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ON THE CARDINALITY OF STAR OPERATIONS ON A PSEUDO-VALUATION DOMAIN

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ABSTRACT. Let R be a pseudo-valuation domain with residue field k, and let V be the associated valuation domain of R with residue field L. The purpose of this article is to compute the cardinality $|\mathrm{Star}\,(R)|$ (respectively $|\mathrm{SStar}\,(R)|$) of star (respectively semistar) operations on R. It depends upon the relation between the residue fields of R and V. We will show that $|\mathrm{Star}\,(R)| < \infty$ if and only if $\dim_k L = 1, 2, 3$, or L is a finite field, and that $|\mathrm{SStar}\,(R)| < \infty$ if and only if $|\mathrm{Star}\,(R)| < \infty$ and $\dim R < \infty$.

1. Introduction. Let R be an integral domain with quotient field K, $\mathcal{F}(R)$ the set of nonzero fractional ideals of R, and $\overline{\mathcal{F}}(R)$ the set of nonzero R-submodules of K.

A mapping $* : \mathcal{F}(R) \to \mathcal{F}(R)$, $I \mapsto I^*$, is called a *star-operation* on R if the following conditions hold for all $a \in K \setminus \{0\}$ and $I, J \in \mathcal{F}(R)$:

(i) $(a)^* = (a)$; $(aI)^* = aI^*$;

(ii) $I \subseteq I^*$; if $I \subseteq J$, then $I^* \subseteq J^*$; and

(iii) $(I^*)^* = I^*$.

A fractional ideal $I \in \mathcal{F}(R)$ is called a $*$-ideal if $I^* = I$.

The best known examples of a star-operation are the d-operation and the v-operation. The d-operation is the identity mapping $I \mapsto I_d = I$ and the v-operation is defined by $I \mapsto I_v = (I^{-1})^{-1} = \cap\{Rx \mid x \in K,\, I \subseteq Rx\}$. A v-ideal is often called a divisorial ideal. It is easy to see that, for each star-operation $*$ on R and each fractional ideal $I \in \mathcal{F}(R)$, $I \subseteq I^* \subseteq I_v$. As an immediate consequence, if $d = v$, i.e., each nonzero

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ideal of R is divisorial, then R has only one star-operation. Such a domain is called a divisorial domain.

The reader can refer to [**6**, Sections 32 and 34] for the basic properties of star-operations and the v-operation, and [**4, 9**] for the divisorial domains.

As a generalization of the notion of a star-operation, Okabe and Matsuda introduced the notion of a semistar-operation on R [**12**].

A mapping $* : \overline{\mathcal{F}}(R) \to \overline{\mathcal{F}}(R)$, $E \mapsto E^*$, is called a *semistar-operation* on R if the following conditions hold for all $a \in K \setminus \{0\}$ and $E, F \in \overline{\mathcal{F}}(R)$:

(I) $(aE)^* = aE^*$;

(II) $E \subseteq E^*$; if $E \subseteq F$, then $E^* \subseteq F^*$; and

(III) $(E^*)^* = E^*$.

We denote by $\mathrm{Star}\,(R)$ the set of star-operations on R and by $\mathrm{SStar}\,(R)$ the set of semistar-operations on R.

In this note we are interested in the cardinalities $|\mathrm{Star}\,(R)|$ and $|\mathrm{SStar}\,(R)|$ of $\mathrm{Star}\,(R)$ and $\mathrm{SStar}\,(R)$, respectively.

We compile in Theorem 1.1 the results that appear in the literature concerning the cardinalities $|\mathrm{Star}\,(R)|$ and $|\mathrm{SStar}\,(R)|$.

Theorem 1.1. (1) [**3**, Theorem 6.2]. *Let R be a Noetherian domain. Then $|\mathrm{Star}\,(R)| = 1$ if and only if R is a one-dimensional Gorenstein domain.*

(2) [**9**, Lemma 5.2] and [**1**, Proposition 12]. *Let V be a nontrivial valuation domain with maximal ideal M.*

(a) *If M is principal, then $|\mathrm{Star}\,(V)| = 1$.*

(b) *If M is not principal, then $|\mathrm{Star}\,(V)| = 2$.*

(3) [**10**, Theorem 4]. *Let V be an n-dimensional valuation domain. Then $n + 1 \leq |\mathrm{SStar}\,(V)| \leq 2n + 1$.*

(4) [**9**, Theorem 5.1]. *Let R be an integrally closed domain. Then $|\mathrm{Star}\,(R)| = 1$ if and only if R is an h-local Prüfer domain in which maximal ideals are finitely generated.*

(5) [**10**, Theorem 5]. *Let R be an integrally closed domain. If $|\mathrm{SStar}\,(R)| < \infty$, then R is a semiquasiocal Prüfer domain. Furthermore, if, in addition, R is quasilocal, then R is a valuation domain.*

(6) [**11**, Proposition 2.1]. *Let R be a conducive domain with $|\mathrm{SStar}\,(R)| < \infty$. Then $|\mathrm{SStar}\,(R)| = \sum_T |\mathrm{Star}\,(T)|$, where T ranges over the overrings of R.*

An integral domain R is called a *conducive domain* if, for each overring T of R other than K, the conductor $(R : T) = \{x \in K \mid xT \subseteq R\}$ is nonzero, or equivalently, if $(R : V) \neq (0)$ for some valuation overring V of R. Examples of conducive domains include all domains of $(D+M)$-type, all valuation domains and, more generally, all pseudo-valuation domains [**5**].

The result in Theorem 1.1 (6) by Mimouni and Samman naturally leads to the following question: When are the cardinalities $|\mathrm{Star}(R)|$ and $|\mathrm{SStar}(R)|$ on a conducive domain R finite?

We will give a complete answer to the question in the case R is a pseudo-valuation domain (Theorem 2.15).

2. Main results. An integral domain R is called a *pseudo-valuation domain* (or, for short, a PVD) if R has a valuation overring V such that $\mathrm{Spec}\,(R) = \mathrm{Spec}\,(V)$ as sets [**7, 8**]. Such a valuation overring V is uniquely determined and it is called the associated valuation domain of R. Obviously, every valuation domain is a pseudo-valuation domain.

To avoid the trivial cases, we assume that R is a pseudo-valuation domain which is not a valuation domain. Then it is characterized as the pullback of a diagram:

$$\begin{tikzcd} & k \arrow[d] \\ V \arrow[r, "\varphi"] & L \end{tikzcd}$$

in which V is a nontrivial valuation domain with maximal ideal M, L is the residue field of V, k is a proper subfield of L, the horizontal map φ is the canonical surjection, and the vertical map is the canonical injection [**2**, Proposition 2.6].

Throughout this section, we preserve the above notation and let K be the quotient field of R (and V).

Lemma 2.1. *Let I be a nonzero nondivisorial ideal of R. Then an element $a \in K \setminus \{0\}$ exists such that $R \subsetneq a^{-1}I \subsetneq V$.*

Proof. Let I be a nonzero nondivisorial ideal of R. Then by [**7**, Proposition 2.11 and Theorem 2.13], I is not an ideal of V and IV is a principal ideal of V. We can write $IV = aV$ for some nonzero element $a \in I$. Thus, we have $R \subsetneq a^{-1}I \subsetneq V$. $\square$

Lemma 2.2. *Let I be a divisorial ideal of R lying between R and V. Then $I = R$ or $I = V$.*

Proof. By [**7**, Corollary 2.15], I is a principal ideal of R or I is an ideal of V.

Assume that $I = xR$ for some x. Then from the inclusions $R \subseteq xR \subseteq V$, it follows that $xV = V$, and hence that $x^{-1} \in V \setminus M$. Also, since $x^{-1} \in R$, $x^{-1} \in R \setminus M$. Therefore, $I = xR = R$.

Now assume that I is an ideal of V. Then, from the inclusions $R \subseteq I \subseteq V$, it follows that $I = IV = V$. $\square$

Theorem 2.3. *If $\dim_k L = 2$, then $|\mathrm{Star}\,(R)| = 1$, i.e., R is a divisorial domain.*

Proof. Suppose that R has a nonzero nondivisorial ideal I. Then by Lemma 2.1, $R \subsetneq a^{-1}I \subsetneq V$ for some $a \in K \setminus \{0\}$. Therefore, $\varphi(a^{-1}I)$ is a k-vector space properly between k and L. This contradicts that $\dim_k L = 2$. $\square$

Proposition 2.4. *Let the relation $\sim$ be defined on $\mathcal{F}(R)$ by*

$$I \sim J \quad \text{if and only if } I = \alpha J \text{ for some } \alpha \in K \setminus \{0\}.$$

(1) *The relation $\sim$ is an equivalence relation on $\mathcal{F}(R)$.*

(2) *If $I \sim J$, then either both of I and J are divisorial or neither is.*

(3) *Let the relation $\sim$ be defined on the set of k-vector spaces between k and L by*

$$U \sim W \quad \textit{if and only if } U = \beta W \textit{ for some } \beta \in L \setminus \{0\}.$$

Then $\sim$ is an equivalence relation, and $U \sim W$ if and only if $\varphi^{-1}(U) \sim \varphi^{-1}(W)$.

(4) *If I is a nondivisorial ideal, then $I \sim J$ for some J lying properly between R and V. In this case, we define* $\operatorname{rank} I$ *to be the dimension of the k-vector space $\varphi(J)$. Then it is well-defined, i.e., it is independent of the choice of J.*

Proof. (1), (2), and the first statement of (3) are immediate consequences of the definition of $\sim$. The second statement of (3) follows from the observation that, if I and J are fractional ideals of R between R and V, then $I \sim J$ if and only if $I = \alpha J$ for some $\alpha \in V \setminus M$.

(4) The first part follows from Lemma 2.1. For the second, let J_1, J_2 be fractional ideals of R properly between R and V such that $I \sim J_1$ and $I \sim J_2$. Then $J_1 \sim J_2$ and hence $J_1 = \alpha J_2$ for some $\alpha \in V \setminus M$. Therefore, $\varphi(J_1) = \beta\varphi(J_2)$, where $\beta = \varphi(\alpha) \in L \setminus \{0\}$. It is obvious that $\dim_k \varphi(J_1) = \dim_k \varphi(J_2)$. $\square$

Now we assume that $\dim_k L = n \geq 3$. For each integer m with $2 \leq m < n$, let l_m be the cardinality of distinct equivalence classes of $\mathcal{F}(R)$ of rank m. Note that, by Lemma 2.2 and Proposition 2.4, l_m is equal to the cardinality of distinct equivalence classes of the set of k-vector spaces between k and L of dimension m.

Theorem 2.5. *If $\dim_k L = n \geq 3$, then $2^{l_2} + 2^{l_3} + \cdots + 2^{l_{n-1}} - (n-3) \leq |\mathrm{Star}\,(R)| \leq 2^{l_2+\cdots+l_{n-1}}$.*

Proof. "Upper bound." Let $*$ be a star operation on R. If I is a divisorial ideal of R, then $I^* = I$. If I is nondivisorial, then $I = aJ$ for some $a \in K \setminus \{0\}$ and some $J \in \mathcal{F}(R)$ with $R \subsetneq J \subsetneq V$, and hence $I^* = aJ^*$. Thus, $*$ is completely determined by $\mathcal{F}_*(R) \cap X$, where $\mathcal{F}_*(R)$ is the set of $*$-ideals of R and X is a complete set of class representatives under the equivalence relation $\sim$ on the set of nonzero nondivisorial fractional ideals of R. In fact, if $I \in \mathcal{F}(R)$ with $R \subsetneq I \subsetneq V$, then $I^* = (\cap\{aJ \mid aJ \supseteq I,\, a \in K \setminus \{0\},\, J \in \mathcal{F}_*(R) \cap X\}) \cap I_v$ (cf. [**6**, Proposition 32.4] or [**1**, Lemma 3]). Therefore, $|\mathrm{Star}\,(R)| \leq |\mathcal{P}(X)| = 2^{|X|} = 2^{l_2+\cdots+l_{n-1}}$.

"Lower bound." For each integer m with $2 \leq m < n$, let X_m be a complete set of class representatives under the equivalence relation $\sim$ on the set of fractional ideals of R of rank m. Then it is obvious that $|X_m| = l_m \geq 1$ and that $X = \cup_{m=2}^{n-1} X_m$ is a complete set of equivalence class representatives on the set of nonzero nondivisorial fractional ideals of R.

For each subset S of X_m, define a mapping $*_S : \mathcal{F}(R) \to \mathcal{F}(R)$ as follows:

$$I \longmapsto I^{*_S} = \begin{cases} I & \text{if } I \text{ is divisorial or } I \sim J \\ & \text{for some } J \in (\bigcup_{i=2}^{m-1} X_i) \cup S \\ I_v & \text{otherwise.} \end{cases}$$

Claim 1. $*_S$ is a star operation on R.

All of the properties for $*_S$ to be a star operation are easily verified except the property that $I_1 \subseteq I_2$ implies $I_1^{*_S} \subseteq I_2^{*_S}$.

Since $I^{*_S} \subseteq I_v$ for all $I \in \mathcal{F}(R)$ and $I_1 \subseteq I_2$ implies $(I_1)_v \subseteq (I_2)_v$, we need only consider the case that $I_1^{*_S} = (I_1)_v \neq I_1$ and $I_2^{*_S} = I_2 \neq (I_2)_v$.

Suppose $(I_1)_v = (I_2)_v$. Then an element $a \in I_1 \setminus \{0\}$ exists such that $R \subsetneq a^{-1}I_1 \subsetneq a^{-1}I_2 \subsetneq a^{-1}(I_2)_v = V$ by Lemmas 2.1 and 2.2. Therefore, $\mathrm{rank}\, I_2 > \mathrm{rank}\, I_1 \geq m$, and hence $I_2 \not\sim J$ for any $J \in (\cup_{i=1}^{m-1} X_i) \cup S$. Therefore, ${I_2}^{*_S} = (I_2)_v$, which is a contradiction to our assumption. Thus, we have $(I_1)_v \subsetneq (I_2)_v$. Then, by [**7**, Proposition 2.14], $(I_1)_v \subseteq (I_2)_v M = I_2 V M = I_2 M \subseteq I_2$. Therefore, ${I_1}^{*_S} = (I_1)_v \subseteq I_2 = I_2^{*_S}$.

Claim 2. Fix an integer m, then each distinct subset S of X_m gives a distinct star operation $*_S$ on R.

Let S, T be distinct subsets of X_m. We may assume that $S \not\subseteq T$. Choose $I \in S \setminus T$. Then I is nondivisorial, $I^{*_S} = I$, and $I^{*_T} = I_v$. Therefore, $*_S \neq *_T$.

Claim 3. Let $2 \leq i < j < n$. For subsets S of X_i and T of X_j, $*_S = *_T$ if and only if $j = i+1$, $S = X_i$, and $T = \varnothing$.

Note that $(\cup_{m=2}^{j-1} X_m) \cup T = (\cup_{m=2}^{i-1} X_m) \cup (\cup_{m=i}^{j-1} X_m) \cup T \supseteq (\cup_{m=2}^{i-1} X_m) \cup S$. Suppose that the inclusion is proper. Choose an ideal $I \in (\cup_{m=2}^{j-1} X_m) \cup T \setminus (\cup_{m=2}^{i-1} X_m) \cup S$. Then $I^{*_T} = I \neq I_v = I^{*_S}$, and hence $*_T \neq *_S$. Therefore, $*_S = *_T$ if and only if $(\cup_{m=2}^{j-1} X_m) \cup T = (\cup_{m=2}^{i-1} X_m) \cup (\cup_{m=i}^{j-1} X_m) \cup T = (\cup_{m=2}^{i-1} X_m) \cup S$, which holds only when $j = i+1$, $S = X_i$, and $T = \varnothing$.

Therefore, it follows that $|\mathrm{Star}\,(R)| \geq |\mathcal{P}(X_2)| + |\mathcal{P}(X_3)| + \cdots + |\mathcal{P}(X_{n-1})| - (n-3) = 2^{l_2} + 2^{l_3} + \cdots + 2^{l_{n-1}} - (n-3)$. □

Theorem 2.6. *If* $\dim_k L = 3$, *then* $|\mathrm{Star}\,(R)| = 2$. *In fact,* $\mathrm{Star}\,(R) = \{d, v\}$ *and the ideals of rank* 2 *are precisely the nondivisorial ideals of* R.

Proof. By the above theorem, it suffices to show that $l_2 = 1$.

Let $L = k(x)$, and let $\mathrm{irr}\,(x,k) = X^3 + e_2X^2 + e_1X + e_0$, $e_i \in k$. Fix $U = k + kx$, and let $W = k + ky$ be a k-vector space of dimension 2 between k and L. We may assume that $y = a_1x + a_2x^2$, $a_1, a_2 \in k$. If $a_2 = 0$, then $a_1 \neq 0$ and $U = W$. Assume that $a_2 \neq 0$. Then we may assume that $y = cx + x^2$ for some $c \in k$. It is easy to check that $U = (e_2 - c + x)W$.

Thus, every two-dimensional k-vector space between k and L is equivalent to each other under the equivalence relation $\sim$ defined in Proposition 2.4. This implies that $l_2 = 1$. □

Lemma 2.7. *Let* T *be an overring of* R *contained in* V. *Define a mapping* $*_T : \mathcal{F}(R) \to \mathcal{F}(R)$ *by* $*_T(I) = I$ *if* I *is divisorial,* IT *if* I *is not divisorial. Then*

(1) $*_T$ *is a star operation. In particular,* $*_R = d$ *and* $*_V = v$.

(2) *If* T_1 *and* T_2 *are distinct rings properly between* R *and* V, *then* $*_{T_1} \neq *_{T_2}$.

Proof. (1) We will only show that, if $I \subseteq J$, then $I^{*_T} \subseteq J^{*_T}$; the other axioms are easily verified.

It suffices to consider the case when I is nondivisorial and J is divisorial. If J is not principal, then $J = JV$ by [**7**, Corollary 2.15]. Therefore, $I^{*_T} = IT \subseteq JT \subseteq JV = J = J^{*_T}$. Assume that J is principal, say bR. Then, since $I \subsetneq J = bR$, $I \subseteq bM$. Therefore, $I^{*_T} = IT \subseteq IV \subseteq bMV = bM \subseteq bR = J = J^{*_T}$.

(2) By Lemma 2.2, T_1 and T_2 are not divisorial. Since T_1 and T_2 are distinct, we may assume that $T_2 \not\subseteq T_1$. Then $(T_1)^{*_{T_1}} = T_1 \subsetneq T_1T_2 = (T_1)^{*_{T_2}}$. □

Theorem 2.8. (1) *If* $\dim_k L = n \geq 3$, *then* $|\mathrm{Star}\,(R)| \geq n-1$. *In particular, if* $\dim_k L = \infty$, *then* $|\mathrm{Star}(R)| = \infty$.

(2) *If* L *is not a simple extension field of* k, *then* $|\mathrm{Star}\,(R)| = \infty$.

Proof. (1) Since for each m with $2 \leq m < n$, $l_m \geq 1$, it follows from Theorem 2.5 that $|\mathrm{Star}\,(R)| \geq 2^{l_2} + 2^{l_3} + \cdots + 2^{l_{n-1}} - (n-3) \geq n-1$.

(2) Assume that L is a finite extension field of k. By Artin's theorem, L is a simple extension of k if and only if there are only finitely many intermediate fields. Therefore, if L is not a simple extension of k, then there are infinitely many rings between R and V. Therefore, $|\mathrm{Star}\,(R)| = \infty$ by Lemma 2.7. □

Remark 2.9. Let $\dim_k L = n$. In Theorems 2.3 and 2.6, we have shown that when $n = 2$ or 3, $|\mathrm{Star}\,(R)| = n-1$. From the above theorem, we know that when $n \geq 4$, $|\mathrm{Star}\,(R)| \geq n-1$. It would be interesting to know when the equality holds. We can show that it never happens. In fact, if $n \geq 4$, then $|\mathrm{Star}\,(R)| \geq n+1$. By the above theorem, we may assume that L is a finite simple extension of k, say $k(x)$. Then it is easy to see that $k + kx, k + kx^2, \ldots, k + kx^{[n/2]}$ are not equivalent, where $[n/2]$ is the largest integer less than or equal to $n/2$. Therefore, $l_2 \geq [n/2]$, and hence $|\mathrm{Star}\,(R)| \geq 2^{l_2} + 2^{l_3} + \cdots + 2^{l_{n-1}} - (n-3) \geq 2^{l_2} + 2(n-3) - (n-3) = 2^{l_2} + n - 3 \geq 2^{[n/2]} + n - 3 \geq 2^2 + n - 3 = n+1$.

Theorem 2.10. *If* $\dim_k L = n \geq 4$ *and* $|k| = \infty$, *then* $|\mathrm{Star}\,(R)| = \infty$.

Proof. By Theorem 2.8, we may assume that L is a finite simple extension of k, say $k(x)$. By Theorem 2.5, it suffices to show that $l_2 = \infty$.

Case I. $n \geq 5$. Let $U = k + k(x + f_1x^2)$ and $W = k + k(x + f_2x^2)$, where f_1, f_2 are distinct nonzero elements of k. We claim that $U \not\sim W$.

Suppose not; then $k + k(x + f_1x^2) = \beta(k + k(x + f_2x^2))$ for some $\beta \in L \setminus \{0\}$. That is, an element $\beta \neq 0$ exists such that $\beta \in U$ and $\beta(x + f_2x^2) \in U$. We can write $\beta = a + b(x + f_1x^2)$, $\beta(x + f_2x^2) = c + d(x + f_1x^2)$, where $a, b, c, d \in k$. Then $ax + (af_2 + b)x^2 + b(f_1 + f_2)x^3 + bf_1f_2x^4 = c + dx + df_1x^2$. Since $\deg(x, k) \geq 5$, we have $bf_1f_2 = 0$, $b(f_1 + f_2) = 0$, $af_2 + b = df_1$, $a = d$, $c = 0$. Since f_1 and f_2 are distinct nonzero elements of k, $b = 0$ and $a = d = 0$. Therefore, we have $\beta = 0$, a contradiction.

Thus, if $|k| = \infty$, then for each $m > 1$, we can choose distinct nonzero elements $f_1, \dots, f_m$ of k. Let $U_i = k + k(x + f_ix^2)$. Then $U_1, \dots, U_m$ are not equivalent, and hence $l_2 \geq m$. Therefore, $l_2 = \infty$.

Case II. $n = 4$. Let $\operatorname{irr}(x, k) = X^4 - e_3X^3 - e_2X^2 - e_1X - e_0$, $e_i \in k$. Let $U = k + k(x + f_1x^2)$ and $W = k + k(x + f_2x^2)$, where f_1, f_2 are distinct elements of k such that $(1 + f_1e_3)f_2 \neq -f_1$. We claim that $U \not\sim W$. Suppose not; then $k + k(x + f_1x^2) = \beta(k + k(x + f_2x^2))$ for some $\beta \in L \setminus \{0\}$. We can write $\beta = a + b(x + f_1x^2)$, $\beta(x + f_2x^2) = c + d(x + f_1x^2)$, where $a, b, c, d \in k$. Then $ax + (af_2 + b)x^2 + b(f_1 + f_2)x^3 + bf_1f_2x^4 = c + dx + df_1x^2$, i.e., $bf_1f_2e_0 + (a + bf_1f_2e_1)x + (af_2 + b(1 + f_1f_2e_2))x^2 + b(f_1 + f_2 + f_1f_2e_3)x^3 = c + dx + df_1x^2$. Since $\deg(x, k) = 4$, we have $b(f_1 + f_2 + f_1f_2e_3) = 0$, $af_2 + b(1 + f_1f_2e_2) = df_1$, $a + bf_1f_2e_1 = d$, $bf_1f_2e_0 = c$. Since $f_1 + f_2 + f_1f_2e_3 \neq 0$ and $f_1 \neq f_2$, $b = 0$ and $a = d = 0$. Therefore, we have $\beta = 0$, a contradiction.

Thus, if $|k| = \infty$, then for each $m > 1$, we can choose inductively distinct elements $f_1, \dots, f_m$ of k such that for each i with $1 < i \leq m$, $(1 + f_je_3)f_i \neq -f_j$ for all $j < i$. Let $U_i = k + k(x + f_ix^2)$. Then $U_1, \dots, U_m$ are not equivalent, and hence $l_2 \geq m$. Therefore, $l_2 = \infty$. ◻

Corollary 2.11. *Let* $\dim_k L = n \geq 4$. *Then* $|\mathrm{Star}(R)| < \infty$ *if and only if* $n < \infty$ *and* $|k| < \infty$, *i.e.,* L *is a finite field.*

Proof. The "only if" part follows from Theorem 2.8 (1) and Theorem 2.10. For the "if" part, let X be a complete set of class representatives under the equivalence relation $\sim$ on the set of nonzero nondivisorial fractional ideals of R as in the proof of Theorem 2.5. Then by Proposition 2.4, $|X| \leq |\mathcal{P}(L)|$, and hence $|\mathrm{Star}(R)| \leq 2^{|X|} \leq 2^{2^{|L|}} < \infty$. ◻

Lemma 2.12. *Let L be a simple extension of k of degree 4. Then $l_3 = 1$.*

Proof. By Proposition 2.4 (3), it suffices to show every three-dimensional k-vector space between k and L is equivalent under the equivalence relation $\sim$.

Let $L = k(x)$, and let $\operatorname{irr}(x, k) = X^4 - e_3X^3 - e_2X^2 - e_1X - e_0$, $e_i \in k$.

Fix $U = k + kx + kx^2$, and let W be a three-dimensional k-vector space between k and L. We can write $W = k + k(ax + bx^2 + cx^3) + k(a'x + b'x^2 + c'x^3)$, $a, a', b, b', c, c' \in k$. If $c = c' = 0$, then $W \subseteq U$ and hence $W = U$. Assume that $c \neq 0$. Then since $k(ax + bx^2 + cx^3) = k((a/c)x + (b/c)x^2 + x^3)$, we may assume that $c = 1$. Thus $W = k + k(ax + bx^2 + x^3) + k(a'x + b'x^2 + c'x^3) = k + k(ax + bx^2 + x^3) + k((a' - c'a)x + (b' - c'b)x^2)$. Since $\dim_k W = 3$, $a' - c'a \neq 0$ or $b' - c'b \neq 0$, and hence $W = k + k(cx + x^2) + k(ax + bx^2 + x^3)$ or $k + k(x + cx^2) + k(ax + bx^2 + x^3)$ for some $c \in k$.

First consider the case $W = k + k(cx + x^2) + k(ax + bx^2 + x^3)$.

We want to find $\beta = d_0 + d_1x + d_2x^2$, where $d_0, d_1, d_2 \in k$, not all zero, such that $\beta(cx + x^2) \in U$ and $\beta(ax + bx^2 + x^3) \in U$. These two conditions are equivalent to the following equations:

$$\begin{aligned} &d_1 + d_2(c + e_3) = 0 \\ &d_0 + d_1(b + e_3) + d_2(a + e_2 + be_3 + e_3^2) = 0. \end{aligned}$$

It is clear that nontrivial solutions $(d_0, d_1, d_2) \in k^3$ exist satisfying the above homogeneous system of linear equations.

Similarly, for the second case $W = k + k(x + cx^2) + k(ax + bx^2 + x^3)$, we can also find an element $\beta \in L \setminus \{0\}$ such that $U = \beta W$. □

Proposition 2.13. *Let L be a simple extension of k of degree 4. Then $|\operatorname{Star}(R)| = 2^{l_2} + 1$ (which is finite if and only if $|k| < \infty$).*

Proof. Let I be a nondivisorial ideal of R of rank 3 between R and V, and let $I_1, \ldots, I_{l_2}$ be nonequivalent nondivisorial ideals of R of rank 2 between R and V.

Let $*$ be a star operation on R. Then $*$ is completely determined by the images of $I, I_1, \ldots, I_{l_2}$. Note that I^* must be equal to I or $V (= I_v)$.

Case I. $I^* = I$. We claim that $I_i^* = I_i$ for all $i = 1, \ldots, l_2$, and hence that $* = d$.

Since $\operatorname{rank} I_i = 2$, I_i is contained in an ideal of rank 3 between R and V. Since $l_3 = 1$ (Lemma 2.12), it is of the form aI for some $a \in V \setminus M$. Therefore, $R \subsetneq I_i \subseteq I_i^* \subseteq aI^* = aI \subsetneq V$, whence $I_i^* = I_i$ or $I_i^* = aI$.

Choose an element $\alpha \in V \setminus aI$. Then $I_i + \alpha R$ is an ideal of rank 3 between R and V, and hence it is of the form bI for some $b \in V \setminus M$. Note that $aI \neq bI$. Suppose that $I_i^* = aI$. Then $aI = I_i^* \subseteq (bI)^* = bI^* = bI$, which implies $aI = bI$, a contradiction. Therefore, $I_i^* = I_i$.

Case II. $I^* = V$. We claim that $I_i^* = I_i$ or $I_i^* = V$.

Since $R \subsetneq I_i \subseteq I_i^* \subseteq (I_i)_v = V$, we have $I_i^* = I_i$, $\operatorname{rank} I_i^* = 3$, or $I_i^* = V$. Suppose that $\operatorname{rank} I_i^* = 3$. Then $I_i^* = aI$ for some $a \in V \setminus M$. Since $*$ is a star operation, $aI = I_i^* = (I_i^*)^* = (aI)^* = aI^* = aV = V$, a contradiction.

Thus we have $|\operatorname{Star}(R)| \leq 2^{l_2} + 1$. Also, by Theorem 2.5 and Lemma 2.12, $|\operatorname{Star}(R)| \geq 2^{l_2} + 2^{l_3} - (\dim_k L - 3) = 2^{l_2} + 2^1 - (4-3) = 2^{l_2} + 1$. Therefore, we can conclude that $|\operatorname{Star}(R)| = 2^{l_2} + 1$. □

Example 2.14. If $\dim_k L = 4$ and $k = \mathbf{Z}_2$, then $|\operatorname{Star}(R)| = 2^3 + 1 = 9$:

Let $x \in L$ be a zero of the irreducible polynomial $X^4 + X^3 + 1$ over $\mathbf{Z}_2$. Then $L = k(x)$. In order to show that $l_2 = 3$, it suffices to show that the cardinality of distinct equivalence classes of the set W_2 of two dimensional k-vector spaces between k and L is 3. Note that $W_2 = \{k + k(ax + bx^2 + cx^3) \mid a, b, c \in k, \text{ not all zero}\}$.

Through a simple computation, we have three distinct equivalence classes $k + kx \sim k + kx^3 \sim k + k(x^2 + x^3)$, $k + kx^2 \sim k + k(x + x^2) \sim k + k(x + x^2 + x^3)$, $k + k(x + x^3)$. In fact, $k + kx = (1 + x)(k + kx^3) = x(k + k(x^2 + x^3))$, $k + kx^2 = x^2(k + k(x + x^2)) = (1 + x^2)(k + k(x + x^2 + x^3))$, and $k + kx$, $k + kx^2$, $k + k(x + x^3)$ are not equivalent.

In order to have a quick reference for the future, we summarize the main results of this section.

Theorem 2.15. *Let R be a pseudo-valuation domain with residue field k, and let V be the associated valuation domain of R with residue field L.*

(1) *R is a divisorial domain if and only if R is a divisorial valuation domain, or* $\dim_k L = 2$.

(2) $|\mathrm{Star}\,(R)| < \infty$ *if and only if* $\dim_k L = 1, 2, 3$, *or L is a finite field.*

(3) $|\mathrm{SStar}\,(R)| < \infty$ *if and only if* $|\mathrm{Star}\,(R)| < \infty$ *and* $\dim R < \infty$.

Proof. (1) See Theorems 1.1, 2.3 and 2.8.

(2) See Theorem 2.3, Theorem 2.6 and Corollary 2.11.

(3) Each overring of R is comparable to V, and hence by Theorem 1.1 (6), we have

$$\begin{aligned}|\mathrm{SStar}\,(R)| &= \sum_{R\subseteq T\subseteq K} |\mathrm{Star}\,(T)| \\ &= \sum_{R\subseteq T\subsetneq V} |\mathrm{Star}\,(T)| + \sum_{V\subseteq T\subseteq K} |\mathrm{Star}\,(T)| \\ &= \sum_{R\subseteq T\subsetneq V} |\mathrm{Star}\,(T)| + |\mathrm{SStar}\,(V)|.\end{aligned}$$

Therefore,

$$\begin{aligned}|\mathrm{SStar}\,(R)| < \infty \Longleftrightarrow &\sum_{R\subseteq T\subsetneq V} |\mathrm{Star}\,(T)| < \infty \\ &\qquad \text{and } |\mathrm{SStar}\,(V)| < \infty \\ \Longleftrightarrow &\ |\mathrm{Star}\,(R)| < \infty \text{ and } \dim R < \infty.\end{aligned}$$

By Theorem 1.1 (3), $|\mathrm{SStar}\,(V)| < \infty$ if and only if $\dim V < \infty$, i.e., $\dim R < \infty$. Therefore, in the above second equivalence, the direction ($\Rightarrow$) is obvious. Conversely, if $|\mathrm{Star}\,(R)| < \infty$, then $\dim_k L = 1, 2, 3$, or L is a finite field. In this case, there are only finitely many rings between R and V, and they are all pseudo-valuation domains whose residue fields are between k and L. Therefore, $\sum_{R\subseteq T\subsetneq V} |\mathrm{Star}\,(T)| < \infty$. $\square$

REFERENCES

1. D.D. Anderson and D.F. Anderson, *Examples of star operations on integral domains*, Comm. Algebra **18** (1990), 1621–1643.

2. D.F. Anderson and D.E. Dobbs, *Pairs of rings with the same prime ideals*, Canad. J. Math. **32** (1980), 362–384.

3. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

4. S. Bazzoni, *Divisorial domains*, Forum Math. **12** (2000), 397–419.

5. D.E. Dobbs and R. Fedder, *Conducive integral domains*, J. Algebra **86** (1984), 494–510.

6. R. Gilmer, *Multiplicative ideal theory*, Dekker, New York, 1972.

7. J.R. Hedstrom and E.G. Houston, *Pseudo-valuation domains*, Pacific J. Math. **75** (1978), 137–147.

8. ———, *Pseudo-valuation domains* II, Houston J. Math. **4** (1978), 199–207.

9. W. Heinzer, *Integral domains in which each non-zero ideal is divisorial*, Mathematica **15** (1968), 164–170.

10. R. Matsuda and T. Sugatani, *Semistar-operations on integral domains*, II, Math. J. Tayama Univ. **18** (1995), 155–161.

11. A. Mimouni and M. Samman, *On the cardinality of semistar operations on integral domains*, Comm. Algebra **33** (2005), 3311–3321.

12. A. Okabe and R. Matsuda, *Semistar-operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1–21.

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