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NONLINEAR ELLIPTICITY ON UNBOUNDED DOMAINS

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ABSTRACT. With $\Omega \subset \mathbf{R}^N$ an unbounded open connected set, and

$$Lu = -\sum_{j=1}^{N} D_j p_j D_j u + qu,$$

new results for the equation

$$Lu = \rho\lambda_1 u - \alpha\rho u^- + \rho g\,(x, u) + h$$

will be obtained where $\alpha > 0$, λ_1 is the principal eigenvalue associated with the elliptic operator L and $L\phi_1 = \rho\lambda_1\phi_1$. The results presented will constitute a *five-way* improvement over previous results on the subject. In particular, ρ and p_j will not be assumed to be integrable on Ω. Also, it is possible that $q(x) \to \infty$ as $|x| \to \infty$. Examples will be given using the Schrödinger operator, the Hermite operator and the Laguerre operator.

1. Introduction. Let $\Omega \sqsubseteq \mathbf{R}^N$, $N \geq 1$, be a domain (i.e., open connected set) which may be unbounded, and let ρ and q denote functions in $C(\Omega)$ and $p_1, \ldots, p_N$ denote functions in $C^1(\Omega)$. Assume that $\rho > 0$, $q \geq 0$ and $p_j > 0$ in Ω, for $j = 1, \ldots, N$.

We deal with the elliptic operator

$$Lu = -\sum_{j=1}^{N} D_j\,[p_j D_j u] + qu + \rho a_0 u \tag{1.1}$$

where $a_0 \in L^\infty(\Omega)$ and $a_0 \geq 0$.

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Also, let $\Gamma \subset \partial\Omega$ be a fixed closed set. (Γ may be the empty set and q may be identically zero.) We introduce the real pre-Hilbert space:

$$
\begin{aligned}
(1.2)\quad & C^1_{p,q,\rho}(\Omega,\Gamma) \\
&= \Big\{ u \in C\left(\overline{\Omega}\right) \cap C^1(\Omega) : u(x) = 0 \text{ for all } x \in \Gamma; \\
&\qquad \int_\Omega \left[\sum_{j=1}^N p_j (D_j u)^2 + (\rho + q) u^2 \right] dx < \infty \Big\}
\end{aligned}
$$

where $p = (p_1, \ldots, p_N)$. In $C^1_{p,q,\rho}(\Omega,\Gamma)$, we have the inner product

$$
(1.3)\qquad \langle u, v \rangle_{p,q,\rho} = \int_\Omega \left[\sum_{j=1}^N p_j D_j u D_j v + (\rho + q) uv \right] dx.
$$

$H^1_{p,q,\rho}(\Omega,\Gamma)$ will be the real Hilbert space that we obtain by completing $C^1_{p,q,\rho}(\Omega,\Gamma)$ with respect to the norm

$$
\|u\|_{p,q,\rho} = \langle u, u \rangle^{1/2}_{p,q,\rho}
$$

by the method of Cauchy sequences. $L^2_\rho(\Omega)$ will be the real Hilbert space with the inner product

$$
\langle u, v \rangle_\rho = \int_\Omega uv\rho \, dx \text{ where } \|u\|^2_\rho = \langle u, u \rangle_\rho.
$$

In a similar manner, we have the spaces $L^2_q(\Omega)$ and $L^2_{p_j}(\Omega)$, $j = 1, \ldots, N$. Hence, we see from (1.3) that

$$
(1.4)\qquad \langle u, v \rangle_{p,q,\rho} = \sum_{j=1}^N \langle D_j u, D_j v \rangle_{p_j} + \langle u, v \rangle_q + \langle u, v \rangle_\rho.
$$

Also, in the sequel, sometimes we shall write $H^1_{p,q,\rho}$ for $H^1_{p,q,\rho}(\Omega,\Gamma)$, $C^1_{p,q,\rho}$ for $C^1_{p,q,\rho}(\Omega,\Gamma)$ and L^2_ρ for $L^2_\rho(\Omega)$.

Next, we introduce the two-form corresponding to the elliptic operator Lu defined in (1.1), namely,

$$\mathcal{L}(u,v) = \int_\Omega \left[\sum_{j=1}^{N} p_j D_j u D_j v + (a_0\rho + q)uv \right] dx \tag{1.5}$$

for $u, v \in H^1_{p,q,\rho}$.

We have the possibility of (1.1) being singular because the p_j's may tend to zero on all or part of $\partial\Omega$, or Ω may be unbounded, or both. These two possibilities give rise to singular differential operators (see [**2**, pages 661–662]). The example we provide involving the Laguerre operator in Section 6 illustrates both possibilities.

We shall assume that Ω, Γ and the operator L satisfy the following conditions (O_1)–(O_3) which we shall refer to as $\mathbf{V}_L^{\diamond}(\Omega,\Gamma)$-conditions.

(O_1) *There exists a complete orthonormal system* $\{\phi_n\}_{n=1}^\infty$ *in* $L^2_\rho(\Omega)$. *Also,*

$$\phi_n \in H^1_{p,q,\rho}(\Omega,\Gamma) \cap C^1(\Omega) \quad \text{for all } n.$$

(O_2) *There exists a sequence of eigenvalues* $\{\lambda_n\}_{n=1}^\infty$ *with*

$$0 \le \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \longrightarrow \infty$$

such that $\mathcal{L}(\phi_n, v) = \lambda_n \langle \phi_n, v\rangle_\rho$ *for all* $v \in H^1_{p,q,\rho}(\Omega,\Gamma)$.

(O_3) λ_1 *is a simple eigenvalue and* $\phi_1(x) > 0$ *for all* $x \in \Omega$.

We give three examples of $\mathbf{V}_L^{\diamond}(\Omega,\Gamma)$-conditions later on, but, for starters, using the Hermite functions, the following remark is clear (see [**6**, pages 244 and 416]).

Remark 1. It is apparent with $\Omega = \mathbf{R}^2$, $p_1 = p_2 = 1$, $\rho = 1$, $q = x_1^2 + x_2^2$, $a_0 = 0$ and

$$Lu = -\Delta u + \left(x_1^2 + x_2^2\right) u,$$

where Δ is the Laplacian that Ω, with Γ the empty set, and L fit the definition of satisfying the

$$\mathbf{V}_L^{\diamond}(\Omega,\Gamma) - \text{conditions.}$$

Remark 2. Because of a well-known result due to Molchanov, [**9**, pages 239–245], the $x_1^2 + x_2^2$ in Remark 1 may be replaced by a $q(x)$ such that

$$\lim_{|x|\to\infty} q(x) = \infty.$$

So,

$$Lu = -\Delta u + q(x)\, u,$$

and we see the $\mathbf{V}_L^{\diamond}(\Omega, \Gamma)$-conditions cover the familiar Schrödinger operator.

We have more to say about the Schrödinger operator in Section 4 below.

$\mathbf{V}_L^{\diamond}$-conditions differ from the $\mathbf{V}_L$-conditions introduced in [**10**, page 328] in the following three ways: (i) Ω may be unbounded; (ii) ρ and p_j are not assumed to be integrable on Ω; (iii) $\phi_n \in L^\infty(\Omega)$ is not assumed. Also, there is a $q(x)$ in the definition of $L(u)$ defined here which is not in the $L(u)$ that is used in [**10**]. But the example where $\Omega \subset \mathbf{R}^2$ is a rectangle given in [**10**, pages 329–330] satisfies the $\mathbf{V}_L^{\diamond}(\Omega, \Gamma)$-conditions stated here.

We study the following problem:

$$\begin{cases} Lu = \lambda_1 \rho u - \alpha \rho u^- + \rho g(x, u) + h, \\ u \in H^1_{p,q,\rho}(\Omega, \Gamma), \end{cases} \tag{1.6}$$

where $\alpha > 0$, $h \in H^1_{p,q,\rho}(\Omega, \Gamma)^*$ (the dual of $H^1_{p,q,\rho}(\Omega, \Gamma)$) and $u^- = \max(0, -u)$ in Ω.

We shall assume the following three conditions for g:

$(g-1)$ $g(x,s)$ is a Caratheodory real-valued function, i.e., for each $s \in \mathbf{R}$, the function $x \mapsto g(x,s)$ is measurable in Ω, and for almost every $x \in \Omega$, the map $s \mapsto g(x,s)$ is continuous on $\mathbf{R}$.

$(g-2)$ There exists a $b \in L^2_\rho(\Omega)$ with $b \geq 0$ almost everywhere in Ω such that

$$|g(x,s)| \leq b(x) \text{ for almost every } x \in \Omega \text{ and } s \geq 0.$$

$(g-3)$ For every $\varepsilon > 0$, there exists a $b_\varepsilon \in L^2_\rho(\Omega)$ with $b_\varepsilon \geq 0$ almost everywhere in Ω such that

$$|g(x,s)| \leq \varepsilon |s| + b_\varepsilon(x) \text{ for a.e. } x \in \Omega \text{ and } s < 0.$$

By a weak solution to problem (1.6), we shall mean a function $u \in H^1_{p,q,\rho}(\Omega,\Gamma)$ for which

$$\mathcal{L}(u,v) = \langle \lambda_1 u - \alpha u^- + g(\cdot,u), v\rangle_\rho + h(v) \tag{1.7}$$

for all $v \in H^1_{p,q,\rho}(\Omega,\Gamma)$.

Also, we shall set $G(x,t) = \int_0^t g(x,s)\,ds$, and suppose that the following solvability condition holds.

$$\lim_{t\to\infty} \left\{ \int_\Omega G(x, t\phi_1(x))\,\rho(x)\ dx + h(t\phi_1) \right\} = +\infty. \tag{1.8}$$

We now state the main result of this paper.

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^N$, $N \geq 1$, be a domain with $\Gamma \subset \partial\Omega$ a closed set, and let ρ and q denote functions in $C(\Omega)$ and $p_1, \dots, p_N$ denote functions in $C^1(\Omega)$. Assume that $\rho > 0$, $q \geq 0$ and $p_j > 0$ in Ω for $j = 1, \dots, N$. Also, let the operator L be defined by* (1.1), *and assume that Ω, Γ and L satisfy the $\mathbf{V}_L^\diamond(\Omega,\Gamma)$ -conditions. Suppose also that* $(g-1)$–$(g-3)$ *holds, that $h \in H^1_{p,q,\rho}(\Omega,\Gamma)^*$, that $\alpha > 0$, and that the solvability condition* (1.8) *holds. Then problem* (1.6) *has a weak solution $u \in H^1_{p,q,\rho}(\Omega,\Gamma)$.*

We see that Theorem 1.1 is a five-way improvement over the corresponding Theorem 1.1 in [**10**]. First of all, we do not assume that $\Omega \subset \mathbf{R}^N$ is a bounded domain. Secondly, we do not assume that $\rho \in L^1(\Omega)$ or that $p_j \in L^1(\Omega)$ for $j = 1, \dots, N$. Thirdly, we do not assume that $\phi_n \in L^\infty(\Omega)$ where the ϕ_n are the eigenfunctions that arise in the definition of the $\mathbf{V}_L^\diamond(\Omega,\Gamma)$-conditions. Fourthly, we do not assume that

$$\lim_{t\to\infty} g(x,t) = g_+(x) \ \text{ for a.e. } x \in \Omega. \tag{1.9}$$

Also, we do not assume that

$$2G(x,s) - g(x,t) \geq -b^{**}(x)\,|t|$$

for almost every $x \in \Omega$ and $t \leq 0$ where $b^{**}(x) \in L^2(\Omega)$ and $b^{**}(x) \geq 0$ almost everywhere in Ω.

Fifthly, the solvability condition given in (1.8) above is more general than the solvability condition given in (13) of [**10**], i.e., if (1.9) is assumed, then the solvability condition (13) which is the following

$$\int_{\Omega} g_{+}(x)\,\phi_1(x)\,\rho(x)\;dx + h(\phi_1) > 0,$$

implies the solvability condition (1.8) (as an easy calculation shows). So, indeed, Theorem 1.1 above is a five-way improvement on Theorem 1.1 of [**10**].

In Sections 4, 5 and 6 of this paper, we will give examples of Ω, Γ and L, which are covered by the $\mathbf{V}_L^{\diamond}(\Omega,\Gamma)$-conditions stated here but not by the $\mathbf{V}_L(\Omega,\Gamma)$-conditions in [**10**].

If we assume that, in addition, to $(g-1)$–$(g-3)$ and (1.9) the nonlinearity g also satisfies

$$(g-4) \qquad g(x,s) < g_{+}(x) \quad \text{for a.e. } x \in \Omega \text{ and } s \in \mathbf{R},$$

then the solvability condition is also necessary for obtaining a weak solution $u \in H^1_{p,q,\rho}(\Omega,\Gamma)$ of problem (1.6). We state this fact in the following theorem.

Theorem 1.2. *In addition to the assumptions of Theorem* 1.1, *suppose also that g satisfies* (1.9) *and* $(g-4)$. *Then the solvability condition* (1.8) *is both necessary and sufficient for obtaining a weak solution $u \in H^1_{p,q,\rho}(\Omega,\Gamma)$ to problem* (1.6).

We now give a proof of the necessary condition in Theorem 1.2. The sufficiency part is an immediate corollary of Theorem 1.1.

Proof that (1.8) *is a necessary condition.* Suppose that $u \in H^1_{p,q,\rho}(\Omega,\Gamma)$ is a weak solution of (1.6). So, in particular, (1.7) holds with $v = \phi_1$. Consequently,

$$\int_{\Omega} g(x,u)\,\phi_1(x)\,\rho(x)\;dx + h(\phi_1) \geq 0.$$

But then it follows from (1.9), $(g-2)$, and $(g-4)$ that

$$\int_\Omega g_+(x)\,\phi_1(x)\,\rho(x)\,dx + h(\phi_1) = \gamma, \tag{1.10}$$

where $\gamma > 0$.

Next, using the Lebesgue dominated convergence in conjunction with $(g-2)$ and (1.9), we see that

$$\lim_{t\to\infty} \frac{1}{t} \int_\Omega G(x, t\phi_1)\,\rho(x)\,dx = \int_\Omega g_+(x)\,\phi_1(x)\,\rho(x)\,dx.$$

Therefore, it follows from (1.10) that

$$\lim_{t\to\infty} \frac{1}{t} \left[\int_\Omega G(x, t\phi_1)\,\rho(x)\,dx + th(\phi_1) \right] = \gamma > 0,$$

and condition (1.8) is established. □

2. Preliminary lemmas. Throughout this section, we will assume $\Omega \subseteq \mathbf{R}^N$, $N \geq 1$, is a domain (which may be unbounded), that ρ and q denote functions in $C(\Omega)$, that $p_1, \dots, p_N$ denote functions in $C^1(\Omega)$, and that $\rho > 0$, $q \geq 0$ and $p_j > 0$ in Ω, for $j = 1, \dots, N$. We will also assume that Lu is given by (1.1) and that the $\mathbf{V}_L^\diamond(\Omega, \Gamma)$-conditions hold. Furthermore, throughout this section and Section 3, we will assume that

$$a_0(x) \geq 1 \text{ almost everywhere in } \Omega, \tag{2.1}$$

where a_0 is given in (1.1). There is no loss in generality in making this assumption because a weak solution of the problem

$$\begin{cases} Lu + \rho u = (\lambda_1 + 1)\rho u - \alpha\rho u^- + \rho g(x,u) + h, \\ u \in H^1_{p,q,\rho}(\Omega, \Gamma), \end{cases}$$

is also a weak solution of the problem (1.6).

Because of our assumption (2.1), we see from (1.4) and (1.5) that

$$\langle u, u\rangle_{p,q,\rho} \leq \mathcal{L}(u,u) \quad \text{for all } u \in H^1_{p,q,\rho}(\Omega, \Gamma). \tag{2.2}$$

As a consequence of this inequality and the fact that $a_0 \in L^\infty(\Omega)$, we have the following lemma.

Lemma 2.1. *Let* $\mathcal{L}(u,v)$ *be as defined in* (1.5) *where* (2.1) *holds, and let* $\langle u,v\rangle_{p,q,\rho}$ *be as defined in* (1.4). *Then* $\mathcal{L}(u,v)$ *and* $\langle u,v\rangle_{p,q,\rho}$ *are equivalent inner products for* $H^1_{p,q,\rho}(\Omega,\Gamma)$.

The following are important consequences of the $\mathbf{V}^\diamond_L(\Omega,\Gamma)$-conditions and Lemma 2.1:

(i) For $v \in L^2_\rho(\Omega)$, let $\widehat{v}(n) = \langle v,\phi_n\rangle_\rho$ for all n. Then, for $v,w \in L^2_\rho(\Omega)$,

$$\langle v,w\rangle_\rho = \sum_{n=1}^\infty \widehat{v}\,(n)\,\widehat{w}\,(n)\,; \tag{2.3}$$

(ii) $\lambda_1 \geq 1$ and $\{\phi_n/\sqrt{\lambda_n}\}_{n=1}^\infty$ constitutes a complete orthonormal system for $H^1_{p,q,\rho}(\Omega,\Gamma)$ with respect to the inner product $\mathcal{L}(\cdot,\cdot)$.

Consequently,

$$\mathcal{L}(v,w) = \sum_{n=1}^\infty \lambda_n \widehat{v}\,(n)\,\widehat{w}\,(n)\,, \tag{2.4}$$

for all $v,w \in H^1_{p,q,\rho}(\Omega,\Gamma)$.

Also, we will need the following lemma.

Lemma 2.2. *Let* $\mathcal{L}(u,v)$ *be as defined in* (1.5) *where* (2.1) *holds, and assume the* $\mathbf{V}^\diamond_L(\Omega,\Gamma)$*-conditions hold. Also, assume that* $v \in L^2_\rho(\Omega)$. *Put* $\widehat{v}(n) = \langle v,\phi_n\rangle_\rho$ *for all* n. *Then* $v \in H^1_{p,q,\rho}(\Omega,\Gamma)$ *if and only if*

$$\sum_{n=1}^\infty \lambda_n \left|\widehat{v}\,(n)\right|^2 < \infty.$$

Proof of Lemma 2.2. The proof of the only if part follows from (2.4). The proof of the if part is essentially the same as the proof given in [**10**, page 336], and we leave the details to the reader. □

The proof of the next lemma is exactly the same as that given in [**11**, page 38].

Lemma 2.3. *Assume the* $\mathbf{V}_L^{\diamond}(\Omega,\Gamma)$*-conditions hold. Then* $H^1_{p,q,\rho}(\Omega,\Gamma)$ *is compactly imbedded in* $L^2_\rho(\Omega)$.

Next, we set

$$\begin{aligned} I(u) = \frac{\mathcal{L}(u,u)}{2} - \frac{\lambda_1\langle u,u\rangle_\rho}{2} + \frac{\alpha\langle u^-,u\rangle_\rho}{2} \\ - \int_\Omega \rho G(x,u)\,dx - h(u) \end{aligned}$$

for $u \in H^1_{p,q,\rho}(\Omega,\Gamma)$, where $G(x,s)$ is defined above in (1.8) and $h \in H^1_{p,q,\rho}(\Omega,\Gamma)^*$.

Lemma 2.4. *Assume the conditions in the hypothesis of Theorem* 1.1 *hold. Then*

$$\lim_{|t|\to\infty} I(t\phi_1) = -\infty.$$

Proof of Lemma 2.4. Suppose first that $t > 0$. Then, it follows from the definition of $I(u)$ and the fact that $\phi_1 > 0$ that

$$I(t\phi_1) = -\int_\Omega \rho G(x,t\phi_1)\,dx - h(t\phi_1).$$

Solvability condition (1.8) then implies that

$$\lim_{t\to\infty} I(t\phi_1) = -\infty.. \tag{2.5}$$

Next, suppose that $t < 0$. Then,

$$\begin{aligned} I(t\phi_1) &= -2^{-1}t^2\alpha\int_\Omega \phi_1^2\rho\,dx - \int_\Omega \rho G(x,t\phi_1)\,dx - h(t\phi_1) \\ &= -2^{-1}t^2\alpha - \int_\Omega \rho G(x,t\phi_1)\,dx - th(\phi_1), \end{aligned} \tag{2.6}$$

for $t < 0$. Now, for $\varepsilon > 0$ and $t < 0$, it follows from $(g-3)$ that

$$|G(x, t\phi_1)| \le b_\varepsilon(x)\,\phi_1\,|t| + \varepsilon t^2 \phi_1^2$$

for almost every $x \in \Omega$ where $b_\varepsilon \in L^2_\rho(\Omega)$. Hence, we obtain from this last inequality that

$$\lim_{t\to-\infty} t^{-2}\left|\int_\Omega \rho G(x, t\phi_1)\,dx\right| \le \varepsilon,$$

and consequently, since ε is arbitrary that

$$\lim_{t\to-\infty} t^{-2}\left|\int_\Omega \rho G(x, t\phi_1)\,dx\right| = 0.$$

This last fact together with (2.6) in turn implies that

$$\lim_{t\to-\infty} \frac{I(t\phi_1)}{t^2} = -2^{-1}\alpha.$$

The limit in (2.5) together with this last limit shows that

$$\lim_{|t|\to\infty} I(t\phi_1)\,, = -\infty,$$

and the proof of Lemma 2.4 is complete. $\square$

Next, let V designate the closed subspace of $H^1_{p,q,\rho}(\Omega,\Gamma)$ as follows:

$$V = \left\{ v \in H^1_{p,q,\rho}(\Omega,\Gamma) :\ \mathcal{L}(v,\phi_1) = 0 \right\}. \tag{2.7}$$

For $\gamma > 0$, we also introduce the following closed subset A_γ of $H^1_{p,q,\rho}(\Omega,\Gamma)$:

$$A_\gamma = \Big\{ u \in H^1_{p,q,\rho}(\Omega,\Gamma) : u = \gamma\left[\mathcal{L}(v,v)\right]^{1/2}\phi_1 + v \quad \text{where } v \in V \Big\}. \tag{2.8}$$

$J(u)$ will be the functional defined on $H^1_{p,q,\rho}(\Omega,\Gamma)$ as follows:

$$J(u) = \frac{\mathcal{L}(u,u)}{2} - \frac{\lambda_1\langle u,u\rangle_\rho}{2} + \frac{\alpha\langle u^-,u\rangle_\rho}{2}. \tag{2.9}$$

We prove the following lemma for $J(u)$.

Lemma 2.5. *Assume the conditions in the hypothesis of Theorem* 1.1 *hold, and that* (2.1) *is true. Then there exist positive constants* γ_1 *and* β_1 *such that*

$$J(u) \geq \beta_1 \mathcal{L}(u,u) \quad \text{for } u \in A_{\gamma_1}. \tag{2.10}$$

Proof of Lemma 2.5. Using (O_2) of the $\mathbf{V}_L^{\diamond}(\Omega,\Gamma)$-conditions, we select $n_1 \geq 3$ such that

$$\lambda_n - \lambda_1 \geq 6(\alpha+3) \quad \text{for } n \geq n_1. \tag{2.11}$$

Then, given $u \in H^1_{p,q,\rho}(\Omega,\Gamma)$, with

$$u = w + v$$

where $v \in V$, with V defined in (2.7) and $w = \widehat{u}(1)\phi_1$, we see from (O_3) that

$$v = \sum_{n=2}^{\infty} \widehat{u}(n)\,\phi_n.$$

We set

$$v_1 = \sum_{n=2}^{n_1} \widehat{u}(n)\,\phi_n \quad \text{and} \quad v_2 = \sum_{n=n_1+1}^{\infty} \widehat{u}(n)\,\phi_n, \tag{2.12}$$

and obtain from (2.11) that

$$\begin{aligned} \mathcal{L}(v_2,v_2) - \lambda_1 \langle v_2,v_2\rangle_\rho &= \sum_{n=n_1+1}^{\infty} (\lambda_n - \lambda_1)\,|\widehat{u}(n)|^2 \\ &\geq 6(\alpha+3)\,\langle v_2,v_2\rangle_\rho. \end{aligned} \tag{2.13}$$

Also, from the first equality in (2.12), we obtain that

$$\begin{aligned} |v_1(x)| &= \left| \sum_{n=2}^{n_1} \widehat{u}(n)\,\phi_n(x) \right| \\ &\leq \langle v_1,v_1\rangle_\rho^{1/2} \left| \sum_{n=2}^{n_1} |\phi_n(x)|^2 \right|^{1/2}. \end{aligned} \tag{2.14}$$

Since each $\phi_n \in L^2_\rho(\Omega)$, we can find Ω_1 such that $\overline{\Omega}_1$ is compactly imbedded in Ω and

$$\int_{\Omega\setminus\overline{\Omega}_1} \sum_{n=2}^{n_1} |\phi_n(x)|^2 \rho dx \le (\lambda_2 - \lambda_1)/32\alpha.$$

So we see from (2.14) that

$$\int_{\Omega\setminus\overline{\Omega}_1} |v_1(x)|^2 \rho\, dx \le < v_1, v_1 >_\rho (\lambda_2 - \lambda_1)/32\alpha. \tag{2.15}$$

Next, since $\overline{\Omega}_1$ is compactly imbedded in Ω, we see that there are an $\varepsilon_0 > 0$ and an $R_1 > 0$ such that

$$\phi_1(x) \ge \varepsilon_0 \quad \text{and} \quad \sum_{n=2}^{n_1} |\phi_n(x)|^2 \le R_1^2 \quad \text{for all } x \in \overline{\Omega}_1. \tag{2.16}$$

Consequently, we obtain from (2.14) that

$$|v_1(x)| \le \mathcal{L}(v_1, v_1)^{1/2} R_1/\lambda_2^{1/2} \quad \text{for all } x \in \overline{\Omega}_1.$$

We choose

$$\gamma_1 = R_1/(\varepsilon_0 \lambda_2^{1/2}).$$

Then, it follows from this last inequality that

$$\gamma_1 \mathcal{L}(v, v)^{1/2} \phi_1(x) + v_1(x) \ge 0 \quad \text{for all } x \in \overline{\Omega}_1, \tag{2.17}$$

because

$$\begin{aligned} \gamma_1 \mathcal{L}(v, v)^{1/2} \phi_1(x) &\ge R_1 \mathcal{L}(v_1, v_1)^{1/2} \varepsilon_0/(\varepsilon_0 \lambda_2^{1/2}) \\ &\ge \mathcal{L}(v_1, v_1)^{1/2} R_1/\lambda_2^{1/2} \\ &\ge -v_1(x) \end{aligned}$$

for all $x \in \overline{\Omega}_1$.

Next, with γ_1 chosen as above, we have that

$$u(x) = \gamma_1 \mathcal{L}(v, v)^{1/2} \phi_1(x) + v_1(x) + v_2(x)$$

for $u \in A_{\gamma_1}$. Hence, it follows from (2.17) that

$$\begin{aligned} u^-(x) &\le v_2^-(x) \quad \text{for all } x \in \overline{\Omega}_1 \\ &\le (v_1+v_2)^-(x) \quad \text{for all } x \in \Omega \backslash \overline{\Omega}_1 \end{aligned} \tag{2.18}$$

for $u \in A_{\gamma_1}$.

We infer from (2.9) and this last of inequalities that

$$\begin{aligned} 2J(u) &\ge \sum_{n=2}^{\infty} (\lambda_n - \lambda_1) |\widehat{u}(n)|^2 - \alpha \int_{\Omega_1} \left(v_2^-\right)^2 \rho\, dx \\ &\quad - \alpha \int_{\Omega\backslash\Omega_1} [(v_1+v_2)^-]^2 \rho\, dx \\ &\ge \sum_{n=2}^{\infty} (\lambda_n - \lambda_1) |\widehat{u}(n)|^2 - \alpha \int_{\Omega_1} v_2^2 \rho\, dx \\ &\quad - 2\alpha \int_{\Omega\backslash\Omega_1} (v_1^2+v_2^2) \rho\, dx \\ &\ge \sum_{n=2}^{\infty} (\lambda_n - \lambda_1) |\widehat{u}(n)|^2 - 2\alpha \int_{\Omega} v_2^2 \rho\, dx \\ &\quad - 2\alpha \int_{\Omega\backslash\Omega_1} v_1^2 \rho\, dx \end{aligned} \tag{2.19}$$

for $u \in A_{\gamma_1}$.

Next, we see that

$$2^{-1} \sum_{n=n_1+1}^{\infty} (\lambda_n - \lambda_1) |\widehat{u}(n)|^2 - 2\alpha \int_{\Omega} v_2^2 \rho\, dx \ge 0, \tag{2.20}$$

because from (2.11) and (2.12), we obtain that the left-hand side of this last inequality majorizes

$$3(\alpha+3) \sum_{n=n_1+1}^{\infty} |\widehat{u}(n)|^2 - 2\alpha \sum_{n=n_1+1}^{\infty} |\widehat{u}(n)|^2 \ge 0.$$

Likewise, we see that

$$2^{-1} \sum_{n=2}^{n_1} (\lambda_n - \lambda_1) |\widehat{u}(n)|^2 - 2\alpha \int_{\Omega\backslash\Omega_1} v_1^2 \rho\, dx \ge 0, \tag{2.21}$$

because from (2.12) and (2.15), we have that the left-hand side of the inequality in (2.21) majorizes

$$2^{-1}\sum_{n=2}^{n_1}(\lambda_n-\lambda_1)\left|\widehat{u}\,(n)\right|^2-\frac{\lambda_2-\lambda_1}{16}\sum_{n=2}^{n_1}\left|\widehat{u}\,(n)\right|^2\geq 0.$$

We consequently conclude from (2.19), (2.20) and (2.21) that

$$4J\,(u)\geq\sum_{n=2}^{\infty}(\lambda_n-\lambda_1)\left|\widehat{u}\,(n)\right|^2.$$

By (O3) in the $\mathbf{V}_L^\diamond(\Omega,\Gamma)$-conditions, λ_1 is a simple eigenvalue. Furthermore, $\lambda_n\to\infty$. Consequently, there exists a $\beta>0$ such that

$$(\lambda_n-\lambda_1)/4\geq\beta\lambda_n\quad\text{for } n\geq 2.$$

We infer from these last two inequalities and (2.12) that

$$J\,(u)\geq\beta\sum_{n=2}^{\infty}\lambda_n\left|\widehat{u}\,(n)\right|^2=\beta\mathcal{L}(v,v).$$

On the other hand, we see from the definition of A_{γ_1} that

$$\mathcal{L}(u,u)=\left(\gamma_1^2\lambda_1+1\right)\mathcal{L}(v,v).$$

We conclude from the previous inequality that

$$J\,(u)\geq\beta_1\mathcal{L}(u,u)$$

for $u\in A_{\gamma_1}$, where $\beta_1=\beta/(\gamma_1^2\lambda_1+1)$. $\square$

Next, we establish the following lemma.

Lemma 2.6. *Assume the conditions in the hypothesis of Theorem* 1.1 *hold and that* (2.1) *is true. Then, with* γ_1 *as in Lemma* 2.5 *and*

$$I\,(u)=J\,(u)-\int_\Omega G(x,u)\rho\,dx-h\,(u)\,,\tag{2.22}$$

the following holds:

$$\lim_{\|u\|_{p,q,\rho}\to\infty} I\left(u\right) = +\infty \quad \text{for } u \in A_{\gamma_1}.$$

Proof of Lemma 2.6. With $\beta_1 > 0$ as in Lemma 2.5, we invoke $(g-2)$ and $(g-3)$ and choose $\varepsilon = \beta_1\lambda_1$. Then

$$\left|g\left(x,s\right)\right| \le \varepsilon\left|s\right| + b_\varepsilon\left(x\right) + b\left(x\right)$$

for $s \in \mathbf{R}$ and almost every $x \in \Omega$, where b_ε, $b \in L^2_\rho(\Omega)$. With $G(x,s) = \int_0^s g(x,t)\,dt$, we see that

$$\left|G\left(x,s\right)\right| \le \varepsilon\left|s\right|^2/2 + \left(b_\varepsilon\left(x\right) + b\left(x\right)\right)\left|s\right|.$$

Consequently,

$$\int_\Omega \left|G\left(x,u\right)\right|\rho\,dx \le \beta_1\lambda_1\left\|u\right\|_\rho^2/2 + \left\|b_\varepsilon + b\right\|_\rho\left\|u\right\|_\rho.$$

But then it follows from (2.22) that there is a positive constant c_1 such that

$$I\left(u\right) \ge J\left(u\right) - \beta_1\mathcal{L}(u,u)/2 - c_1\mathcal{L}(u,u)^{1/2},$$

where we made use of the fact that $\lambda_1\|u\|_\rho^2 \le \mathcal{L}(u,u)$.

Next, we use Lemma 2.5 in conjunction with this last inequality involving $I(u)$ and obtain

$$I\left(u\right) \ge \beta_1\mathcal{L}(u,u)/2 - c_1\mathcal{L}(u,u)^{1/2},$$

for $u \in A_{\gamma_1}$. This last fact implies that

$$\lim_{\mathcal{L}(u,u)\to\infty} I\left(u\right) = +\infty \quad \text{for } u \in A_{\gamma_1},$$

which completes the proof of the lemma because, according to Lemma 2.1, $\mathcal{L}(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_{p,q,\rho}$ are equivalent inner products on $H^1_{p,q,\rho}(\Omega,\Gamma)$. □

In order to prove Theorem 1.1, it will be necessary to show that $I(u)$ satisfies the (PS)-condition, which is that $I \in C^1(H^1_{p,q,\rho},\mathbf{R})$ and the following hold:

Let $\{u_n\}_{n=1}^{\infty} \subset H^1_{p,q,\rho}(\Omega, \Gamma)$, and suppose that

$$\text{(2.23)} \qquad \begin{array}{rl} \text{(i)} & \{I(u_n)\}_{n=1}^{\infty} \text{ is a uniformly bounded sequence,} \\ \text{(ii)} & I'(u_n) \to 0 \text{ in norm as } n \to \infty. \end{array}$$

Then, there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ and a $u \in H^1_{p,q,\rho}(\Omega, \Gamma)$ such that

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{p,q,\rho} = 0. \tag{2.24}$$

We next prove the following lemma about $I(u)$ satisfying the (PS)-condition.

Lemma 2.7. *Assume the conditions in the hypothesis of Theorem* 1.1 *hold and that* (2.1) *is true. Then, with $J(u)$ defined by* (2.9) *and*

$$I(u) = J(u) - \int_\Omega G(x,u)\rho\, dx - h(u) \quad \textit{for } u \in H^1_{p,q,\rho}(\Omega, \Gamma),$$

$I(u)$ satisfies the (PS)*-condition.*

Proof of Lemma 2.7. Using (2.9), we compute the Gateau derivative of $I(u)$ and obtain

$$\begin{aligned} I'(u)(z) = \mathcal{L}(u,z) - \lambda_1 \langle u, z\rangle_\rho + \alpha\langle u^-, z\rangle_\rho \\ - \langle g(\cdot, u), z\rangle_\rho - h(z) \end{aligned} \tag{2.25}$$

for $u, z \in H^1_{p,q,\rho}(\Omega, \Gamma)$.

It is easy to see from this last computation that, as an element in $H^1_{p,q,\rho}(\Omega, \Gamma)^*$, $I'(u)$ is continuous with respect to u for $u \in H^1_{p,q,\rho}(\Omega, \Gamma)$. Hence, $I \in C^1(H^1_{p,q,\rho}, \mathbf{R})$.

Next, suppose that $\{u_n\}_{n=1}^{\infty} \subset H^1_{p,q,\rho}(\Omega, \Gamma)$ and that (2.23) (i) and (ii) hold. To complete the proof of the lemma, it is sufficient to show that there exists a subsequence such that

$$\left\{\|u_{n_k}\|_{p,q,\rho}\right\}_{k=1}^{\infty} \text{ is uniformly bounded.} \tag{2.26}$$

To see that this is indeed the case, suppose (where for ease of notation we use the full sequence)

$$\|u_n\|_{p,q,\rho} \leq c \quad \text{for all } n, \tag{2.27}$$

and (2.23) (i) and (ii) hold. Then, from Lemma 2.3 and the well-known Hilbert space theory, it follows that there is a subsequence (which once again we take to be the full sequence) and a $u^\diamond \in H^1_{p,q,\rho}(\Omega, \Gamma)$ such that

$$\begin{array}{rll} \text{(i)} & u_n \rightharpoonup u^\diamond & \text{in } H^1_{p,q,\rho}, \\ \text{(ii)} & u_n \to u^\diamond & \text{in } L^2_\rho(\Omega), \\ \text{(iii)} & u_n \to u^\diamond & \text{a.e. in } \Omega, \\ \text{(iv)} & |u_n(x)| \leq f(x) & \text{a.e. in } \Omega \text{ for all } n, \end{array} \tag{2.28}$$

where $f \in L^2_\rho(\Omega)$.

From (2.23) (ii) and (2.27), we see that

$$I'(u_n)(u_n - u^\diamond) \to 0 \quad \text{as } n \to \infty.$$

Hence, using $(g-2)$ and $(g-3)$ in conjunction with (2.28), we can then infer from (2.25) that

$$\mathcal{L}(u_n, u_n - u^\diamond) \longrightarrow 0 \quad \text{as } n \to \infty.$$

On the other hand, from (2.28), we have that

$$\mathcal{L}(u^\diamond, u_n - u^\diamond) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Putting these last two facts together gives

$$\mathcal{L}(u_n - u^\diamond, u_n - u^\diamond) \longrightarrow 0 \quad \text{as } n \to \infty,$$

which is our desired result because $\mathcal{L}(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle_{p,q,\rho}$ are equivalent inner products.

So, to complete the proof of the lemma, it remains to show that (2.23) (i) and (ii) imply that (2.26) is valid. We now do this.

Suppose then that we are given a sequence $\{u_n\}_{n=1}^\infty \subset H^1_{p,q,\rho}(\Omega, \Gamma)$ for which (2.23) (i) and (ii) are valid and for which the condition in

(2.26) does not hold. Then, since $I \in C^1(H^1_{p,q,\rho}, \mathbf{R})$, without loss of generality, we can assume

$$u_n \in C^1_{p,q,\rho}(\Omega, \Gamma) \quad \text{for all } n,$$

and that

$$\lim_{n\to\infty} \|u_n\|_{p,q,\rho} = \infty. \tag{2.29}$$

We will show that this last fact leads to contradiction of the solvability condition (1.8). In order to do this, we set

$$U_n(x) = u_n(x) / \|u_n\|_{p,q,\rho} \tag{2.30}$$

and see that $\|U_n\|_{p,q,\rho} = 1$ for every n. Hence, as before, from Lemma 2.3 and the well-known Hilbert space theory, it follows that there is a subsequence (which once again we take to be the full sequence) and a $U \in H^1_{p,q,\rho}(\Omega, \Gamma)$ such that

$$\begin{array}{rlll} & \text{(i)} & U_n \rightharpoonup U & \text{in } H^1_{p,q,\rho}, \\ & \text{(ii)} & U_n \to U & \text{in } L^2_\rho(\Omega), \\ (2.31) & \text{(iii)} & U_n \to U & \text{a.e. in } \Omega, \\ & \text{(iv)} & |U_n(x)| \le F(x) & \text{a.e. in } \Omega \text{ for all } n, \end{array}$$

where $F \in L^2_\rho(\Omega)$.

Next, we observe from (2.23) (ii) that

$$\lim_{n\to\infty} I'(u_n)(\phi_1) / \|u_n\|_{p,q,\rho} = 0,$$

and consequently, from (2.25) that

$$\lim_{n\to\infty} \left[\alpha\langle U_n^-, \phi_1\rangle_\rho - \frac{\langle g(\cdot, u_n), \phi_1\rangle_\rho}{\|u_n\|_{p,q,\rho}} \right] = 0.$$

It is easy to see from $(g-2)$ and $(g-3)$ that

$$\lim_{n\to\infty} \frac{\langle g(\cdot, u_n), \phi_1\rangle_\rho}{\|u_n\|_{p,q,\rho}} = 0.$$

So we conclude from (2.31) that

$$\int_{\Omega} U^{-}(x)\,\phi_1(x)\,\rho(x)\,dx = 0.$$

But $\phi_1(x) > 0$ everywhere in Ω. Hence, we obtain from this last fact that

$$U(x) \geq 0 \quad \text{a.e. in } \Omega. \tag{2.32}$$

Next, from (2.30), we see that

$$\|U_n\|_{p,q,\rho} = 1 \quad \text{for all } n.$$

Therefore, from (2.23) (ii), we obtain that

$$\lim_{n\to\infty} I'(u_n)(U_n)\,/\,\|u_n\|_{p,q,\rho} = 0.$$

Hence, from (2.25), we get that

$$\begin{aligned}\lim_{n\to\infty} \{\mathcal{L}(U_n, U_n) - \lambda_1 \langle U_n, U_n\rangle_\rho + \alpha\langle U_n^{-}, U_n\rangle_\rho \\ - [\langle g(\cdot, u_n), U_n\rangle_\rho + h(U_n)]/\,\|u_n\|_{p,q,\rho}\} = 0.\end{aligned} \tag{2.33}$$

From $(g-2)$, $(g-3)$ and (2.31) (iv), we obtain once again that

$$\lim_{n\to\infty} \frac{\langle g(\cdot, u_n), U_n\rangle_\rho}{\|u_n\|_{p,q,\rho}} = 0.$$

Likewise, we see that

$$\lim_{n\to\infty} \frac{h(U_n)}{\|u_n\|_{p,q,\rho}} = 0.$$

Also, from (2.31) (iii), (iv) and (2.32), we see that

$$\lim_{n\to\infty} \langle U_n^{-}, U_n\rangle_\rho = 0.$$

We conclude from these last three facts and (2.33) that

$$\lim_{n\to\infty}[\mathcal{L}(U_n, U_n) - \lambda_1\langle U_n, U_n\rangle_\rho] = 0. \tag{2.34}$$

Next, we write

$$U_n = W_n + V_n \quad \text{and} \quad U = W + V \tag{2.35}$$

where $\langle\phi_1, V_n\rangle_\rho = 0$, $\langle\phi_1, V\rangle_\rho = 0$, and

$$W_n = c_n\phi_1 \quad \text{and} \quad W = c^\diamond\phi_1 \tag{2.36}$$

with c_n, $c^\diamond$ real constants.

It follows from (2.1) that

$$\mathcal{L}(U_n, U_n) \geq 1$$

and from (2.34) that

$$\lim_{n\to\infty}\mathcal{L}(V_n, V_n) = 0.$$

Also, from (2.31) (ii), we see that

$$\lim_{n\to\infty} c_n = c^\diamond.$$

Hence, we infer from (2.30), (2.31) and (2.32) that $V = 0$ and

$$U = c^\diamond\phi_1 \quad \text{with} \quad c^\diamond > 0. \tag{2.37}$$

Also, from (2.31) (i) and (2.34), we obtain that

$$\lim_{n\to\infty}\mathcal{L}(U_n - U, U_n - U) = 0. \tag{2.38}$$

Next, we set

$$u_n = w_n + v_n \quad \text{where} \quad \langle w_n, v_n\rangle_\rho = 0 \tag{2.39}$$

and $w_n = \widehat{u}_n(1)\phi_1$ with $\widehat{u}_n(1) = \langle u_n, \phi_1\rangle_\rho$.

Also, we set

$$\widetilde{u}_n = \gamma_1 \left[\mathcal{L}(v_n, v_n)\right]^{1/2} \phi_1 + v_n \tag{2.40}$$

where γ_1 is defined in Lemma 2.5.

From (2.39) and (2.40), we infer that

$$u_n = \widetilde{c}_n \phi_1 + \widetilde{u}_n. \tag{2.41}$$

where

$$\widetilde{c}_n = \left\{\widehat{u}_n(1) - \gamma_1 \left[\mathcal{L}(v_n, v_n)\right]^{1/2}\right\}. \tag{2.42}$$

We claim

$$\left\{\widetilde{c}_n / \|u_n\|_{p,q,\rho}\right\}_{n=1}^{\infty} \quad \text{is a uniformly bounded sequence.} \tag{2.43}$$

Suppose the claim is false. Then, without loss in generality, we can assume that

$$\lim_{n\to\infty} \|u_n\|_{p,q,\rho} / |\widetilde{c}_n| = 0 \text{ and } \lim_{n\to\infty} |\widetilde{c}_n| = \infty.$$

But then it follows from (2.41) that

$$\lim_{n\to\infty} \left\| \frac{\widetilde{c}_n}{|\widetilde{c}_n|}\phi_1 + \frac{\widetilde{u}_n}{|\widetilde{c}_n|} \right\|_{p,q,\rho} = 0. \tag{2.44}$$

We see from (2.40)

$$\frac{\widetilde{u}_n}{|\widetilde{c}_n|} \in A_{\gamma_1} \quad \text{for all } n,$$

where A_{γ_1} is defined in (2.08). However, A_{γ_1} is a closed set in $H^1_{p,q,\rho}$. Also, at least one of the following two cases prevail:

(i) $\widetilde{c}_n > 0$ for a countable number of n;

(ii) $\widetilde{c}_n < 0$ for a countable number of n.

If case (i) holds, we conclude from (2.44) that $-\phi_1 \in A_{\gamma_1}$. But, from the very definition of A_{γ_1} this cannot be true. Likewise, if case (ii) holds, we arrive at a contradiction. Hence, the claim in (2.43) is indeed valid.

From (2.43), we see that there is a subsequence (which for ease of notation, we take to be the full sequence) such that

$$\lim_{n\to\infty} \widetilde{c}_n / \left\| u_n \right\|_{p,q,\rho} = c^{\#}. \tag{2.45}$$

Next, we obtain from (2.37) and (2.38) that

$$\lim_{n\to\infty} \left\| u_n / \left\| u_n \right\|_{p,q,\rho} - c^{\diamond} \phi_1 \right\|_{p,q,\rho} = 0.$$

Hence, we infer from (2.41) that

$$\lim_{n\to\infty} \left\| \frac{\widetilde{c}_n \phi_1}{\left\| u_n \right\|_{p,q,\rho}} + \frac{\widetilde{u}_n \cdot}{\left\| u_n \right\|_{p,q,\rho}} - c^{\diamond} \phi_1 \right\|_{p,q,\rho} = 0.$$

But then, from (2.45), we have that

$$\lim_{n\to\infty} \left\| \left(c^{\#} - c^{\diamond} \right) \phi_1 + \frac{\widetilde{u}_n \cdot}{\left\| u_n \right\|_{p,q,\rho}} \right\|_{p,q,\rho} = 0.$$

However, once again we see that $\widetilde{u}_n \cdot / \| u_n \|_{p,q,\rho} \in A_{\gamma_1}$. Since A_{γ_1} is a closed set in $H^1_{p,q,\rho}$, it follows that

$$- \left(c^{\#} - c^{\diamond} \right) \phi_1 \in A_{\gamma_1}.$$

But, from the very definition of A_{γ_1}, this last fact implies that

$$c^{\#} = c^{\diamond}.$$

Consequently, we obtain the following from (2.29), (2.37) and (2.45):

$$\begin{aligned} &\text{(i)} \quad \text{there is an } n_0 > 0, \text{ such that } \widetilde{c}_n > 0 \text{ for } n > n_0; \\ &\text{(ii)} \quad \lim_{n\to\infty} \widetilde{c}_n = \infty. \end{aligned} \tag{2.46}$$

From this last inequality, we see from (2.41) that

$$u_n^- (x) \leq \widetilde{u}_n^- (x) \quad \text{for all } x \in \Omega \text{ and } n > n_0. \tag{2.47}$$

Also, we obtain that

$$\langle u_n^-, \widetilde{u}_n\rangle_\rho \geq -\langle \widetilde{u}_n^-, \widetilde{u}_n^-\rangle_\rho \text{ for } n > n_0. \tag{2.48}$$

Next, we claim that

$$\left\{\|\widetilde{u}_n\|_{p,q,\rho}\right\}_{n=1}^{\infty} \quad \text{is a uniformly bounded sequence.} \tag{2.49}$$

Suppose the above claim is false. Then, without loss of generality, we can assume that

$$\lim_{n\to\infty} \|\widetilde{u}_n\|_{p,q,\rho} = \infty. \tag{2.50}$$

Also, from (2.25) and (2.41), we have that

$$\begin{aligned} I'(u_n)(\widetilde{u}_n) = \mathcal{L}(\widetilde{u}_n, \widetilde{u}_n) - \lambda_1\langle \widetilde{u}_n, \widetilde{u}_n\rangle_\rho + \alpha < u_n^-, \widetilde{u}_n\rangle_\rho \\ - \langle g(\cdot, u_n), \widetilde{u}_n\rangle_\rho - h(\widetilde{u}_n), \end{aligned}$$

and consequently from (2.48) that

$$\begin{aligned} I'(u_n)(\widetilde{u}_n) \geq \mathcal{L}(\widetilde{u}_n, \widetilde{u}_n) - \lambda_1\langle \widetilde{u}_n, \widetilde{u}_n\rangle \\ + \alpha\langle \widetilde{u}_n^-, \widetilde{u}_n\rangle_\rho - \langle g(\cdot, u_n), \widetilde{u}_n\rangle_\rho - h(\widetilde{u}_n) \end{aligned}$$

for $n > n_0$. But then it follows from this last inequality, (2.9) and Lemma 2.5 that

$$I'(u_n)(\widetilde{u}_n) \geq 2\beta_1\mathcal{L}(\widetilde{u}_n, \widetilde{u}_n) - \langle g(\cdot, u_n), \widetilde{u}_n\langle_\rho - h(\widetilde{u}_n) \tag{2.51}$$

for $n > n_0$ where $\beta_1 > 0$.

Next, with

$$F_n = \{x \in \Omega : u_n(x) \leq 0\},$$

we see from $(g-2)$ that

$$|\langle g(\cdot, u_n), \widetilde{u}_n\rangle_\rho| \leq \|b\|_\rho \|\widetilde{u}_n\|_\rho + \int_{F_n} |g(x, u_n)| |\widetilde{u}_n| \rho\, dx.$$

However, it follows from $(g-3)$ and (2.47) that, given $\varepsilon > 0$,

$$\int_{F_n} |g(x,u_n)| \, |\widetilde{u}_n| \, \rho \, dx \leq \varepsilon \int_{F_n} \left|\widetilde{u}_n^-\right|^2 \rho \, dx + \|b_\varepsilon\|_\rho \, \|\widetilde{u}_n\|_\rho$$

for $n > n_0$.

We conclude from these last two inequalities and (2.50) that

$$\lim_{n\to\infty} |\langle g(\cdot,u_n), \widetilde{u}_n\rangle_\rho| \, / \, \|\widetilde{u}_n\|^2_{p,q,\rho} \leq \varepsilon.$$

Since ε is an arbitrary positive number, we have that

$$\lim_{n\to\infty} |\langle g(\cdot,u_n), \widetilde{u}_n\rangle_\rho| \, / \, \|\widetilde{u}_n\|^2_{p,q,\rho} = 0.$$

Observing that $\mathcal{L}(\widetilde{u}_n, \widetilde{u}_n) \geq \|\widetilde{u}_n\|^2_{p,q,\rho}$, this last fact together with (2.51) gives

$$\liminf_{n\to\infty} I'(u_n)(\widetilde{u}_n) \, / \, \|\widetilde{u}_n\|^2_{p,q,\rho} \geq 2\beta_1,$$

where $\beta_1 > 0$.

On the other hand, it follows from (2.23) (ii) that

$$\lim_{n\to\infty} I'(u_n)(\widetilde{u}_n) \, / \, \|\widetilde{u}_n\|^2_{p,q,\rho} = 0.$$

We have arrived at a contradiction. Hence, the claim (2.49) is indeed true.

Next, we observe that (2.46) and (2.49) along with $(g-2)$ and $(g-3)$ imply that

(2.52)
$$\left\{\int_\Omega [G(x,u_n) - G(x,\widetilde{c}_n\phi_1)]\rho \, dx\right\}_{n=1}^\infty \text{ is a uniformly bounded sequence.}$$

To see that this last assertion actually is true, we obtain from the definition of $G(x,s)$ and (2.41) that

$$G(x,u_n) - G(x,\widetilde{c}_n\phi_1) = \int_{\tilde{c}_n\phi_1(x)}^{\tilde{c}_n\phi_1(x)+\tilde{u}_n(x)} g(x,s) \, ds. \tag{2.53}$$

From (2.46) and (O_3), we see that $\widetilde{c}_n\phi_1(x) > 0$ for $n > n_0$. So one of the following three cases arise:

$$\begin{array}{rl} \text{(i)} & \widetilde{u}_n(x) \geq 0; \\ \text{(ii)} & \widetilde{u}_n(x) < 0 \text{ and } |\widetilde{u}_n(x)| \leq \widetilde{c}_n\phi_1(x); \\ \text{(iii)} & \widetilde{u}_n(x) < 0 \text{ and } \widetilde{c}_n\phi_1(x) < -\widetilde{u}_n(x). \end{array}$$

If case (i) or (ii) holds, then it is clear from $(g-2)$ and (2.53) that

$$|G(x,u_n) - G(x,\widetilde{c}_n\phi_1)| \leq |b(x)|\,|\widetilde{u}_n(x)|.$$

If case (iii) holds, then choosing $\varepsilon = 1$ in $(g-3)$ shows that

$$|G(x,u_n) - G(x,\widetilde{c}_n\phi_1)| \leq |b(x)|\,|\widetilde{u}_n(x)| + |\widetilde{u}_n(x)|^2 + |b_1(x)|\,|\widetilde{u}_n(x)|,$$

where both b and b_1 are in $L^2_\rho(\Omega)$.

The assertion in (2.52) follows immediately from these last two inequalities and (2.49).

We will now complete the proof that (2.29) is false by showing that (2.29) leads to a contradiction.

From the condition in hypothesis (2.23) (i), we see that

$$\{I(u_n)\}_{n=1}^{\infty} \quad \text{is a uniformly bounded sequence.} \tag{2.54}$$

where, using (2.41),

$$\begin{aligned} I(u_n) = J(u_n) &- \left[\int_\Omega [G(x,u_n) - G(x,\widetilde{c}_n\phi_1)]\,\rho\,dx + h(\widetilde{u}_n)\right] \\ &- \left[\int_\Omega G(x,\widetilde{c}_n\phi_1)\,\rho dx + h(\widetilde{c}_n\phi_1)\right], \end{aligned} \tag{2.55}$$

where

$$2J(u_n) = \mathcal{L}(u_n,u_n) - \lambda_1\langle u_n,u_n\rangle_\rho + \alpha\langle u_n^-,u_n\rangle_\rho.$$

Using (2.41), a computation shows

$$\mathcal{L}(u_n,u_n) - \lambda_1\langle u_n,u_n\rangle_\rho = \mathcal{L}(\widetilde{u}_n,\widetilde{u}_n) - \lambda_1\langle\widetilde{u}_n,\widetilde{u}_n\rangle_\rho.$$

Likewise, using (2.47), we see that

$$\left|\langle u_n^-, u_n\rangle_\rho\right| \le \left|\langle \widetilde{u}_n^-, \widetilde{u}_n\rangle_\rho\right|,$$

for $n > n_0$.

We conclude from these last two facts along with (2.49) that

$$\{J(u_n)\}_{n=1}^\infty \quad \text{is a uniformly bounded sequence.} \tag{2.56}$$

Next, since h is a bounded linear functional on $H^1_{p,q,\rho}$, we obtain from (2.49) along with (2.52) that

$$\left\{\int_\Omega [G(x,u_n) - G(x,\widetilde{c}_n\phi_1)]\,\rho\,dx + h(\widetilde{u}_n)\right\}_{n=1}^\infty \quad \text{is a uniformly bounded sequence.}$$

This last fact in conjunction with (2.54), (2.55) and (2.56) gives that

$$\left\{\int_\Omega G(x,\widetilde{c}_n\phi_1)\,\rho\,dx + h(\widetilde{c}_n\phi_1)\right\}_{n=1}^\infty \quad \text{is a uniformly bounded sequence.}$$

On the other hand, we are assuming the solvability condition (1.8). So we obtain from (2.46) (ii),

$$\lim_{n\to\infty}\left[\int_\Omega G(x,\widetilde{c}_n\phi_1)\,\rho\,dx + h(\widetilde{c}_n\phi_1)\right] = \infty.$$

We have arrived at a contradiction. Hence, (2.29) is not valid, and (2.26) is indeed true. This fact completes the proof of the lemma. □

3. Proof of Theorem 1.1. Throughout this section, we will assume that

$$a_0(x) \ge 1 \quad \text{almost everywhere in } \Omega, \tag{3.1}$$

where $a_0(x)$ is given in (1.1). As shown in the first paragraph of Section 2, there is no loss in generality in making this assumption.

Because of assumption (3.1), we see from (1.4) and (1.5) that

$$\langle u, v\rangle_{p,q,\rho} \quad \text{and} \quad \mathcal{L}(u,v) \quad \text{for all } u, v \in H^1_{p,q,\rho}$$

are equivalent inner products on $H^1_{p,q,\rho}$.

We will prove Theorem 1.1 by showing that $I(u)$, defined in Lemma 2.7, has a critical point, i.e., a function $u_0 \in H^1_{p,q,\rho}$ such that

$$I'(u_0)(u) = 0 \quad \text{for all } u \in H^1_{p,q,\rho}, \tag{3.2}$$

where $I'(u)$ is defined in (2.25).

If u_0 satisfies (3.2), then $I(u_0)$ is called a critical value of I. To be explicit, we will establish Theorem 1.1 by showing that I has at least one critical value η. We accomplish this by means of linking theory.

Given a Banach space X, let Y be a finite-dimensional subspace of X, and let $E \subset X$ be a closed set. For $r > 0$, set

$$Y_r = \{y \in Y : \|y\| \le r\} \quad \text{and} \quad \partial Y_r = \{y \in Y : \|y\| = r\}. \tag{3.3}$$

Suppose that there exists an $r_1 > 0$ such that

$$\partial Y_{r_1} \cap E = \varnothing.$$

Let

$$\Psi = \{\psi \in C(X,X) : \psi(y) = y \text{ for all } y \in \partial Y_{r_1}\}. \tag{3.4}$$

Then we say ∂Y_{r_1} and E link, provided

$$\psi(Y_{r_1}) \cap E \neq \varnothing \quad \text{for all } \psi \in \Psi.$$

The following theorem prevails (see [**11**, page 127]) about sets ∂Y_{r_1} and E which link.

Theorem A. *Suppose X is a Banach space, $Y \subset X$ is a finite-dimensional subspace and $E \subset X$ is a closed set. With Y_r and ∂Y_r as in* (3.3), *suppose also that there exists an $r_1 > 0$ such that $\partial Y_{r_1} \cap E = \varnothing$*

and that ∂Y_{r_1} and E link. Furthermore, suppose that $I \in C^1(X, \mathbf{R})$, that I satisfies the (PS)*-condition, and that*

$$\inf_{u\in E} I(u) > \sup_{u\in\partial Y_{r_1}} I(u)\,. \tag{3.5}$$

Then with Ψ designated by (3.4)*, the number*

$$\eta = \inf_{\psi\in\Psi}\ \sup_{u\in Y_{r_1}} I(\psi(u)) \tag{3.6}$$

defines a critical value of I.

We prove Theorem 1.1 by showing that the conditions in the hypotheses of Theorem A are met for $I(u)$ where

$$I(u) = J(u) - \int_\Omega G(x,u)\rho\, dx - h(u)\,,$$

when we take $X = H^1_{p,q,\rho}$, $E = A_{\gamma_1}$ and

$$Y = \{t\phi_1 : -\infty < t < \infty\}\,. \tag{3.7}$$

We recall $J(u)$ is defined in (2.9) and A_{γ_1} is defined in Lemma 2.5. Also, we let

$$V = \left\{v \in H^1_{p,q,\rho} : \mathcal{L}(v, \phi_1) = 0\right\}. \tag{3.8}$$

First of all, we observe, with $I'(u)$ as given in (2.25), that $I(u)$ is indeed in $C^1(H^1_{p,q,\rho}, \mathbf{R})$. Also, it follows from Lemma 2.7 that $I(u)$ does satisfy the (PS)-condition.

Next, we see from Lemma 2.6 that there exists and $r_0 > 0$ such that

$$I(u) \geq 1 \quad \text{for all } u \in A_{\gamma_1} \text{ and } \quad \|u\|_{p,q,\rho} \geq r_0.$$

Also, it is easy to obtain from (2.9) and from $(g-2)$ and $(g-3)$ that there is a $K > 0$ such that

$$|I(u)| \leq K \quad \text{for all } u \in A_{\gamma_1} \text{ and } \|u\|_{p,q,\rho} \leq r_0.$$

Hence,

$$I(u) \geq -K \quad \text{for all } u \in A_{\gamma_1}. \tag{3.9}$$

From Lemma 2.4, we see that there is an $r_1 > 0$ such that both

$$I(r_1\phi_1) \leq -K - 1 \quad \text{and} \quad I(-r_1\phi_1) \leq -K - 1. \tag{3.10}$$

We set

$$Y_{r_1} = \{t\phi_1 : -r_1 \leq t \leq r_1\} \quad \text{and} \quad \partial Y_{r_1} = \{r_1\phi_1, -r_1\phi_1\}$$

and observe from (3.9) and (3.10) that

$$\inf_{u \in A_{\gamma_1}} I(u) > \sup_{u \in \partial Y_{r_1}} I(u).$$

So condition (3.5) of Theorem A is met where $A_{\gamma_1} = E$.

Consequently, to prove Theorem 1.1 by means of Theorem A, it only remains to show that ∂Y_{r_1} and A_{γ_1} link. It is clear that

$$\partial Y_{r_1} \cap A_{\gamma_1} = \varnothing.$$

So, to show the linkage property, we need only show that, given $\psi \in \Psi$ there exists a

$$t_0 \text{ such that } \psi(t_0\phi_1) \in A_{\gamma_1} \text{ where } -r_1 < t_0 < r_1, \tag{3.11}$$

and where

$$\Psi = \left\{\psi \in C\left(H^1_{p,q,\rho}, H^1_{p,q,\rho}\right) : \psi(r_1\phi_1) = r_1\phi_1 \right. \\ \left. \text{and } \psi(-r_1\phi_1) = -r_1\phi_1\right\}. \tag{3.12}$$

To show that (3.11) is true, we proceed as follows.

We set

$$\sigma(t) = \psi(t\phi_1) \quad \text{for } -r_1 \leq t \leq r_1.$$

Then

$$\sigma \in C\left([-r_1, r_1], H^1_{p,q,\rho}\right) \quad \text{with } \sigma(-r_1) = -r_1\phi_1 \\ \text{and } \sigma(r_1) = r_1\phi_1. \tag{3.13}$$

If there exists a $t_1 \in (-r_1, r_1)$ such that

$$\sigma(t_1) = 0,$$

we are done because $0 \in A_{\gamma_1}$ and (3.11) is true with $t_0 = t_1$.

So we can assume for the rest of the proof that

$$\mathcal{L}(\sigma(t), \sigma(t)) \neq 0 \quad \text{for } t \in (-r_1, r_1). \tag{3.14}$$

Next, we set

$$f(t) = \mathcal{L}(\sigma(t), \phi_1)/\lambda_1. \tag{3.15}$$

Then $f \in C([-r_1, r_1], \mathbf{R})$ with $f(-r_1) = -r_1$ and $f(r_1) = r_1$.

Let t_2 be the last value in the interval $(-r_1, r_1)$ such that $f(t) = 0$. Therefore,

$$f(t_2) = 0 \quad \text{and} \quad f(t) > 0 \quad \text{for } t_2 < t \leq r_1. \tag{3.16}$$

Set

$$\mu(t) = [\mathcal{L}(\sigma(t) - f(t)\phi_1, \sigma(t) - f(t)\phi_1)]^{1/2} / f(t)$$

for $t_2 < t \leq r_1$. Then $\mu \in C((t_2, r_1], \mathbf{R})$ with $\mu(r_1) = 0$.

Also, it follows from (3.14) and (3.16) that

$$\lim_{t \to t_2+} \mu(t) = +\infty.$$

Now, γ_1, given by Lemma 2.5, is a positive number. Consequently, there is a t_3 with $t_2 < t_3 < r_1$ such that

$$\mu(t_3) = 1/\gamma_1.$$

Hence,

$$f(t_3) = \gamma_1 [\mathcal{L}(\sigma(t_3) - f(t_3)\phi_1, \sigma(t_3) - f(t_3)\phi_1)]^{1/2}. \tag{3.17}$$

Also, we see from (3.8) and (3.15) that

$$\sigma(t_3) = f(t_3)\phi_1 + v. \tag{3.18}$$

where $v \in V$. We conclude from (3.17) that

$$f(t_3) = \gamma_1 [\mathcal{L}(v,v)]^{1/2}.$$

But then it follows from (3.18) and the fact that $\sigma(t) = \psi(t\phi_1)$ that

$$\psi(t_3\phi_1) = \gamma_1 [\mathcal{L}(v,v)]^{1/2} \phi_1 + v.$$

This establishes (3.11) with $t_0 = t_3$. Consequently, ∂Y_{r_1} and A_{γ_1} link, and the proof of Theorem 1.1 is complete. □

4. The Schrödinger operator. We elucidate on Remark 2 concerning the Schrödinger operator

$$Lu = -\Delta u + q(x)\, u \tag{4.1}$$

where $q \in C(\mathbf{R}^N)$, $N \geq 1$, $q \geq 0$, and

$$\lim_{|x|\to\infty} q(x) = \infty.$$

In this case, referring to Lu in (1.1), we have that $p_j = 1$ for $j = 1, \dots, N$, $a_0 = 0$ and $\rho = 1$.

To show that Theorem 1.1 holds for the Schrödinger operator in (4.1), we have to establish that the $\mathbf{V}_L^\diamond(\Omega,\Gamma)$-conditions are valid with $\Omega = \mathbf{R}^N$ and $\Gamma = \varnothing$. To do this, we have to show that $(O_1), (O_2)$ and (O_3) hold.

Using the ideas of Molchanov, as set forth in the book of Naimark [**9**, pages 239–245], it is an easy matter to demonstrate that

$$H^1_{p,q,\rho} \subset\subset \mathbf{L}^2\left(\mathbf{R}^N\right),$$

i.e., $H^1_{p,q,\rho}$ is compactly imbedded in $\mathbf{L}^2(\mathbf{R}^N)$. Then, using the theory of compact, symmetric, strictly positive operators on $\mathbf{L}^2(\mathbf{R}^N)$, we obtain the existence of $\{\phi_n\}_{n=1}^\infty$ and $\{\lambda_n\}_{n=1}^\infty$ such that (O_1) and (O_2) hold.

To see that λ_1 is a simple eigenvalue and strictly positive and that

$$\phi_1(x) > 0 \quad \text{for all } x \in \mathbf{R}^N,$$

we use the well-established techniques from the Calculus of Variations as set forth in [**7**, pages 150–151]. We also need Harnack's inequality as stated in [**5**, page 199].

Also, using the Calderon-Zygmund theory as given in [**5**, page 230], it is an easy matter to show that

$$\phi_n \in C^1\left(\mathbf{R}^N\right) \quad \text{for all } n \geq 1.$$

Consequently, $(O_1), (O_2)$ and (O_3) hold and Theorem 1.1 is valid for the Schrödinger operator L given in (4.1) above.

Also, Theorem 1.2, giving a necessary and sufficient condition for the existence of a weak solution to (1.6) when L is the Schrödinger operator, is valid.

It turns out that the study of nonlinear equations of the form

$$Lu = f\left(x, u\right) \tag{4.2}$$

where L is the Schrödinger operator is a topic of great current interest. Four current articles are by Yin and Zhang [**13**], Ding and Szulkin [**4**], Kryszewski and Sulkin [**8**] and Ambrosetti [**1**]. L is now of the form

$$L\left(u\right) = -\Delta u + V\left(x\right) u \tag{4.3}$$

In our result,

$$f\left(x, u\right) = \lambda_1 u - \alpha\rho u^- + g\left(x, u\right) + h.$$

In [**13**], in a paper entitled, *Bound states of nonlinear Schrödinger equations with potentials tending to zero at infinity*,

$$f\left(x, u\right) = K\left(x\right) u^p,$$

with $1 < p < (N+2)/(N-2)$. Also, the authors put an ε^2 before the Δ in (4.3), and they assume $K(x)$ and $V(x)$ are nonnegative functions which are positive in a smooth bounded domain $A \subset \mathbf{R}^N$ with the decay rate of $V(x)$ at infinity also being important. For $N \geq 3$, they obtain positive solutions of (4.2) when ε is small. For $N \geq 5$, they need less restrictions on the $V(x)$.

In [**4**], in a paper entitled, *Existence and number of solutions for a class of semilinear Schrödinger equations*,

$$f\,(x,u) = |u|^{p-2}\,u$$

with $2 < p < 2^*$ where $2^* = 2N/N-2$ if $N \geq 3$ and $2^* = +\infty$ if $N = 2$ or 1. Also, the authors put λ before the $V(x)$ in (4.3) and assume $V \geq 0$. In addition, they assume $V^{-1}(0)$ is nonempty and $\{x \in \mathbf{R}^N : V(x) < b\}$ has finite measure for at least one $b > 0$. They then show that the number of solutions to (4.2) increase as $\lambda \to \infty$.

In [**8**] in a paper entitled, *Generalized linking theorem with an application to a semilinear Schrödinger equation*, the authors assume $f(x,u) \in C(\mathbf{R}^N \times \mathbf{R})$ and periodic of period one with respect to each x_j-variable. Also, they assume $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly with respect to x and that $V(x) \in C(\mathbf{R}^N)$ and is periodic of period one with respect to each x_j-variable. Furthermore, they assume $|f(x,u)| \leq c(1 + |u|^{p-1})$ where $2 < p < 2^*$ and that there is a $\gamma > 2$ such that, for all $x \in \mathbf{R}^N$ and $u \in \mathbf{R}\backslash\{0\}$,

$$0 < \gamma F\,(x,u) \leq uf\,(x,u)\,,$$

where $F(x,u) = \int_0^u f(x,\xi)\,d\xi$. Also, they assume that 0 lies in the spectral gap of L. Then the authors show that, under these assumptions, that (4.2) has an infinitely many distinct nontrivial solutions.

In [**1**], in a survey article entitled, *Mathematical analysis–systems of nonlinear Schrödinger equations, A survey*, the author surveys results on Bound and Ground State solutions in $W^{1,2}(\mathbf{R}^N) \times W^{1,2}(\mathbf{R}^N)$ to nonlinear Schrödinger systems of the form

$$\begin{cases} -\Delta u + u = u^3 + \lambda F_u\,(u,v)\,, \\ -\Delta v + v = v^3 + \lambda F_v\,(u,v)\,. \end{cases}$$

Also surveyed are solutions in $W^{1,2}(\mathbf{R}^N) \times W^{1,2}(\mathbf{R}^N)$ to the nonlinear coupled non-autonomous system

$$(*) \qquad \begin{cases} -\Delta u + u = (1 + a\,(x))\,u^3 + \lambda u, \\ -\Delta v + v = (1 + b\,(x))\,v^3 + \lambda v, \end{cases}$$

where $a, b \in L^\infty(\mathbf{R}^N)$ and $\lim_{|x|\to\infty} a(x) = 0$, $\lim_{|x|\to\infty} b(x) = 0$. Also,

$$\inf_{\mathbf{R}^N}\,(1 + a\,(x)) > 0, \qquad \inf_{\mathbf{R}^N}\,(1 + b\,(x)) > 0.$$

The author then states the following result:

If $a(x) + b(x) \geq 0$, then for all $0 < \lambda < 1$, $(*)$ has a positive Ground State.

Each of the above manuscripts has an extensive bibliography of papers devoted to solutions of the nonlinear Schrödinger equation

$$Lu = f(x, u)$$

where L is the Schrödinger operator (4.3).

5. The Hermite operator. In this section, for simplicity, we will work in dimension $N = 2$, but everything that is established is also valid for $N = 1$ and $N \geq 3$.

We take $\Omega = \mathbf{R}^2$, $p_j = e^{-(x_1^2+x_2^2)}$ for $j = 1, 2$, and $\rho = e^{-(x_1^2+x_2^2)}$. Also, we take $q = 0$ and $a_0 = 0$. Then Lu in (1.1) becomes

$$Lu = -D_1(e^{-\left(x_1^2+x_2^2\right)}D_1u) - D_2(e^{-\left(x_1^2+x_2^2\right)}D_2u). \tag{5.1}$$

To show that Theorem 1.1 holds for the Hermite operator in (5.1), we have to establish that the $\mathbf{V}_L^{\diamond}(\Omega, \Gamma)$-conditions are valid with $\Omega = \mathbf{R}^2$ and $\Gamma = \varnothing$. To do this, we have to show that (O_1), (O_2) and (O_3) hold.

As is well known, the Hermite polynomials are given by

$$H_n(t) = (-1)^n e^{t^2} d^n e^{-t^2} / dt^n$$

for $n = 0, 1, 2\ldots$, and satisfy the differential equation

$$\left[e^{-t^2} H_n'(t)\right]' = -2ne^{-t^2} H_n(t). \tag{5.2}$$

Also, it is well known that

$$\left\{H_n(t) \Big/ \left(2^n n! \pi^{1/2}\right)^{1/2}\right\}_{n=0}^{\infty}$$

forms a complete orthonormal system over $\mathbf{R}^1$ with respect to the weight e^{-t^2}, i.e.,

$$\left(2^n n!\pi^{1/2}\right)^{-1/2}\left(2^m m!\pi^{1/2}\right)^{-1/2}\int_{-\infty}^{\infty} H_n(t)\,H_m(t)\,e^{-t^2}dt=\delta_{nm},$$

where δ_{nm} is the Kronecker-δ.

For all this, see either [**3**, pages 91–93] or [**6**, pages 244, 416].

We set

$$\Phi_{mn}(x)=H_m(x_1)\,H_n(x_2)\left(2^m m!\pi^{1/2}\right)^{-1/2}\left(2^m n!\pi^{1/2}\right)^{-1/2}, \tag{5.3}$$

for $m,n=0,1,\ldots$, and observe that

$$\int_{\mathbf{R}^2}\Phi_{m_1n_1}(x)\,\Phi_{m_2n_2}(x)\,e^{-|x|^2}dx=\begin{cases}0 & (m_1,n_1)\neq(m_2,n_2)\\ 1 & (m_1,n_1)=(m_2,n_2).\end{cases}$$

Hence, with $\phi_1=\Phi_{00}$, $\phi_2=\Phi_{10}$, $\phi_3=\Phi_{01}$, $\phi_4=\Phi_{20}$, etc., we see that $\{\phi_k\}_{k=1}^{\infty}$ is a complete orthonormal system in $L^2_\rho(\mathbf{R}^2)$ where $\rho=e^{-|x|^2}$. Likewise, we see from (5.3) that

$$\phi_k\in H^1_{p,q,\rho}\left(\mathbf{R}^2,\varnothing\right)\cap C^1\left(\mathbf{R}^2\right)\quad\text{for all } k.$$

Therefore, (O_1) holds.

With Lu defined in (5.1), an easy computation using (5.2) and (5.3) shows that

$$L\left(\Phi_{mn}(x)\right)=(2m+2n)\,e^{-|x|^2}\,\Phi_{mn}(x)$$

for $m,n=0,1,\ldots$. Consequently, after integrating by parts, we obtain from this last equality that

$$\int_{\mathbf{R}^2}\left[D_1uD_1\Phi_{mn}+D_2uD_2\Phi_{mn}\right]e^{-|x|^2}dx = (2m+2n)\int_{\mathbf{R}^2}u\Phi_{mn}e^{-|x|^2}dx, \tag{5.4}$$

where $u\in H^1_{p,q,\rho}$.

So, with $\lambda_1=0$, $\lambda_2=2$, $\lambda_3=2$, $\lambda_4=4$, etc., it follows from (5.4) that

$$\mathcal{L}(\phi_k, u) = \lambda_k \int_{\mathbf{R}^2} u\phi_k e^{-|x|^2} dx \quad \text{for all } k,$$

where $u \in H^1_{p,q,\rho}$. Consequently, (O_2) holds.

With $\lambda_1 = 0$ and $\phi_1 = (\pi)^{-1/2}$, it is clear that λ_1 is a simple eigenvalue and that $\phi_1(x) > 0$ for all $x \in \mathbf{R}^2$. So (O_3) is valid, and the $\mathbf{V}_L^{\diamond}(\mathbf{R}^2, \varnothing)$-condition is completely established.

Consequently, this example, using the Hermite operator in $\mathbf{R}^2$, is covered by Theorem 1.1 above. However, this example is not covered by Theorem 1.1 in [**10**] for several reasons. One is because $\mathbf{R}^2$ is not a bounded domain. Another reason is that the Φ_{mn} given in (5.3) above are such that $\Phi_{mn} \notin L^\infty(\mathbf{R}^2)$. $\square$

6. The Laguerre operator. As is well known, the Laguerre polynomials are given by

$$P_n(t) = e^t d^n t^n e^{-t}/dt^n, \tag{6.1}$$

and are polynomials of degree n.

We will call

$$L_1 u = -D_1[x_1 e^{-x_1} D_1 u(x_1)], \tag{6.2}$$

the one-dimensional Laguerre operator and use it in conjunction with the usual second differential operator to present another example of an elliptic operator on an unbounded domain which is covered by Theorem 1.1.

It is well known that

$$L_1 P_n(x_1) = n e^{-x_1} P_n(x_1) \quad \text{for all } n. \tag{6.3}$$

It is also well known that $\{P_n(t)/n!\}_{n=o}^{\infty}$ forms a complete orthonormal system over $(0, \infty)$ with respect to the weight e^{-t}, i.e.,

$$(n!)^{-2} \int_0^\infty e^{-t} P_m(t) P_n(t)\, dt = \delta_{mn}, \tag{6.4}$$

where δ_{nm} is the Kronecker-δ.

For all these matters about the Laguerre polynomials, see [**3**, pages 93–97].

We will present our example in $\mathbf{R}^2$, but similar examples can be produced in $\mathbf{R}^N$, $N \geq 3$.

We take Ω to be the half-infinite strip

$$\Omega = \{x : 0 < x_1 < \infty,\ 0 < x_2 < \pi\}\,, \tag{6.5}$$

and

$$\Gamma = \{(x_1, 0) : 0 \leq x_1 < \infty\} \cup \{(x_1, \pi) : 0 \leq x_1 < \infty\}\,. \tag{6.6}$$

Also, we take $p_1 = x_1 e^{-x_1}$, $p_2 = e^{-x_1}$, $\rho = e^{-x_1}$, $q = 0$ and $a_0 = 0$. So, L defined in (1.1) becomes

$$Lu = -D_1[x_1 e^{-x_1} D_1 u\,(x)] - D_2[e^{-x_1} D_2 u\,(x)]. \tag{6.7}$$

With $P_n(x_1)$, the nth Laguerre polynomial given by (6.1), we set

$$\Phi_{mn}\,(x) = (2)^{1/2}\, P_m\,(x_1) \sin n x_2 / \pi^{1/2} m!$$

for $m = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$. Then it follows from (6.4) that

$$\int_0^\infty \int_0^\pi e^{-x_1} \Phi_{m_1 n_1}\,(x)\, \Phi_{m_2 n_2}\,(x)\; dx_1\, dx_2 = \begin{cases} 0 & (m_1, n_1) \neq (m_2, n_2) \\ 1 & (m_1, n_1) = (m_2, n_2). \end{cases} \tag{6.8}$$

As we will show, the eigenvalues are of the form $(m + n^2)$. Consequently, upon setting $\phi_1 = \Phi_{01}$, $\phi_2 = \Phi_{11}$, $\phi_3 = \Phi_{21}$, $\phi_4 = \Phi_{02}$, $\phi_5 = \Phi_{31}$, $\phi_6 = \Phi_{12}$, etc., we see from (6.5) and (6.6) that $\phi_k \in H^1_{p,q,\rho}(\Omega, \Gamma)$ and that

$$\{\phi_k\}_{k=1}^\infty \text{ is a complete orthonormal system in } L^2_\rho\,(\Omega)\,.$$

Hence, (O_1) holds for this example.

With Lu defined in (6.7), an easy computation using (6.2) and (6.3) shows that

$$L\left(\Phi_{mn}\left(x\right)\right)=\left(m+n^2\right)e^{-x_1}\Phi_{mn}\left(x\right)$$

for $m=0,1,\ldots$ and $n=1,2,\ldots$. Consequently, after integrating by parts, we obtain from this last equality that

(6.9)

$$\int_\Omega\left[x_1D_1uD_1\Phi_{mn}+D_2uD_2\Phi_{mn}\right]e^{-x_1}\,dx=\left(m+n^2\right)\int_\Omega u\Phi_{mn}e^{-x_1}\,dx$$

for $u\in H^1_{p,q,\rho}(\Omega,\Gamma)$.

So, on setting

$$\mathcal{L}(u,v)=\int_\Omega\left[x_1D_1uD_1v+D_2uD_2v\right]e^{-x_1}\,dx, \tag{6.10}$$

for $u,v\in H^1_{p,q,\rho}(\Omega,\Gamma)$ and $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$, $\lambda_4=4$, $\lambda_5=4$, $\lambda_6=5$, etc., we see from (6.9) that

$$\mathcal{L}(u,\phi_k)=\lambda_k\langle u,\phi_k\rangle_\rho\quad\text{for all } k \text{ and } u\in H^1_{p,q,\rho}\left(\Omega,\Gamma\right).$$

Hence, (O_2) holds for this example. Observing that $\lambda_1=1$, $\lambda_2=2$ and $\phi_1(x)=(2/\pi)^{1/2}\sin x_2$, we see that λ_1 is a simple eigenvalue and $\phi_1(x)$ is positive for $x\in\Omega$. Therefore, (O_3) also holds.

We conclude that the $\mathbf{V}^\diamond_L\left(\Omega,\Gamma\right)$-conditions are valid for this example. So, Theorem 1.1 holds for the elliptic operator $L(u)$ defined in (6.7) above with Ω the half-open strip given in (6.5) and Γ the two half-closed lines defined in (6.6).

Once again, we see that this example involving the Laguerre operator is not covered by [**10**, Theorem 1.1] for two reasons. The first is because Ω is not a bounded domain, and the second is because $\phi_k\notin L^\infty(\Omega)$. $\square$

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REFERENCES

1. A. Ambrosetti, *Mathematical analysis–Systems of nonlinearSchrödinger equations, A survey*, Att. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Linc. Mat. Appl. **20** (2009), 99–110.

2. W.E. Boyce and R.C. Diprima, *Elementary differential equations and boundary value problems*, Sixth ed., John Wiley & Sons, New York, 1997.

3. R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. I, Interscience, New York, 1966.

4. Y. Ding and A. Szulkin, *Existence and number of solutions for a class of semilinear Schrödinger equations*, *Contributions to nonlinear analysis*, 221–231, Prog. Nonlin. Diff. Eqns. Appl. **66**, Birkhäuser, Basel, 2006.

5. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd edition, Springer-Verlag, New York, 1983.

6. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1965.

7. S. Kesavan, *Functional analysis and applications*, John Wiley & Sons, New York, 1989.

8. W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to a semilinear Schrödinger equation*, Adv. Differ. Eqns. **3** (1998), 441–472.

9. M.A. Naimark, *Linear diferential operators*, Part II, Frederick Ungar, New York, 1968.

10. A. Rumbos and V.L. Shapiro, *Jumping nonlinearities and weighted Sobolev spaces*, J. Differ. Eqns. **214** (2005), 326–357.

11. V.L. Shapiro, *Singular quasilinearity and higher eigenvalues*, Mem. Amer. Math. Soc. **726**, Providence, 2001.

12. M. Struwe, *Variational methods*, fourth edition, Spinger-Verlag, Berlin, 2008.

13. H Yin, and P. Zhang, *Bound states of nonlinear Schrödinger equations with potentials tending to zero at infinity*, J. Differ. Eqns. **247** (2009), 618–647.

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