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SPECTRAL PROPERTIES OF THE SIMPLE LAYER POTENTIAL TYPE OPERATORS

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ABSTRACT. We establish the exact asymptotical behavior of singular values of the simple layer potential type operators.

1. Introduction and notation. It is well known that any function $f \in C^2(\overline{\Omega})$ $(\Omega \subset \mathbf{R}^n, n \geq 2)$ may be expressed as

$$f(x) = \int_\Omega u(x-y) \triangle f(y)\, dy + \int_{\partial\Omega} f(y) \frac{\partial u(x-y)}{\partial n_y}\, dS_y - \int_{\partial\Omega} \frac{\partial f}{\partial n_y} u(x-y)\, dS_y \quad (x \in \Omega),$$

where

$$u(x) = \begin{cases} -1/(n-2)\,\sigma_n\,|x|^{n-2}, & n > 2 \\ -1/2\pi \ln 1/|x|, & n = 2 \end{cases}$$

and $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of $(n-1)$-dimensional sphere.

Here, Ω is a domain in $\mathbf{R}^n$, $\triangle$ is the Laplace operator, $\partial/\partial n_y$ denotes the derivative in the direction of external normal to $\partial\Omega$ with respect to y and dS_y denotes the area element.

Operators

$$g \longmapsto \int_\Omega u(x-y)\, g(y)\, dy,$$

$$g \longmapsto \int_{\partial\Omega} -u(x-y)\, g(y)\, dS_y$$

and

$$g \longmapsto \int_{\partial\Omega} \frac{\partial u(x-y)}{\partial n_y}\, g(y)\, dS_y$$

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are called, respectively, operators of volume potential, simple layer potential and double layer potential type.

Spectral properties of volume potential type (and more general) operators on $L^2(\Omega)$ have been investigated in a number of papers (see for example [**1**–**5, 7, 16**]).

We use the potentials of the simple layer and double layer to solve boundary problems.

We seek the solution of the inner Dirichlet problem in the form of a potential of double layer. Moreover, we seek the solution of the external Dirichlet problem in the form of a potential of double layer plus a constant. Finally, we seek the solution of the Neumann problem (inner or external) in the form of a potential of simple layer.

In this way, solving the boundary problem is reduced to integral equations with polar kernels.

As for a review of the classical methods of the potential simple and double layer for harmonic function in domains with smooth boundary, see [**9, 14**].

In the paper [**10**], a general and rigorous account of Poincare's variational problem (which is related to solvability of Dirichlet problem) is given using method of modern analysis. More precisely, the authors interpreted Poincare's variational principle as a nonselfadjoint eigenvalue problem for angle operator between two distinct pairs of subspaces of potentials.

An example of use of the method of potentials (applied to equations of the second order with nonconstant coefficients) may be seen in [**11**].

To the author's knowledge, spectral properties of the simple layer potential type operators (acting from $L^2(\partial\Omega)$ to $L^2(\Omega)$) haven't been studied so far.

In this paper, the exact asymptotic of the singular values of the simple layer potential type operators will be established (for $n = 2$), as well as the connection with the length of the boundary $\partial\Omega$ of domain Ω.

For the rest of this paper, $|\partial\Omega|$ will denote the length of the boundary $\partial\Omega$ of domain ($\Omega \subset \mathbf{C}$), while $L^2(\partial\Omega)$ will denote the class of functions f such that $\int_{\partial\Omega} |f|^2 |\,dz| < \infty$ ($|dz|$ is the arclength measure on $\partial\Omega$).

Finally, dA will denote Lebesgue measure in Ω.

If H_1 and H_2 are Hilbert spaces and $T : H_1 \to H_2$ is a compact operator, then $T^*T : H_1 \to H_1$ is compact, selfadjoint and nonnegative and $|T| = \sqrt{T^*T}$ is a unique nonnegative square root of operator T^*T.

Eigenvalues of operator $|T|$, i.e., $\lambda_n(|T|)$, are called the singular values of operator T. Hence, $s_n(T) = \lambda_n(|T|)$ (where $s_n(T)$ denotes singular values of operator T). For properties of singular values in more detail see, for example, [**8, 15**].

Let $\mathbf{N}_t(S)$ denote the singular values distribution function of operator S, i.e.,

$$\mathbf{N}_t(S) = \sum_{s_n(S) \geq t} 1 \quad (t > 0).$$

The symbol $\int_K S(x,y) \cdot d\mu$ will denote an integral operator on $L^2(K, \mu)$, with kernel $S(x,y)$.

2. Main result.

Theorem 1. *Let Ω be a bounded, simply connected domain in* $\mathbf{C}$ *with an analytic boundary, and let*

$$T : L^2(\partial\Omega) \longrightarrow L^2(\Omega)$$

be an operator defined by

$$Tf(z) = \frac{1}{2\pi} \int_{\partial\Omega} \ln|z - \xi| \; f(\xi) \; |d\xi|.$$

Then

$$s_n(T) \sim \left(\frac{|\partial\Omega|}{2\pi n}\right)^{3/2}, \quad n \to \infty.$$

(*Here, symbol $a_n \sim b_n$ denotes the fact that* $\lim_{n\to\infty}(a_n/b_n) = 1$).

Note that operator T assigns harmonic functions on $\mathbf{C} \setminus \partial\Omega$ to functions in space $C(\partial\Omega)$ ([**14**]).

Furthermore, if $f \in L^2(\Omega)$, then

$$\int_\Omega \ln|z - \xi| \; f(\xi) \; dA(\xi) \in C\left(\overline{\Omega}\right).$$

Remark 1. Eigenvalues and eigenfunctions of the operator of the logarithmic potential on the unit disc were found in [**1, 2**]. Also, in the case of arbitrary bounded domain $\Omega \subset \mathbf{C}$, two-sided estimates of the growth of singular values of that operator were given. In addition, asymptotic behavior of singular values of the product of the Bergman projection and the operator of logarithmic potential was studied.

An exact asymptotic of $s_n(T)$, where T is the operator from Theorem 1, is given. The operator T acts from $L^2(\partial\Omega)$ into $L^2(\Omega)$ and the growth order of $s_n(T)$ is precisely between the growth order's singular values of logarithmic potential operator and singular values of its product with Bergman projection.

In the presence of corners the investigation of $s_n(T)$, using our method, is more difficult because the derivative of conformal map $\varphi : D \to \Omega$ (D is unit disc) has singularities on $\partial\Omega$ in that case. The presence of corner manifests (perhaps) in the higher-order terms of asymptotics of $s_n(T)$.

Note that the assumption of the analytic boundary can be relaxed; we choose to work with the analytic boundary to make exposition simpler. However, using Kellog's theorem, one can prove the same result for C^k boundary (k large enough).

Several lemmas will be needed for the proof of Theorem 1.

Lemma 1 [**3**]. *Let H be a compact operator, $r > 0$, and let for any $\varepsilon > 0$ there exist decomposition $H = H'_\varepsilon + H''_\varepsilon$ where H'_ε and H''_ε are compact operators for which the following holds*:

1. $\lim\limits_{n\to\infty} n^r s_n(H'_\varepsilon) = C(H'_\varepsilon)$
2. $\overline{\lim\limits_{n\to\infty}}\, n^r s_n(H''_\varepsilon) \le \varepsilon$.

Then, there exists $\lim_{\varepsilon\to 0^+} C(H'_\varepsilon) = C(H)$ *and, furthermore,*

$$\lim_{n\to\infty} n^r s_n\,(H) = C\,(H)\,.$$

Lemma 2. *If $H : L^2(0,a) \to L^2(0,a)$ is an operator defined by*

$$H\,f\,(x) = \int_0^a \,(x+y)^2 \ln\,(x+y)\ f\,(y)\ dy,$$

then

$$\lim_{n\to\infty} n^3 s_n(H) = 0.$$

Proof. Applying integration by parts, one obtain

$$H = R_8 + 2H_1 J^2$$

where R_8 is an operator on $L^2(0,a)$ of rank ≤ 8,

$$H_1, J : L^2(0,a) \longrightarrow L^2(0,a)$$
$$H_1 f(x) = \int_0^a \ln(x+y)\, f(y)\, dy \quad \text{and}$$
$$Jf(x) = \int_0^x f(y)\, dy.$$

Since ([**6**]), $\lim_{n\to\infty} n s_{n+1}(H_1) = 0$ and, since $s_n(J) \sim \text{const} \cdot 1/n$, from the property of singular values of the product of operators and Ky-Fan theorem [**8**] it follows that

$$\lim_{n\to\infty} n^3 s_n(H) = 0. \qquad \square$$

Lemma 3. *If* $p \in C[-\pi,\pi]$, $p(x) > 0$ *on* $[-\pi,\pi]$, *then for operator* $D : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ *defined by*

$$Df(x) = \int_{-\pi}^{\pi} p(x)\, p(y)\, (x-y)^2 \ln|x-y|\, f(y)\, dy,$$

the following holds:

$$s_n(D) \sim \frac{2}{\pi^2 n^3} \left(\int_{-\pi}^{\pi} p(x)^{2/3}\, dx \right)^3, \quad n \to \infty.$$

Proof. Let us first demonstrate that

$$s_n\left(\int_{-1}^{1} (x-y)^2 \ln|x-y| \cdot dy \right) \sim \frac{16}{n^3 \pi^2}. \tag{1}$$

It is known ([**13**]) that

$$\int_{\mathbf{R}} e^{it\cdot\xi} K_0(|\xi|)\, d\xi = \frac{\pi}{\sqrt{1+t^2}}. \tag{2}$$

(Here $K_0(z) = -I_0(z)\ln(z/2) + \sum_{k=0}^{\infty}[(z/2)^{2k}/(k!)^2]\cdot\Psi(k+1)$ is a McDonald function, $I_0(z) = \sum_{k=0}^{\infty}[(z/2)^{2k}/(k!)^2]$, $\Psi(n+1) = \sum_{k=1}^{n}\frac{1}{k} - \gamma$, γ is an Euler constant, $\Psi(1) = -\gamma$.)

From (2), it follows that

$$\int_{-\infty}^{\infty} e^{it\cdot\xi}\xi^2 K_0(|\xi|)\, d\xi = -\frac{d^2}{dt^2}\left(\pi\left(1+t^2\right)^{-1/2}\right) \sim -\frac{2\pi}{t^3}, t \to +\infty.$$

Applying [**7**, Theorem 1] to integral operator $\int_{-1}^{1} k(x-y)\cdot dy$ with kernel $k(t) = t^2 K_0(|t|)$, Lemma 2 and the Ky-Fan theorem, one obtains (1).

(Theorem 1 from [**7**] says that if k is an even function which decreases rapidly at ∞ and $K(\xi) = \int_{\mathbf{R}} e^{it\xi}\, k(t)\, dt$ is a decreasing function with the asymptotic $|K(\xi)| \sim L(\xi)/\xi^r$, $\xi \to +\infty$ ($r \in \mathbf{N}$, L is a slowly varying function) and if

$$s_n\left(\int_0^2 k(x+y)\cdot dy\right) = o\left(\frac{L(n)}{n^r}\right),$$

then

$$s_n\left(\int_{-1}^{1} k(x-y)\cdot dy\right) \sim \frac{L(n)}{(n\pi/2)^r}, \quad n \to \infty.)$$

From (1), it follows that

$$s_n\left(\int_{\triangle} (x-y)^2 \ln|x-y|\cdot dy\right) \sim \frac{2|\triangle|^3}{n^3\pi^2}, \quad n \to \infty, \tag{3}$$

where $\triangle$ is an interval and $|\triangle|$ its length.

Let $\triangle_i = [-\pi + (2\pi/N)(i-1), -\pi + (2\pi/N)i]$, $\xi_i \in \triangle_i$, $i = 1, 2, \ldots, N$. Let P_i ($1 \le i \le N$) denote linear operators on $L^2[-\pi,\pi]$ defined by

$$P_i f(x) = \mathcal{X}_{\triangle_i}(x)\cdot f(x)$$

($\mathcal{X}_S$-characteristic function of set S.)

Furthermore, let $D_i : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ $(1 \le i \le N)$ denote operators defined by

$$D_i f(x) = \int_{-\pi}^{\pi} p(\xi_i)^2 (x-y)^2 \ln|x-y| f(y)\, dy.$$

Then

$$D = \sum_{i=1}^{N} P_i D_i P_i + \sum_{i=1}^{N} P_i (D - D_i) P_i + \sum_{\substack{i \neq j \\ i,j=1}}^{N} P_i D P_j.$$

Obviously,

$$\begin{aligned} &s_{2n}(P_i(D-D_i)P_i) \\ &\quad = s_{2n}\bigg(\int_{\triangle_i} p(x)(p(y)-p(\xi_i))(x-y)^2 \ln|x-y| \cdot dy \\ &\qquad\qquad + \int_{\triangle_i} p(\xi_i)(p(x)-p(\xi_i))(x-y)^2 \ln|x-y| \cdot dy\bigg). \end{aligned}$$

Let $\varepsilon > 0$. Choose N large enough such that $|p(x) - p(\xi_i)| < \varepsilon$ for all $x \in \triangle_i$ and all $1 \le i \le N$. Keeping this in mind, one obtains

$$s_{2n}(P_i(D-D_i)P_i) \le 2 \cdot \varepsilon \, \|p\|_\infty \cdot s_n\bigg(\int_{\triangle_i} (x-y)^2 \ln|x-y| \cdot dy\bigg). \tag{4}$$

From (3) and (4), it follows that

$$s_{2n}(P_i(D-D_i)P_i) \le 2 \cdot \varepsilon \cdot \|p\|_\infty \cdot \frac{C' |\triangle_i|^3}{n^3}$$

where constant C' does not depend on n nor on $i \in \{1, 2, \dots, N\}$.

From the last inequality, it follows that

$$s_n(P_i(D-D_i)P_i) \le C \frac{|\triangle_i|^3}{n^3} \varepsilon \tag{5}$$

where the constant C does not depend on n nor on $i \in \{1, 2, \dots, N\}$.

Since operators $P_i(D-D_i)P_i$ $(1 \le i \le N)$ are orthogonal, it follows that

$$\mathbf{N}_t\left(\sum_{i=1}^{N} P_i\left(D-D_i\right)P_i\right)=\sum_{i=1}^{N}\mathbf{N}_t\left(P_i\left(D-D_i\right)P_i\right). \tag{6}$$

From (5) it follows that

$$\mathbf{N}_t\left(P_i\left(D-D_i\right)P_i\right)\le\left(\frac{C\varepsilon}{t}\right)^{1/3}\left|\triangle_i\right|,\quad i=1,2,\dots,N,$$

and so, from (6), one obtains

$$\mathbf{N}_t\left(\sum_{i=1}^{N} P_i\left(D-D_i\right)P_i\right)\le\left(\frac{C\varepsilon}{t}\right)^{1/3}\sum_{i=1}^{N}\left|\triangle_i\right|=2\pi\left(\frac{C\varepsilon}{t}\right)^{1/3}.$$

Putting in this inequality $t=s_n(\sum_{i=1}^{N}P_i(D-D_i)P_i)$, one obtains

$$s_n\left(\sum_{i=1}^{N} P_i\left(D-D_i\right)P_i\right)\le\frac{8\pi^3C\varepsilon}{n^3}. \tag{7}$$

According to Lemma 2, it follows that for all $i,j\in\{1,2,\dots,N\}$ such that $|i-j|=1$, the following holds

$$\lim_{n\to\infty} n^3 s_n\left(P_iDP_j\right)=0. \tag{8}$$

Indeed, if for example $i=j+1$, then the singular values of the operator P_iDP_{i+1} are equal to the singular values of the operator

$$\int_{\Delta_i} p\left(x\right)p\left(y\right)\left(x-y\right)^2\ln\left|x-y\right|\cdot dy: L^2\left(\Delta_i\right)\longrightarrow L^2\left(\Delta_{i+1}\right).$$

A change of variables $x=(2\pi/N)i-x_1$, $y=(2\pi/N)i+y_1$, $x_1,y_1\in[0,(2\pi/N)]$ (keeping in mind that the mappings $f\mapsto f((2\pi/N)i\mp x)$ are isometries between $L^2(\Delta_i)(L^2(\Delta_{i+1}))$ and $L^2(0,(2\pi/N)))$ reduces the previous operator to the form for application of Lemma 2.

If $|i-j|\ge 2$, then singular values of operators P_iDP_j have an exponential decrease (Birman-Solomyak theorem [**4**, Theorem 11.3, page 78]) and so, from (8) and based on properties of singular values of the sum of operators, one obtains

$$\lim_{n\to\infty} n^3 s_n\left(\sum_{\substack{i\neq j\\ i,j=1}}^{N} P_i D P_j \right) = 0. \tag{9}$$

Let

$$E_N'' = \sum_{i=1}^{N} P_i \left(D - D_i\right) P_i + \sum_{\substack{i\neq j\\ i=1}}^{N} P_i D P_j.$$

Then, from (7) and (8), it follows that

$$\overline{\lim_{n\to\infty}}\, n^3 s_n\left(E_N''\right) \leq 8\pi^3 C \cdot \varepsilon. \tag{10}$$

Let

$$E_N' = \sum_{i=1}^{N} P_i D_i P_i$$

Operators $P_i D_i P_i$ $(1 \leq i \leq N)$ are mutually orthogonal and so

$$\mathbf{N}_t\left(E_N'\right) = \sum_{i=1}^{N} \mathbf{N}_t\left(P_i D_i P_i\right).$$

From (3), it follows that

$$\mathbf{N}_t\left(P_i D_i P_i\right) \sim \left(\frac{2p\left(\xi_i\right)^2 \left|\triangle_i\right|^3}{\pi^2 t} \right)^{1/3}, \quad t \to 0^+$$

and so we obtain

$$\lim_{t\to 0^+} t^{1/3} \mathbf{N}_t\left(E_N'\right) = \sum_{i=1}^{N} \left(\frac{2}{\pi^2}\right)^{1/3} p\left(\xi_i\right)^{2/3} \left|\triangle_i\right| \left(\stackrel{\text{def}}{=} \omega_N\right). \tag{11}$$

Putting $t = s_n(E_N')$ in (11), one obtains

$$\lim_{n\to\infty} n^3 s_n\left(E_N'\right) = \omega_N^3. \tag{12}$$

Since $D = E'_N + E''_N$, from (10) and (12) and applying Lemma 1, we obtain

$$\lim_{n\to\infty} n^3 s_n(D) = \lim_{N\to\infty} \omega_N^3 = \frac{2}{\pi^2}\left(\int_{-\pi}^{\pi} p(x)^{2/3}\,dx\right)^3,$$

which proves Lemma 3. $\square$

Lemma 4. *Let*

$$k_0(t) = \frac{1}{16\pi}\sum_{n\geq 1}\frac{e^{int}+e^{-int}}{n^2(n+1)},$$

$p \in C[-\pi,\pi]$, $p(x) > 0$ *on* $[-\pi,\pi]$ *and* $W_0 : L^2(-\pi,\pi) \to L^2(-\pi,\pi)$

$$W_0 f(x) = \int_{-\pi}^{\pi} p(x)\,p(y)\,k_o(x-y)\,f(y)\,dy.$$

Then

$$s_n(W_0) \sim \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)^{2/3}\,dx\right)^3 \cdot \frac{1}{n^3}, \quad n \to \infty.$$

Proof. Since

$$-\ln\left(2\sin\frac{|t|}{2}\right) = \sum_{n=1}^{\infty}\frac{\cos nt}{n}, |t| < \pi, \quad t \neq 0,$$

(this follows from the expansion

$$\ln\left(1 - re^{i|x|}\right) = -\sum_{n=1}^{\infty} r^n\frac{e^{in|x|}}{n}$$

taking real part from both sides and putting $r \to 1-$), transforming function k_0 (for $|t| < \pi$, $t \neq 0$) we get

$$k_0(t) = \frac{1}{8\pi}\left[-\frac{1}{2} + \frac{5}{4}\cos t - \sum_{n=2}^{\infty}\frac{\cos nt}{n^2(n^2-1)}\right]$$

$$
\begin{aligned}
&+\left[\frac{1-\cos t}{8\pi}\ln\left(\frac{2\sin(|t|/2)}{|t|\,(\pi-(|t|/2))}\right)\right]\\
&+\left[\frac{1-\cos t-\frac{t^2}{2}}{8\pi}\ln|t|\right]\\
&+\frac{1}{8\pi}\left[(1-\cos t)\ln\left(\pi-\frac{|t|}{2}\right)\right]\\
&+\left[\frac{1}{16\pi}t^2\ln|t|\right]\\
=&\sum_{i=1}^{5} r_i\,(t)\,.
\end{aligned}
$$

Let $M_p, W_1, W_2, W_3, W_4, W_5$ be linear operators on $L^2(-\pi,\pi)$ defined by

$$M_p f\,(x) = p\,(x)\, f\,(x)$$

and

$$W_i = \int_{-\pi}^{\pi} r_i\,(x-y)\cdot dy \quad (i=1,2,3,4,5).$$

Then

$$W_0 = \sum_{i=1}^{5} M_p W_i M_p. \tag{13}$$

It is easily checked that

$$s_n\,(W_1) = O\left(\frac{1}{n^4}\right). \tag{14}$$

From the Birman-Solomyak theorem ([**4**, Theorem 11.3, page 78]) on asymptoticness of singular values of operators with analytic kernel it follows that

$$s_n\,(W_2) = O\left(e^{-\delta n}\right), \tag{15}$$

$\delta > 0$ and does not depend on n.

From [**8**, page 157] it follows that

$$s_n(W_3) = O\left(\frac{1}{n^{7/2}}\right), \tag{16}$$

and from Lemma 2, it follows that

$$s_n(W_4) = o\left(n^{-3}\right). \tag{17}$$

Indeed, if $P_+, P_- : L^2(-\pi,\pi) \to L^2(-\pi,\pi)$ are operators defined by $P_+ f = \mathcal{X}_{[0,\pi]} \cdot f$, $P_- f = \mathcal{X}_{[-\pi,0]} \cdot f$, then we have

$$W_4 = P_+W_4P_+ + P_-W_4P_+ + P_+W_4P_- + P_-W_4P_-.$$

For operators $P_+W_4P_+$, $P_-W_4P_-$ we have (according to [**8**, pages 154–157])

$$s_n(P_+W_4P_+) = o\left(n^{-p}\right)$$
$$s_n(P_-W_4P_-) = o\left(n^{-p}\right)$$

for every $p \in \mathbf{N}$.

The isometry $f \mapsto f(-x)$ (between $L^2(0,\pi)$ and $L^2(-\pi,0)$) reduce operators $P_-W_4P_+$, $P_+W_4P_-$ to the suitable form for application to Lemma 2 and so,

$$s_n(P_+W_4P_-) = o\left(n^{-3}\right)$$
$$s_n(P_-W_4P_+) = o\left(n^{-3}\right).$$

Since, according to Lemma 3,

$$s_n(M_pW_5M_p) \sim \frac{1}{(2n\pi)^3}\left(\int_{-\pi}^{\pi} p(x)^{2/3}\,dx\right)^3, \quad n \to \infty,$$

and since operator M_p is bounded, from (13)–(17) and the Ky-Fan theorem, it follows that

$$s_n(W_0) \sim \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} p(x)^{2/3}\,dx\right)^3 \cdot \frac{1}{n^3}, \quad n \to \infty. \qquad \square$$

Let φ be the conformal mapping of $D = \{z : |z| < 1\}$ onto Ω. According to Schwarz's theorem, φ can be analytically extended to the neighborhood $\overline{D}$, and $\varphi'(z) \neq 0$ on $\overline{D}$.

Let

$$\mathcal{A}_3(z,\xi)=\frac{\varphi'(z)\cdot\overline{\varphi'(\xi)}}{4\pi^2}\int_D \ln|t-z|\cdot\ln|t-\xi|\;dA(t)$$

and $G : L^2(\partial D)\to L^2(\partial D)$ be an operator defined by

$$Gf(z)=\int_{\partial D}\sqrt{\varphi'(z)}\sqrt{\overline{\varphi'(\xi)}}\mathcal{A}_3(z,\xi)\,f(\xi)\;|d\xi|\,.$$

Lemma 5. *The following asymptotic holds*:

$$s_n(G)\sim\left(\frac{|\partial\Omega|}{2\pi n}\right)^3,\quad n\to\infty.$$

Proof. Let

$$\Omega_0(z,\xi)=\frac{1}{4\pi^2}\int_D \ln|t-z|\cdot\ln|t-\xi|\;dA(t)$$

and

$$G_0(z,\xi)=\varphi'(z)^{3/2}\,\overline{\varphi'(\xi)}^{3/2}\cdot\Omega_0(z,\xi)\,.$$

Then

$$Gf(z)=\int_{\partial D}G_0(z,\xi)\,f(\xi)\;|d\xi|\,.$$

From this, it follows that

$$V_2G=WV_1, \tag{18}$$

where $V_1,V_2 : L^2(-\pi,\pi)\to L^2(-\pi,\pi)$ are isometries defined by

$$(V_1f)(x)=f\left(e^{ix}\right)\cdot\frac{\overline{\varphi'(e^{ix})}^{3/2}}{|\varphi'(e^{ix})|^{3/2}}$$

and

$$(V_2f)(x)=f\left(e^{ix}\right)\cdot\frac{\left|\varphi'\left(e^{ix}\right)\right|^{3/2}}{\varphi'\left(e^{ix}\right)^{3/2}},$$

and $W : L^2(-\pi,\pi) \to L^2(-\pi,\pi)$ is an operator defined by

$$Wf(x) = \int_{-\pi}^{\pi} p(x)\, p(y)\, \Omega_0\left(e^{ix}, e^{iy}\right) f(y)\, dy$$

where $p(x) = |\varphi'(e^{ix})|^{3/2}$.

From (18) it follows that $s_n(G) = s_n(W)$ and so, keeping in mind that

$$\begin{aligned}\Omega_0\left(e^{ix}, e^{iy}\right) &= \frac{1}{4\pi^2}\int_D \ln\left|t - e^{ix}\right| \cdot \ln\left|t - e^{iy}\right|\, dA(t)\\ &= k_0(x-y)\end{aligned}$$

(since

$$\begin{aligned}\ln\left|t - e^{ix}\right| &= \frac{1}{2}\ln\left(1 - te^{-ix}\right) + \frac{1}{2}\ln\left(1 - \overline{t}e^{ix}\right)\\ &= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(t^n e^{-inx} + \overline{t}^n e^{inx}\right)\end{aligned}$$

and

$$\ln\left|t - e^{iy}\right| = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(t^n e^{-iny} + \overline{t}^n e^{iny}\right), \quad |t| < 1,$$

then multiplying and integrating with respect to $dA(t)$ over D, we obtain previous equality), applying Lemma 4 and substituting $p(x) = |\varphi'(e^{ix})|^{3/2}$ in it, one obtains

$$s_n(G) \sim \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\varphi'\left(e^{ix}\right)\right|\, dx\right)^3 \frac{1}{n^3} = \left(\frac{|\partial\Omega|}{2\pi n}\right)^3, \quad n \to \infty,$$

which proves Lemma 5. $\square$

3. Proof of Theorem 1. It is easily established that operator $T^* : L^2(\Omega) \to L^2(\partial\Omega)$ acts in the following way

$$T^* f(z) = \frac{1}{2\pi}\int_\Omega \ln|\xi - z|\, f(\xi)\, dA(\xi),$$

and so $T^*T : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is an operator defined by

$$T^*Tf(z) = \int_{\partial\Omega} K(z,\xi)\, f(\xi)\, |d\xi|$$

where

$$K(z,\xi)=\frac{1}{4\pi^2}\int_\Omega \ln|t-z|\cdot\ln|t-\xi|\ dA(t).$$

Let $V : L^2(\partial\Omega)\to L^2(\partial D)$ and $S_1 : L^2(\partial D)\to L^2(\partial D)$ be operators defined by

$$Vf(z)=f(\varphi(z))\sqrt{\varphi'(z)}$$

and

$$S_1 f(z)=\int_{\partial D}\sqrt{\varphi'(z)}\sqrt{\overline{\varphi'(\xi)}}K(\varphi(z),\varphi(\xi))\,f(\xi)\ |d\xi|.$$

Since V is an isometry and

$$VT^*T=S_1V,$$

one obtains

$$s_n(T^*T)=s_n(S_1),$$

i.e.,

$$s_n^2(T)=s_n(S_1). \tag{19}$$

Let $M_1, S_2 : L^2(\partial D)\to L^2(\partial D)$ be operators defined by

$$S_2 f(z)=\int_{\partial D}K(\varphi(z),\varphi(\xi))\,f(\xi)\ |d\xi|$$

and

$$M_1 f(z)=\sqrt{\varphi'(z)}f(z).$$

(Operator M_1 is bounded because φ' is analytic function on neighborhood $\overline{D}$.)

Then we have

$$M_1^* f(z)=\sqrt{\overline{\varphi'(z)}}f(z)$$

and

$$S_1=M_1S_2M_1^*. \tag{20}$$

The kernel of operator S_2, i.e., the function $K(\varphi(z),\varphi(\xi))$ may be expanded in the following way:

$$\begin{aligned}
&K\left(\varphi\left(z\right),\varphi\left(\xi\right)\right)\\
&\qquad=\frac{1}{4\pi^2}\int_D \ln\left|\frac{\varphi\left(t\right)-\varphi\left(z\right)}{t-z}\right|\cdot\ln\left|\frac{\varphi\left(t\right)-\varphi\left(\xi\right)}{t-\xi}\right|\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\left|\varphi'\left(t\right)\right|^2 dA\left(t\right)\\
&\qquad\quad+\frac{1}{4\pi^2}\int_D \ln\left|t-z\right|\cdot\ln\left|\frac{\varphi\left(t\right)-\varphi\left(\xi\right)}{t-\xi}\right|\cdot\left|\varphi'\left(t\right)\right|^2 dA\left(t\right)\\
&\qquad\quad+\frac{1}{4\pi^2}\int_D \ln\left|\frac{\varphi\left(t\right)-\varphi\left(z\right)}{t-z}\right|\cdot\ln\left|t-\xi\right|\cdot\left|\varphi'\left(t\right)\right|^2 dA\left(t\right)\\
&\qquad\quad+\frac{1}{4\pi^2}\int_D \ln\left|t-z\right|\cdot\ln\left|t-\xi\right|\cdot\left|\varphi'\left(t\right)\right|^2 dA\left(t\right)\\
&\qquad=K_1\left(z,\xi\right)+K_2\left(z,\xi\right)+K_3\left(z,\xi\right)+K_4\left(z,\xi\right).
\end{aligned}$$

Therefore, $S_2=\sum_{i=1}^4 S_2^{(i)}$ where operators $S_2^{(i)}:L^2(\partial D)\to L^2(\partial D)$ are defined by

$$S_2^{(i)}f\left(z\right)=\int_{\partial D} K_i\left(z,\xi\right)f\left(\xi\right)\,\left|d\xi\right|.$$

Let us demonstrate that operators $S_2^{(i)}$ $(1\le i\le 3)$ have an exponential decrease of singular values.

Note that $S_2^{(1)}=PQ$ where $P:L^2(D)\to L^2(\partial D)$, $Q:L^2(\partial D)\to L^2(D)$ and

$$\begin{aligned}
Pf\left(z\right)&=\frac{1}{4\pi^2}\int_D \ln\left|\frac{\varphi\left(t\right)-\varphi\left(z\right)}{t-z}\right|\cdot\left|\varphi'\left(t\right)\right|^2\cdot f\left(t\right)\,dA\left(t\right)\\
Qf\left(z\right)&=\frac{1}{4\pi^2}\int_{\partial D} \ln\left|\frac{\varphi\left(t\right)-\varphi\left(z\right)}{t-z}\right|\cdot f\left(t\right)\,\left|dt\right|.
\end{aligned}$$

Operator P is bounded, while operator Q (according to the Birman-Solomyak theorem ([**4**, Theorem 11.3, page 78]) has exponential decay of singular values. Therefore, singular values of operator $S_2^{(1)}$ have exponential decay.

In a similar way it may be demonstrated that singular values of operator $S_2^{(2)}$ and $S_2^{(3)}$ also have exponential decay.

Kernel K_4 may be expressed as

$$\begin{aligned} K_4(z,\xi) &= \frac{1}{4\pi^2}\int_D \left(\varphi'(t)-\varphi'(z)\right)\cdot \ln|t-z| \\ &\qquad\qquad \cdot \ln|t-\xi|\cdot \overline{\varphi'(t)}\, dAt \\ &\quad + \frac{1}{4\pi^2}\int_D \varphi'(z)\cdot \ln|t-z|\cdot \ln|t-\xi| \\ &\qquad\qquad \cdot \left(\overline{\varphi'(t)}-\overline{\varphi'(\xi)}\right)\, dA(t) \\ &\quad + \frac{\varphi'(z)\,\overline{\varphi'(\xi)}}{4\pi^2}\int_D \ln|t-z|\cdot \ln|t-\xi|\cdot\, dA(t) \\ &= \mathcal{A}_1(z,\xi)+\mathcal{A}_2(z,\xi)+\mathcal{A}_3(z,\xi)\,. \end{aligned}$$

Then $S_2^{(4)} = A_1 + A_2 + A_3$ where $\mathcal{A}_i : L^2(\partial D) \to L^2(\partial D)$ and

$$A_i f(z) = \int_{\partial D} \mathcal{A}_i(z,\xi)\, f(\xi)\; |d\xi| \quad (1 \le i \le 3).$$

Let us demonstrate that

$$s_n(A_1) = o\left(n^{-3}\right).$$

Note that

$$A_1 = P_1 \cdot M_{\overline{\varphi'}} \cdot Q_1,$$

where

$$\begin{aligned} P_1 &: L^2(D) \longrightarrow L^2(\partial D) \\ Q_1 &: L^2(\partial D) \longrightarrow L^2(D) \\ M_{\overline{\varphi'}} &: L^2(D) \longrightarrow L^2(D) \end{aligned}$$

and

$$\begin{aligned} P_1 f(z) &= \frac{1}{4\pi^2}\int_D \left(\varphi'(t)-\varphi'(z)\right)\ln|t-z|\cdot f(t)\; dA(t) \\ Q_1 f(z) &= \int_{\partial D} \ln|t-z|\cdot f(t)\; |dt| \\ M_{\overline{\varphi'}} f(z) &= \overline{\varphi'(z)}\, f(z)\,. \end{aligned}$$

Let $H_1, H_2 : L^2(D) \to L^2(\partial D)$ and

$$H_1 f(z) = \int_D (t - z) \cdot \ln|t - z| \cdot f(t)\, dA(t)$$
$$H_2 f(z) = \int_D [\varphi'(t) - \varphi'(z) - \varphi''(z)(t - z)]\, f(t)\, dA(t)$$

Clearly,
$$P_1 = \frac{1}{4\pi^2}\left(M_{\varphi\prime\prime} \cdot H_1 + H_2\right)$$
($M_{\varphi\prime\prime}$-operator of multiplying by $\varphi\prime\prime$ on $L^2(\partial D)$).

The singular values of the operator H_1 can be directly calculated, and we obtain $s_n(H_1) = O(n^{-5/2})$ and so

$$s_n\left(M_{\varphi\prime\prime} \cdot H_1\right) = O\left(n^{-5/2}\right).$$

From the Paraska theorem ([**12**]) it follows that

$$s_n(H_2) = o\left(n^{-3/2}\right)$$

which, together with the previous formula, gives

$$s_n(P_1) = o\left(n^{-3/2}\right). \tag{21}$$

The singular values of operator Q_1 can also be directly calculated, and we get
$$s_n(Q_1) = O\left(n^{-3/2}\right).$$

Keeping in mind that $M_{\overline{\varphi'}}$ is a bounded operator and that $s_n(Q_1) = O(n^{-3/2})$ is easily obtained from (21) and the fact that $A_1 = P_1 \cdot M_{\overline{\varphi'}} \cdot Q_1$, it follows that

$$s_n(A_1) = o\left(n^{-3}\right). \tag{22}$$

In a similar way, one demonstrates

$$s_n(A_2) = o\left(n^{-3}\right). \tag{23}$$

Let $R = S_2^{(1)} + S_2^{(2)} + S_2^{(3)} + A_1 + A_2$.

Since operator $S_2^{(1)} + S_2^{(2)} + S_2^{(3)}$ has exponential decay of singular values, from (22) and (23), it follows that

$$s_n(R) = o\left(n^{-3}\right). \tag{24}$$

Since $S_2 = R + A_3$, we have

$$S_1 = M_1 S_2 M_1^* = M_1 R M_1^* + M_1 A_3 M_1^*.$$

Since $M_1 A_3 M_1^* = G$ (operator from Lemma 5) we get

$$S_1 = G + M_1 R M_1^*.$$

Keeping in mind Lemma 5, as well as the fact that M_1 is a bounded operator, from (24), the Ky-Fan theorem and the previous equality, it follows that

$$s_n(S_1) \sim \left(\frac{|\partial\Omega|}{2\pi n}\right)^3, \quad n \to \infty$$

and so, from (19), we obtain

$$s_n(T) \sim \left(\frac{|\partial\Omega|}{2\pi n}\right)^{3/2}, \quad n \to \infty,$$

which proves Theorem 1. □

Theorem 2. *For the operators*

$$\int_{\partial\Omega} \ln|z-\xi| \cdot |d\xi|$$

and

$$\int_{\partial\Omega} |\xi - z|^{\alpha-1} \cdot |d\xi|,$$

which act on $L^2(\partial\Omega)$, the following asymptotic formula holds:

$$s_n\left(\int_{\partial\Omega} \ln|z-\xi|\cdot|d\xi|\right) \sim \frac{|\partial\Omega|}{n}, \quad n \to \infty$$

and

$$s_n\left(\int_{\partial\Omega} |\xi - z|^{\alpha-1}\cdot|d\xi|\right) \sim 2\,\Gamma(\alpha)\left|\cos\frac{\alpha\pi}{2}\right|\cdot|\partial\Omega|^{\alpha}\cdot n^{-\alpha}, \quad n \to \infty$$
$$(\alpha > 0,\ \alpha \neq 1, 3, 5, \dots).$$

Remark 2. The proof of Theorem 2 is similar to the proof of Theorem 1, but simpler.

Using the conformal map $\varphi : D \to \Omega$, study of asymptotics of singular values of the operator from Theorem 2 is reduced to the investigation of spectral properties of integral operators on $L^2(0, 2\pi)$ with kernels of the form $a(x)b(y)\ln|x-y|$ or $a(x)b(y)|x-y|^{\alpha-1}$.

The operator $\int_{\partial\Omega} \ln|z-\xi|\cdot|d\xi|$, from Theorem 2, acts on space $L^2(\partial\Omega)$ while the operator from Theorem 1 acts from $L^2(\partial\Omega)$ to $L^2(\Omega)$, and this is a reason for different asymptotics of their singular values.

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