of symmetries of Riemann surfaces*, Ann. Acad. Sci. Fenn. **23** (1998), 3–24.

10. E. Kozłowska-Walania, *On p-hyperellipticity of doubly symmetric Riemann surfaces*, Publ. Matem. **51** (2007), 291–307.

11. ———, *Non-central fixed point free symmetries of bisymmetric Riemann surfaces*, Osaka J. Mathematics, accepted.

12. ———, *On commutativity and ovals for a pair of symmetries of a Riemann surface*, Colloq. Math. **109** (2007), 61–69.

13. S.M. Natanzon, *Finite groups of homeomorphisms of surfaces and real forms of complex algebraic curves*, Trans. Moscow Math. Soc. **51** (1989), 1–51.

14. ———, *Topological classification of pairs of commuting antiholomorphic involutions of Riemann surfaces*, Uspek. Mat. Nauk. **41** (1986), 191–192.

15. D. Singerman, *Finitely maximal Fuchsian groups*, J. Lond. Math. Soc. **6** (1972), 29–38.

16. ———, *On the structure of non-euclidean crystallographic groups*, Proc. Camb. Phil. Soc. **76** (1974), 233–240.

Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland and Mathematical Institute of the Polish Academy of Sciences, Śniadeckich 8, 00-956 Warszawa, skr. poczt. 21, Poland
Email address: retrakt@math.univ.gda.pl