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# EXISTENCE OF POSITIVE SOLUTIONS FOR THE $p(x)$-LAPLACIAN EQUATION

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ABSTRACT. In this paper, we study the existence of positive solutions for the $p(x)$-Laplacian equation based on the Krasnoselskii fixed point theorem on the cone. Our efforts mainly center on the establishment of the global $C^{1,\alpha}$ estimates on bounded weak solutions and the Harnack inequality which, together with the blow-up argument and Liouville type theorem, plays a key role in the a priori estimates.

**1. Introduction.** In this paper, we consider the following problem

$$\begin{cases} -\Delta_{p(x)}u = f(x,u,\nabla u) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \\ u(x) > 0 & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary, $N \geq 2$, $\Delta_{p(x)}$ is the $p(x)$-Laplacian operator, namely,

$$\Delta_{p(x)}u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right),$$

$p(x)$ and $f$ satisfy some conditions, which will be mentioned later.

For the case $p(x) \equiv$ Constant, there is a rich literature concerning problem (1.1), see e.g., [**2–4, 18, 19, 23, 24**] and the references therein. Azizieh and Clément [**2**] obtained the existence of positive solutions for problem (1.1) with $f$ depending only upon $u$. Later, Ruiz [**19**] and Zou [**24**] extended the results by considering the general case where

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$f$ depends upon $x$, $u$ and $\nabla u$ with disparate conditions, respectively. While, for the case $p(x) \not\equiv$ Constant, most of the current research focuses on problem (1.1) with $f = f(x, u)$, which can be solved by the variational approach and the upper and lower solutions method, see e.g., [**5–8, 11**] and the references therein.

In the present paper, we extend Ruiz's results [**19**] to the $p(x)$-Laplacian, namely, we establish the existence of solutions for problem (1.1). Since the exponent $p(x)$ is not a constant, the methods, which can usually deal with the case $p(x) \equiv$ Constant, are inappropriate. For example, one cannot expect that the first eigenvalue of the $p(x)$-Laplacian is always positive; one cannot use the eigenfunction for the first eigenvalue of the $p(x)$-Laplacian to construct upper and lower solutions, etc. The proofs are more complex than those for the constant case. Furthermore, due to the appearance of $\nabla u$ in $f$, the variational approach is no longer suitable. In this paper, we use the topological method to deal with problem (1.1). Our efforts center on the establishment of the Harnack inequality which, together with the blow-up argument and Liouville type theorem, plays a key role in the a priori estimates. After obtaining the a priori estimates, we can use the Krasnoselskii fixed point theorem on the cone to obtain the existence of solutions for problem (1.1). To verify the conditions which satisfy the fixed point theorem, we need a similar conclusion as that in [**19**]. However, the proof of this conclusion in this paper is quite different from that in [**19**].

This paper is organized as follows. In Section 2, we introduce some necessary preliminaries. In Section 3, we use an iteration technique to establish the Harnack inequality, for solutions of the problem (1.1), which will be used in Section 4 to obtain the $L^\infty$-norm estimates, by applying the Liouville theorem based on the blow-up argument. Next, in Section 5, we prove the existence of positive solutions for problem (1.1) based upon the Krasnoselskii fixed point theorem on the cone. Finally, in the Appendix (Sections 6 and 7), we give the detailed proof of some estimates on the weak solutions, more specifically, the $C^\alpha$ estimates and $C^{1,\alpha}$ estimates, respectively.

**2. Preliminaries.** In this section, we introduce some preliminary definitions on the space $W^{1,p(x)}(\Omega)$ and several preliminary lemmas, which will be used in the following sections. Let $\Omega$ be an open subset in $\mathbf{R}^N$ and $p(x)$ a bounded measurable function defined on $\mathbf{R}^N$ which satisfies

$$1 < p_- = \inf_{\mathbf{R}^N} p(x) \le \sup_{\mathbf{R}^N} p(x) = p_+ < \infty, \quad x \in \mathbf{R}^N.$$

The variable exponent Lebesgue space $L^{p(x)}$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u \mid u : \Omega \to R \text{ is measurable, } \int_\Omega |u|^{p(x)}\, dx < \infty \right\}$$

with the norm

$$\|u\|_{p(x)} = \inf \left\{ \sigma > 0 \mid \int_\Omega \left| \frac{u}{\sigma} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \mid u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. More elementary properties on the space $W^{1,p(x)}(\Omega)$ can be seen in [**13**].

On the space $W^{1,p(x)}(\Omega)$, the following lemma holds.

**Lemma 2.1** [**9**, Lemma 2.5]. *Let $\Omega$ be a domain, $u \in W^{1,p(x)}(\Omega)$, $p(x)$ a bounded measurable function on $\Omega$ which satisfies*:

$$1 < p_- \le p(x) \le p_+ < \infty \quad \textit{and} \quad R^{-\operatorname{osc}\{p;\Omega_R\}} \le L,$$

*where $p_-$, $p_+$ and $L$ are positive constants,*

$$\operatorname{osc}\{p;\Omega_R\} = \sup_{\Omega_R} p(x) - \inf_{\Omega_R} p(x),$$

*and $\Omega_R = \Omega \cap B_R$ for any ball $B_R$ with radius $R$. Then there exist positive constants $\varepsilon$, $R_0$ and $C$ depending only upon $N$, $p_-$, $p_+$ and $L$, such that*

$$\frac{1}{R^N}\int_{B_R}\left|\frac{u-u_R}{R}\right|^{p(x)}dx \le C + C\left(\frac{1}{R^N}\int_{B_R}|\nabla u|^{p(x)/(1+\varepsilon)}dx\right)^{1+\varepsilon}$$

*for any* $B_R \subseteq \Omega$ *with* $R \le R_0$ *and* $\int_{B_R}|\nabla u|^{p(x)}dx \le 1$.

The next three lemmas are taken from [**15**], and they will be used in the next section to prove the Hölder continuity of the functions in the class $\mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$.

**Lemma 2.2** [**15**, Chapter II, Lemma 3.9]. *For any* $u \in W^{1,1}(B_\rho)$ *and arbitrary numbers* $k$ *and* $l$ *with* $l > k$, *the following inequality holds*:

$$(l-k)|B_{l,\rho}|^{1-(1/N)} \le \frac{\beta(N)\rho^N}{|B_\rho \setminus B_{k,\rho}|}\int_{B_{k,\rho}\setminus B_{l,\rho}}|\nabla u|\,dx$$

*where* $B_{k,\rho} := \{x \in B_\rho \mid u(x) > k\}$ *and* $\beta(N) > 1$ *is a constant.*

**Lemma 2.3** [**15**, Chapter II, Lemma 4.7]. *Suppose a sequence* $y_h$, $h = 0, 1, \dots$, *of nonnegative numbers satisfies the recursion relation*

$$y_{h+1} \le cb^h y_h^{1+\varepsilon}, \quad h = 0, 1, \dots,$$

*with some positive constants* $c, \varepsilon$ *and* $b > 1$. *If* $y_0 \le \theta = c^{-1/\varepsilon}b^{-1/\varepsilon^2}$, *then* $y_h \le \theta b^{-h/\varepsilon}$, *and consequently* $y_h \to 0$ *as* $h \to \infty$.

**Lemma 2.4** [**15**, Chapter II, Lemma 4.8]. *Suppose a function* $u$ *is measurable and bounded in some ball* $B_{\rho_0}$ *or* $\Omega_{\rho_0} := \Omega \cap B_{\rho_0}$. *Consider balls* $B_\rho$ *and* $B_{b\rho}$ *which have a common center with* $B_{\rho_0}$, *where* $b > 1$ *is a fixed constant, and suppose that for any* $0 < \rho < b^{-1}\rho_0$, *at least one of the following two inequalities holds*

$$\operatorname{osc}\{u; \Omega_\rho\} \le c_1\rho^\varepsilon, \qquad \operatorname{osc}\{u; \Omega_\rho\} \le \theta\{u; \Omega_{b\rho}\},$$

*where* $c_1, \varepsilon \le 1$ *and* $\theta < 1$ *are positive constants. Then, for any* $\rho \le \rho_0$, *we have the following estimates*

$$\operatorname{osc}\{u; \Omega_\rho\} \le c\rho_0^{-\alpha}\rho^\alpha,$$

*where* $\alpha = \min\{\varepsilon, -\log_b\theta\}$, $c = b^\alpha\max\{c_1\rho_0^\varepsilon, \omega_0\}$ *and* $\omega_0 = \operatorname{osc}\ \{u; \Omega_{\rho_0}\}$.

**3. The Harnack inequality.** Let $\Omega$ be an arbitrary domain in $\mathbf{R}^N$. In this section, we are going to establish the Harnack inequality for the weak solutions of the differential inequality with the form

$$K_1 u^{q(x)} - K_2\big(|\nabla u|^{\lambda(x)} + 1\big) \le -\Delta_{p(x)} u \le K_2\big(u^{q(x)} + |\nabla u|^{\lambda(x)} + 1\big), \quad x \in \Omega, \tag{3.1}$$

where $K_1$ and $K_2$ are positive constants. We say $u$ is a weak solution of the above inequality, if $u \in C^1(\overline{\Omega})$ and for any $\eta \in C_0^\infty(\Omega)$, $\eta \ge 0$, it follows

$$\begin{aligned}\int_\Omega \Big[K_1 u^{q(x)} - K_2\big(|\nabla u|^{\lambda(x)} + 1\big)\Big]\,\eta(x)\,dx &\\ \le \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\eta\,dx &\\ \le K_2 \int_\Omega \Big(u^{q(x)} + |\nabla u|^{\lambda(x)} + 1\Big)\,\eta\,dx.&\end{aligned}$$

Throughout this section (and the next section), we always suppose that the functions $p(x)$, $q(x)$ and $\lambda(x)$ satisfy the following assumption

(H1) $p(x) \in C^1(\overline{\Omega})$, $q(x) \in C^{\alpha_0}(\overline{\Omega})$, $\lambda(x) \in C(\overline{\Omega})$, and

$$\begin{gathered} 1 < p(x) < N, \\ p(x) - 1 < q(x) < \frac{N(p(x)-1)}{N-p(x)}, \\ p(x) - 1 \le \lambda(x) < \frac{p(x)q(x)}{q(x)+1}, \end{gathered}$$

for any $x \in \overline{\Omega}$.

**Lemma 3.1.** *Assume $u \ge 1$ be a weak solution of* (3.1). *Let $B_{2R}$ and $B_R$ be two concentric balls contained in $\Omega$. Denote by $p_1$ and $q_1$ the minimum, $p_2$ and $q_2$ the maximum, of $p(x)$ and $q(x)$ on $\overline{B_{2R}}$, respectively. Take $R_0 = R_0(p,q) \le 1$ small enough, such that $q_1 > p_2 - 1$ for any $R \le R_0$. Then there exists a positive constant $C = C(N,p,q,\lambda,\gamma)$, such that for any $\gamma \in (0, q_1)$ and $\mu \in (0, p_1q_1/(q_1+1))$, the following holds*:

$$\int_{B_R} u^\gamma\,dx \le CR^{N-[\gamma p_2/(q_1-p_2+1)]},$$

*and*

$$\int_{B_R} |\nabla u|^\mu dx \le CR^{N-[(q_1+1)\mu/(q_1-p_2+1)]}.$$

*Proof.* Thanks to the Hölder inequality, we need only to prove the conclusions for the case that $\gamma$ is close enough to $q_1$. Take $\eta(x) \in C_0^\infty(B_{2R})$ such that $0 \le \eta \le 1$ on $B_{2R}$, $\eta \equiv 1$ on $B_R$ and $|\nabla \eta| \le (C(N))/R$ on $B_{2R}$. Let $\phi = \eta^\alpha u^\beta$ be a test function with $\beta < 0$ which is close enough to 0 and $\alpha > 0$ which is large enough. Then

$$\begin{aligned} (3.2) \quad & -\beta \int_{B_{2R}} \eta^\alpha u^{\beta-1} |\nabla u|^p dx + K_1 \int_{B_{2R}} \eta^\alpha u^{q+\beta} dx \\ & \le \int_{B_{2R}} \left( K_2 \eta^\alpha u^\beta |\nabla u|^\lambda + \alpha \eta^{\alpha-1} u^\beta |\nabla u|^{p-1} |\nabla \eta| + K_2 \eta^\alpha u^\beta \right) dx. \end{aligned}$$

By the Young inequality, one can easily conclude that

$$\begin{aligned} K_2 \eta^\alpha u^\beta |\nabla u|^\lambda &\le -\frac{\beta}{3} \eta^\alpha u^{\beta-1} |\nabla u|^p + C \eta^\alpha u^{\beta+[\lambda/(p-\lambda)]}, \\ \alpha \eta^{\alpha-1} u^\beta |\nabla u|^{p-1} |\nabla \eta| &\le -\frac{\beta}{3} \eta^\alpha u^{\beta-1} |\nabla u|^p + C \eta^{\alpha-p} u^{\beta+p-1} |\nabla \eta|^p. \end{aligned}$$

Since $u \ge 1$, putting the previous two inequalities into (3.2), then we have

$$\begin{aligned} (3.3) \quad & \int_{B_{2R}} \eta^\alpha u^{\beta-1} |\nabla u|^p dx + \int_{B_{2R}} \eta^\alpha u^{q+\beta} dx \\ & \le C \int_{B_{2R}} \left( \eta^\alpha u^{\beta+[\lambda/(p-\lambda)]} + \eta^{\alpha-p} u^{\beta+p-1} |\nabla \eta|^p + \eta^\alpha u^\beta \right) dx \\ & \le C \int_{B_{2R}} \left( \eta^\alpha u^{\beta+[\lambda/(p-\lambda)]} + \eta^{\alpha-p} u^{\beta+p-1} |\nabla \eta|^p \right) dx. \end{aligned}$$

Using the Young inequality again, it follows that

$$\begin{aligned} \eta^\alpha u^{\beta+[\lambda/(p-\lambda)]} &\le \frac{\varepsilon}{3} \eta^\alpha u^{\beta+q} + C \eta^\alpha, \\ \eta^{\alpha-p} u^{\beta+p-1} |\nabla \eta|^p &\le \frac{\varepsilon}{3} \eta^\alpha u^{\beta+q} + C \eta^{\alpha-[p(\beta+q)]/[q-p+1]} \\ & \qquad \cdot |\nabla \eta|^{[p(\beta+q)]/[q-p+1]}. \end{aligned}$$

Taking appropriate $\varepsilon$ in the previous two inequalities and putting them into (3.3), then

$$\begin{aligned}(3.4)\quad &\int_{B_{2R}} \eta^\alpha u^{\beta-1}|\nabla u|^p dx + \int_{B_{2R}} \eta^\alpha u^{\beta+q} dx\\ &\quad\le C\int_{B_{2R}} \left(\eta^\alpha + \eta^{\alpha-[p(\beta+q)]/[q-p+1]}|\nabla\eta|^{[p(\beta+q)]/[q-p+1]}\right) dx\\ &\quad\le CR^{N-[(\beta+q_1)p_2]/[q_1-p_2+1]}.\end{aligned}$$

Recalling that $u \ge 1$, it follows from the above inequality that

$$\int_{B_R} u^{\beta+q_1} dx \le \int_{B_{2R}} \eta^\alpha u^{\beta+q} dx \le CR^{N-[(\beta+q_1)p_2]/[q_1-p_2+1]}.$$

Setting $\beta = \gamma - q_1$ in the previous inequality, then it yields

$$\int_{B_R} u^\gamma\, dx \le CR^{N-[\gamma p_2/(q_1-p_2+1)]}.$$

Since $u \ge 1$ and $R \le 1$, it follows from the Young inequality and (3.4) that

$$\begin{aligned}\int_{B_{2R}} \eta^\alpha u^{\beta-1}|\nabla u|^{p_1} dx &\le \int_{B_{2R}} \eta^\alpha u^{\beta-1}(1+|\nabla u|^p)\, dx\\ &\le CR^{N-[(\beta+q_1)p_2/(q_1-p_2+1)]}.\end{aligned}$$

For any $s > \mu/p_1$, which is close enough to $\mu/p_1$, by the Hölder inequality, the previous inequality yields

$$\begin{aligned}\int_{B_R} |\nabla u|^\mu dx &\le \int_{B_{2R}} \eta^\alpha u^{-s}|\nabla u|^\mu u^s dx\\ &\le \left(\int_{B_{2R}} \eta^\alpha u^{-(sp_1/\mu)}|\nabla u|^{p_1} dx\right)^{\mu/p_1}\\ &\quad\cdot\left(\int_{B_{2R}} \eta^\alpha u^{(sp_1)/(p_1-\mu)}\, dx\right)^{(p_1-\mu)/p_1}\\ &\le \left(CR^{N-[(q_1+1-sp_1/\mu)p_2]/[q_1-p_2+1]}\right)^{\mu/p_1}\\ &\quad\cdot\left(CR^{N-[sp_1p_2]/[(p_1-\mu)(q_1-p_2+1)]}\right)^{1-(\mu/p_1)}\\ &= CR^{N-[p_2\mu(q_1+1)]/[p_1(q_1-p_2+1)]}\\ &= CR^{N-[\mu(q_1+1)]/[q_1-p_2+1]}R^{[(p_1-p_2)\mu(q_1+1)]/[p_1(q_1-p_2+1)]}\\ &\le CR^{N-[\mu(q_1+1)]/[q_1-p_2+1]}.\end{aligned}$$

In the last step of the above inequalities, we have used the fact that $R^{[(p_1-p_2)\mu(q_1+1)]/[p_1(q_1-p_2+1)]} \le C$, which can be guaranteed by (H1). The proof is complete. $\square$

The following lemma taken from [**20**] is used to link the integral estimate on the positive and negative power in the Moser iteration procedure.

**Lemma 3.2** (Poincaré, John-Nirenberg). *Let $u \in W^{1,p}(B_R)$, and suppose that*

$$\int_B |\nabla u|^p dx \le K r^{N-p}$$

*for every open ball $B \subseteq B_R$. Then two positive constants $p_0$ and $C$ exist depending only upon $N$, $p$ and $K$, such that*

$$\int_{B_R} e^{p_0 u} dx \cdot \int_{B_R} e^{-p_0 u} dx \le C|B_R|^2.$$

Now, we can state and prove the Harnack inequality.

**Proposition 3.1.** *Assume $u \ge 1$ is a weak solution of* (3.1). *Let $B_{4R}$, $B_{2R}$ and $B_R$ be concentric balls contained in $\Omega$. Then positive constants $C = C(N,p,q,\lambda,\gamma)$ and $R_0 = R_0(p,q)$ exist, such that for any $0 < R < R_0$ the following hold*:

(i) *for any $\gamma > 0$, there exists a $\gamma_1 \in [(\gamma/2), \gamma]$, such that*

$$\sup_{B_R} u \le C\left(\frac{1}{R^N}\int_{B_{2R}} u^{\gamma_1} dx\right)^{1/\gamma_1};$$

(ii) *for any $\gamma < 0$, it follows that*

$$\inf_{B_R} u \ge C\left(\frac{1}{R^N}\int_{B_{2R}} u^{\gamma} dx\right)^{1/\gamma}.$$

*Moreover, if, in addition, $B_{8R} \subseteq \Omega$, then*

$$\sup_{B_R} u \le C \inf_{B_R} u.$$

*Proof.* Take arbitrary $x_0 \in \Omega$ and $0 < R \leq 1$ such that $B_{4R} := B_{4R}(x_0) \subseteq \Omega$, denote by $p_1$ and $q_1$ the minimum, $p_2$ and $q_2$ the maximum of $p(x)$ and $q(x)$ on $\overline{B_{2R}(x_0)}$, respectively, and define

$$v(x) := u(Rx + x_0), \qquad x \in B_4(0).$$

By simple calculations, we conclude that $v(x)$ satisfies the following inequality:

$$\begin{aligned} K_1 R^P v^Q - K_2\big(R^{P-\Lambda}|\nabla v|^\Lambda + R^P\big) - R|\ln R \nabla p(Rx + x_0)||\nabla v|^{P-1} \\ \leq -\Delta_P v \leq K_2\big(R^P v^Q + R^{P-\Lambda}|\nabla v|^\Lambda + R^P\big) \\ + R|\ln R\nabla p(Rx + x_0)||\nabla v|^{P-1}, \quad x \in B_4(0), \end{aligned}$$

where

$$\begin{aligned} P(x) &:= p(Rx + x_0), \\ Q(x) &:= q(Rx + x_0), \\ \Lambda(x) &:= \lambda(Rx + x_0), \quad x \in B_4(0). \end{aligned}$$

Recalling that $R \leq 1$ and $p(x) \in C^1(\overline{\Omega})$, it follows that a positive constant $E = E(\|p\|_{C^1})$ exists, such that $R|\ln R\nabla p(Rx + x_0)| \leq E$ for any $x \in B_4(0)$. Hence, we can rewrite the foregoing inequality as

$$\begin{aligned} (3.5) \quad & K_1 R^P v^Q - K_2\big(R^{P-\Lambda}|\nabla v|^\Lambda + R^P\big) - E|\nabla v|^{P-1} \\ & \leq -\Delta_P v \\ & \leq K_2\big(R^P v^Q + R^{P-\Lambda}|\nabla v|^\Lambda + R^P\big) + E|\nabla v|^{P-1}, \quad x \in B_4(0). \end{aligned}$$

Suppose that $1 \leq r < \rho \leq 2$. Denote by $B_r$ and $B_\rho$ the balls $B_r(0)$ and $B_\rho(0)$, respectively. Let $\eta(x) \in C_0^\infty(B_2(0))$ be the standard cut-off function, such that $0 \leq \eta(x) \leq 1$ on $B_2(0)$, $\eta(x) \equiv 1$ on $B_r$, $\eta \equiv 0$ on $B_\rho^c$ and $|\nabla \eta| \leq 2/(\rho - r)$. Set $K = \max\{K_1, K_2, E\}$. Taking $\phi = \eta^\alpha v^\beta$ as a test function for (3.5) with $\alpha > 0$ large enough and $\beta \neq 0$, we have

$$\begin{aligned} (3.6) \quad & |\beta| \int_{B_\rho} \eta^\alpha v^{\beta-1}|\nabla v|^P\,dx \\ & \leq K \int_{B_\rho} \eta^\alpha v^\beta\big(R^P v^Q + R^{P-\Lambda}|\nabla v|^\Lambda + R^P + |\nabla v|^{P-1}\big)\,dx \\ & \quad + \alpha \int_{B_\rho} \eta^{\alpha-1} v^\beta |\nabla v|^{P-1}|\nabla \eta|\,dx. \end{aligned}$$

By the Young inequality, it follows that

$$
\begin{aligned}
KR^{P-\Lambda}\eta^\alpha v^\beta|\nabla v|^\Lambda &\le \frac{|\beta|}{4}\eta^\alpha v^{\beta-1}|\nabla v|^P\\
&\quad + C|\beta|^{-\Lambda/(P-\Lambda)}R^P\eta^\alpha v^{\beta+\Lambda/(P-\Lambda)},\\
E\eta^\alpha v^\beta|\nabla v|^{P-1} &\le \frac{|\beta|}{4}\eta^\alpha v^{\beta-1}|\nabla v|^P\\
&\quad + C|\beta|^{1-P}\eta^\alpha v^{\beta+P-1},\\
\alpha\eta^{\alpha-1}v^\beta|\nabla v|^{P-1}|\nabla\eta| &\le \frac{|\beta|}{4}\eta^\alpha v^{\beta-1}|\nabla v|^P\\
&\quad + C|\beta|^{1-P}\eta^{\alpha-P}v^{\beta+P-1}|\nabla\eta|^P.
\end{aligned}
$$

Putting the previous three inequalities into (3.6), we obtain

$$
\begin{aligned}
&\frac{|\beta|}{4}\int_{B_\rho}\eta^\alpha v^{\beta-1}|\nabla v|^P dx\\
&\le C\int_{B_\rho}\eta^\alpha R^P\big(|\beta|^{-\Lambda/(P-\Lambda)}v^{\beta+[\Lambda/(P-\Lambda)]}+v^\beta+v^{Q+\beta}\big)\,dx\\
&\quad + C\int_{B_\rho}|\beta|^{1-P}v^{\beta+P-1}\big(\eta^{\alpha-P}|\nabla\eta|^P+\eta^\alpha\big)\,dx.
\end{aligned}
\tag{3.7}
$$

Note that $P-1\le\Lambda/(P-\Lambda)\le Q$, $0\le\eta\le 1$ and $v\ge 1$. Then it follows from (3.7) that

$$
\begin{aligned}
\int_{B_\rho}\eta^\alpha v^{\beta-1}|\nabla v|^P dx &\le C\int_{B_\rho}\eta^\alpha R^P v^{Q+\beta}\big(|\beta|^{-P/(P-\Lambda)}+|\beta|^{-1}\big)\,dx\\
&\quad + C\int_{B_\rho}\eta^{\alpha-P}|\beta|^{-P}v^{\beta+P-1}\big(|\nabla\eta|^P+1\big)\,dx.
\end{aligned}
\tag{3.8}
$$

Since $0<P\le P/(P-\Lambda)<Q+1<q_2+1$, by the Young inequality, this shows

$$
|\beta|^{-P/(P-\Lambda)}+|\beta|^{-1}\le 2|\beta|^{-(q_2+1)}+2,\qquad |\beta|^{-P}\le|\beta|^{-(q_2+1)}+1,
$$

and consequently, by recalling that $v\ge 1$ and $R\le 1$, it follows from

(3.8) that

$$\begin{aligned}\int_{B_\rho}\eta^\alpha v^{\beta-1}|\nabla v|^P dx &\le C\big(|\beta|^{-(q_2+1)}+1\big)\\ &\quad\cdot\bigg[\int_{B_\rho}\eta^\alpha R^P v^{\beta+Q}dx\\ &\qquad+\int_{B_\rho}\eta^{\alpha-P}v^{\beta+P-1}(|\nabla\eta|^P+1)\,dx\bigg]\\ &\le C\big(|\beta|^{-(q_2+1)}+1\big)\\ &\quad\cdot\bigg[R^{p_1}\int_{B_\rho}v^{\beta+q_2}dx\\ &\qquad+(\rho-r)^{-p_2}\int_{B_\rho}v^{\beta+p_2-1}dx\bigg].\end{aligned}$$

Using the Young inequality to the left side of the previous inequality, we obtain

$$\begin{aligned}\int_{B_\rho}\eta^\alpha v^{\beta-1}|\nabla v|^{p_1}dx &\le \int_{B_\rho}\eta^\alpha v^{\beta-1}dx+\int_{B_\rho}\eta^\alpha v^{\beta-1}|\nabla v|^P dx\\ &\le C\big(|\beta|^{-(q_2+1)}+1\big)\\ &\quad\cdot\bigg[R^{p_1}\int_{B_\rho}v^{\beta+q_2}dx+(\rho-r)^{-p_2}\int_{B_\rho}v^{\beta+p_2-1}dx\bigg].\end{aligned}\tag{3.9}$$

Let $p_1+\beta-1=lp_1$. By utilizing the Hölder inequality, it follows from (3.9) that

$$\begin{aligned}\int_{B_\rho}\eta^\alpha v^{\beta-1}|\nabla v|^{p_1}dx &\le C(|\beta|^{-(q_2+1)}+1)\\ &\quad\cdot\bigg[R^{p_1}\bigg(\int_{B_\rho}v^{lp_1t}dx\bigg)^{1/t}\bigg(\int_{B_\rho}v^{(q_2-p_1+1)t'}dx\bigg)^{1/t'}\\ &\qquad+(\rho-r)^{-p_2}\bigg(\int_{B_\rho}v^{lp_1t}dx\bigg)^{1/t}\\ &\qquad\cdot\bigg(\int_{B_\rho}v^{(p_2-p_1)t'}dx\bigg)^{1/t'}\bigg]\end{aligned}\tag{3.10}$$

for any $t > 1$, where $1/t + 1/t' = 1$. Next, we estimate the terms on the right side of inequality (3.10). Taking a positive constant $R_1 = R_1(p, q) \le 1$, such that $R_1$ is smaller than $R_0$ in Lemma 3.1 and $q_1 - p_2 + 1 \ge \varepsilon_1$ for any $0 < R \le R_1$, where $\varepsilon_1$ is a positive constant depending only upon $p(x)$ and $q(x)$. Note that (H1) guarantees the existence of such a $\varepsilon_1$. We now suppose that $0 < R \le R_1$. By using (H1) again, we have

$$R^{p_1 - p_2} = R^{-\mathrm{osc}\{\mathrm{p};\mathrm{B}_{2\mathrm{R}}\}} \le R^{-2\|\nabla p\| R} = e^{-2\|\nabla p\| R \ln R} \le C,$$
$$R^{q_1 - q_2} = R^{-\mathrm{osc}\{\mathrm{q};\mathrm{B}_{2\mathrm{R}}\}} \le R^{-2\|q\|_{C^{\alpha_0}} R} = e^{-2\|q\|_{C^{\alpha_0}} R \ln R} \le C.$$

Applying Lemma 3.1, it follows that

$$\begin{aligned}
R^{p_1} \left( \int_{B_\rho} v^{(q_2 - p_1 + 1)t'} dx \right)^{1/t'}
&= R^{p_1} \left( \frac{1}{R^N} \int_{B_{\rho R}} u^{(q_2 - p_1 + 1)t'} dx \right)^{1/t'} \\
&\le C R^{p_1} \left( \frac{1}{R^N} \int_{B_{2R}} u^{(q_2 - p_1 + 1)t'} dx \right)^{1/t'} \\
&\le C R^{p_1 - [(q_2 - p_1 + 1)p_2]/(q_1 - p_2 + 1)} \\
&\le C R^{[(p_1 - p_2)(q_1 + 1)]/[(q_1 - p_2 + 1)] + [(q_1 - q_2)/(q_1 - p_2 + 1)]} \\
&\le C R^{[(q_1 + 1)/\varepsilon_1](p_1 - p_2) + [(q_1 - q_2)/\varepsilon_1]} \le C,
\end{aligned}$$

and

$$\begin{aligned}
\left( \int_{B_\rho} v^{(p_2 - p_1)t'} dx \right)^{1/t'}
&= \left( \frac{1}{R^N} \int_{B_{\rho R}} u^{(p_2 - p_1)t'} dx \right)^{1/t'} \\
&\le C \left( \frac{1}{R^N} \int_{B_{2R}} u^{(p_2 - p_1)t'} dx \right)^{1/t'} \\
&\le C R^{[(p_1 - p_2)p_2]/(q_1 - p_2 + 1)} \\
&\le C R^{[(p_1 - p_2)p_2/\varepsilon_1]} \le C,
\end{aligned}$$

provided

$$0 < (q_2 - p_1 + 1)t' < q_1 \quad \text{and} \quad 0 < (p_2 - p_1)t' < q_1. \tag{3.11}$$

Hence (3.10) can be simplified as follows
(3.12)

$$\int_{B_\rho} \eta^\alpha v^{\beta-1}|\nabla v|^{p_1} dx \le C(|\beta|^{-(q_2+1)}+1)(\rho-r)^{-p_2}\left(\int_{B_\rho} v^{lp_1 t} dx\right)^{1/t}.$$

Now we prove that condition (3.11) can be satisfied. In fact, we need only verify that $t$ exists, such that

$$0<(q_2-p_1+1)t' \le q_1-\varepsilon_0,$$
$$0<(p_2-p_1)t' \le q_1-\varepsilon_0,$$

and

$$\frac{N}{(N-p_1)t} \ge 1+\varepsilon_0$$

for some suitable $\varepsilon_0>0$, which is small enough and depends only upon $p(x)$ and $q(x)$. Here $1/t+1/t'=1$. Recalling that $q(x)>p(x)-1$ and $p(x)$, $q(x)$ are continuous on $\overline{\Omega}$, we can take a small positive constant $R_0=R_0(p,q)<R_1$, such that $p_2-p_1\le\varepsilon_0\le q_2-p_1+1$ and $q_2-q_1\le\varepsilon_0$ for any $R\le R_0$. Consequently, the problem can be re-changed into verifying the existence of $t$, such that

$$\frac{q_1-\varepsilon_0}{q_1-q_2+p_1-1-\varepsilon_0} \le t \le \frac{N}{(N-p_1)(1+\varepsilon_0)}.$$

Since $q_2-q_1\le\varepsilon_0$, it follows

$$\frac{q_1-\varepsilon_0}{q_1-q_2+p_1-1-\varepsilon_0} \le \frac{q_1-\varepsilon_0}{p_1-1-2\varepsilon_0}.$$

Consequently, if $t$ exists such that

$$\frac{q_1-\varepsilon_0}{p_1-1-2\varepsilon_0} \le t \le \frac{N}{(N-p_1)(1+\varepsilon_0)},$$

then (3.11) can be fulfilled. In fact, by the aid of the continuity of $p(x)$ and $q(x)$ on $\overline{\Omega}$, we only need to let $\varepsilon_0$ and $R_0$ be small enough and verify

$$\frac{q(x)}{p(x)-1} < \frac{N}{N-p(x)}, \quad x\in\overline{\Omega}.$$

And the last inequality is equivalent to

$$q(x) < \frac{N(p(x)-1)}{N-p(x)}, \quad x \in \overline{\Omega},$$

which is included in (H1). Therefore, we can take $t = N/[(N-p_1)(1+\varepsilon_0)]$, such that (3.11) is fulfilled.

Next, we distinguish two cases, that is, $\beta = 1 - p_1$ and $\beta \neq 1 - p_1$, to prove the proposition.

Case I. $\beta = 1 - p_1$. Setting $r = 1$ and $\rho = 2$ in (3.12), then

$$\int_{B_1(0)} |\nabla \ln v|^{p_1} dx \leq C.$$

Utilizing the Hölder inequality applied to the previous inequality, it yields

$$\int_{B_1(0)} |\nabla \ln v| \, dx \leq C,$$

and consequently,

$$\int_{B_R(x_0)} |\nabla \ln u| \, dx \leq C R^{N-1}. \tag{3.13}$$

If we assume in addition that $B_{8R}(x_0) \subseteq \Omega$, then a point $x \in B_{2R}(x_0)$ and $\iota > 0$ exist such that $B = B_\iota(x) \subseteq B_{2R}(x_0)$. Obviously, $B_{4\iota}(x) \subseteq B_{8R}(x_0) \subseteq \Omega$. Noticing that (3.13) holds for any $x_0 \in \Omega$ and $0 < R \leq R_0$ satisfies $B_{4R}(x_0) \subseteq \Omega$, we have

$$\int_B |\nabla \ln u| \, dy = \int_{B_\iota(x)} |\nabla \ln u| \, dy \leq C r^{N-1}.$$

Applying Lemma 3.2 to the foregoing inequality, a positive constant $\gamma_0$ exists, such that

$$\int_{B_{2R}(x_0)} u^{\gamma_0} dx \cdot \int_{B_{2R}(x_0)} u^{-\gamma_0} dx \leq C R^{2N}, \tag{3.14}$$

provided $B_{8R}(x_0) \subseteq \Omega$.

Case II. $\beta \neq 1 - p_1$ and $\beta \neq 0$. Recalling that $p_1 + \beta - 1 = lp_1$ and $|\nabla \eta| \leq [C(N)/(\rho - r)]$, it follows from the Hölder inequality that

$$
\begin{aligned}
(3.15)\quad & \int_{B_\rho} |\nabla(\eta^{\alpha/p_1} v^l)|^{p_1}\,dx \\
&= \int_{B_\rho} \left| \frac{\alpha}{p_1} \eta^{(\alpha/p_1)-1} v^l \nabla \eta + l \eta^{\alpha/p_1} v^{l-1} \nabla v \right|^{p_1} dx \\
&\leq C \left[ \int_{B_\rho} \eta^{\alpha - p_1} v^{lp_1} |\nabla \eta|^{p_1}\,dx + |l|^{p_1} \int_{B_\rho} \eta^{\alpha} v^{\beta - 1} |\nabla v|^{p_1}\,dx \right] \\
&\leq C \left[ (\rho - r)^{-p_1} \left( \int_{B_\rho} v^{lp_1 t} dx \right)^{1/t} + |l|^{p_1} \int_{B_\rho} \eta^{\alpha} v^{\beta - 1} |\nabla v|^{p_1}\,dx \right]
\end{aligned}
$$

for any $t > 1$. Putting (3.12) into (3.15), then it follows from the Sobolev embedding theorem that

(3.16)

$$
\begin{aligned}
\left( \int_{B_r} v^{lp_1^*} dx \right)^{p_1/p_1^*} &\leq \left( \int_{B_\rho} |\eta^{\alpha/p_1} v^l|^{p_1^*} dx \right)^{p_1/p_1^*} \\
&\leq C \int_{B_\rho} |\nabla(\eta^{\alpha/p_1} v^l)|^{p_1}\,dx \\
&\leq C \Big[ (\rho - r)^{-p_1} + |l|^{p_1} (|\beta|^{-(q_2+1)} + 1)(\rho - r)^{-p_2} \Big] \\
&\quad \cdot \left( \int_{B_\rho} v^{lp_1 t} dx \right)^{1/t} \\
&\leq C(1 + |\beta|^{-(q_2+1)})(1 + |l|^{p_1})(\rho - r)^{-p_2} \\
&\quad \cdot \left( \int_{B_\rho} v^{lp_1 t} dx \right)^{1/t},
\end{aligned}
$$

where $t = N/[(N - p_1)(1 + \varepsilon_0)]$. In order to prove the proposition, we need to consider the cases $\gamma > 0$ and $\gamma < 0$, respectively.

Subcase II-1. $\gamma > 0$. Let $\gamma_1 \in [(\gamma/2), \gamma]$ and for any $n \in \mathbf{N}$, denote

(3.17)

$$
\begin{aligned}
r_n &= 1 + \frac{1}{2^{n-1}}, & \beta_n &= \frac{\gamma_1 (N - p_1)}{N} (1 + \varepsilon_0)^n - p_1 + 1, \\
l_n &= \left( \frac{\gamma_1}{p_1} - \frac{\gamma_1}{N} \right) (1 + \varepsilon_0)^n, & a_n &= \gamma_1 (1 + \varepsilon_0)^{n-1}.
\end{aligned}
$$

From the expression of $\beta_n$, we can denote it by $\beta_n(p_1,\gamma_1)$. Taking $l=l_n$, $\beta=\beta_n$, $\rho=r_n$ and $r_n=r_{n+1}$ in (3.16), it yields

$$\left(\int_{B_{r_{n+1}}(0)} v^{a_{n+1}}dx\right)^{p_1/p_1^*} \leq C(1+|\beta_n|^{-(q_2+1)})(1+|l_n|^{p_1})2^{p_2(n+1)}\left(\int_{B_{r_n}(0)} v^{a_n}dx\right)^{1/t}, \tag{3.18}$$

where $t=N/[(N-p_1)(1+\varepsilon_0)]$. From the expression of $\beta_n$, one can easily conclude that there is a positive integer $N^*=N^*(N,p,q,\gamma)$, such that $\beta_n(p_1,\gamma_1)>0$ for any $n\geq N^*$. Noticing that $\beta_n$ is strictly increase with respect to $n$, hence we have

$$\min_{n\in\mathbf{N}}|\beta_n(p_1,\gamma_1)|=\min_{1\leq n\leq N^*}|\beta_n(p_1,\gamma_1)|.$$

Denote $\delta_1=\min_{\overline{\Omega}}(p(x)-1)$ and $\delta_2=\min_{\overline{\Omega}}(N-p(x))$. Recalling that $1<p(x)<N$ on $\overline{\Omega}$, one obtains $\delta_1$, $\delta_2>0$ and $1+\delta_1\leq p(x)\leq N-\delta_2$ for any $x\in\overline{\Omega}$. Denote $G=[1+\delta_1,N-\delta_2]\times[(\gamma/2),\gamma]$. We consider the functions $\beta_n(x,y)$ on $G$, $1\leq n\leq N^*$. Clearly,

$$\beta_n(x,y)=\frac{(1+\varepsilon_0)^n}{N}y(N-x)-x+1.$$

Denote by $O_n$ $(1\leq n\leq N^*)$ the set which consists of all $(x,y)\in G$ with $\beta_n(x,y)=0$. Then one has

$$O_n=\left\{(x,y)\in G\;\middle|\; y=\frac{N(x-1)}{N-x}\left(\frac{1}{1+\varepsilon_0}\right)^n\right\}.$$

For any $1\leq n\leq N^*$, define

$$A_n=\left\{(x,y)\in\mathbf{R}^2\;\middle|\;\left|y-\frac{N(x-1)}{N-x}\left(\frac{1}{1+\varepsilon_0}\right)^n\right|<\frac{\gamma}{8N^*}\right\}.$$

Obviously, by the definition of $O_n$ and $A_n$, it follows that $O_n\subseteq A_n$ and $A_n$ is an open set. Set

$$A=\bigcup_{1\leq n\leq N^*}A_n \quad\text{and}\quad S=G\setminus A.$$

Then $S$ is a compact subset of $G$, and $S \cap O_n = \varnothing$ for any $1 \le n \le N^*$. Thus, by the definition of $O_n$, one obtains $|\beta_n(x,y)| \neq 0$ on $S$, $1 \le n \le N^*$. Consequently, by the continuity of $\beta_n(x,y)$, $C_n = C_n(N,p,q,\gamma)$ exists such that $|\beta_n(x,y)| \ge C_n$ on $S$, $1 \le n \le N^*$. Recalling the definition of $N^*$ and denoting $C_0 = \min\{C_1, C_2, \dots, C_{N^*}\}$, we have

$$\min_{n\in\mathbf{N}} |\beta_n(x,y)| = \min_{1\le n\le N^*} |\beta_n(x,y)| \ge C_0, \quad (x,y) \in S.$$

By the definition of $S$, for any $x \in [1+\delta_1, N-\delta_2]$, one has $S \cap (\{x\} \times [(\gamma/2), \gamma]) \neq \varnothing$. Note that $p_1 \in [1+\delta_1, N-\delta_2]$. Thus, there exists a $\gamma_1 \in [(\gamma/2), \gamma]$, such that $(p_1, \gamma_1) \in S$, and consequently

$$\min_{n\in\mathbf{N}} |\beta_n(p_1,\gamma_1)| = \min_{1\le n\le N^*} |\beta_n(p_1,\gamma_1)| \ge C_0,$$

which implies that $\gamma_1 \in [(\gamma/2), \gamma]$ and a positive constant $C_0 = C_0(N,p,q,\gamma)$ exist such that

$$\min_{n\in\mathbf{N}} |\beta_n(p_1,\gamma_1)| = \min_{n\in\mathbf{N}} \left| \frac{\gamma_1(N-p_1)}{N}(1+\varepsilon_0)^n - p_1 + 1 \right| \ge C_0. \tag{3.19}$$

Combining (3.19) with (3.18), it follows that

$$\left( \int_{B_{r_{n+1}}(0)} v^{a_{n+1}} dx \right)^{p_1/p_1^*} \le C(1+|l_n|^{p_1}) 2^{p_2(n+1)} \left( \int_{B_{r_n}(0)} v^{a_n} dx \right)^{1/t} \tag{3.20}$$

with $t = N/[(N-p_1)(1+\varepsilon_0)]$. Denote $z_n = (\int_{B_{r_n}(0)} v^{a_n} dx)^{1/a_n}$. Then, by using the Young inequality and noticing that $1 < p < N$, (3.20) can be rewritten as

$$(z_{n+1})^{[p_1 a_{n+1}]/p_1^*} \le C(1+|l_n|^{p_1}) 2^{p_2(n+1)} z_n^{a_n/t} \le C(1+|l_n|^N) 2^{N(n+1)} z_n^{a_n/t}.$$

Recalling that $a_n = \gamma_1(1+\varepsilon_0)^{n-1}$, $l_n = [(\gamma_1/p_1) - (\gamma_1/N)](1+\varepsilon_0)^n$ and

$t = N/[(N-p_1)(1+\varepsilon_0)]$, it follows from the above inequality that

$$\begin{aligned} z_{n+1} &\le z_n(2^{nN}C)^{[N(1+\varepsilon_0)^{-n}]/[\gamma_1(N-p_1)]} \\ &\quad\cdot\left[1+\left(\frac{\gamma_1}{p_1}-\frac{\gamma_1}{N}\right)^N(1+\varepsilon_0)^{nN}\right]^{[N(1+\varepsilon_0)^{-n}]/[\gamma_1(N-p_1)]} \\ &\le z_n C^{2nN^2/[\gamma\delta_2(1+\varepsilon_0)^n]}\left[1+\gamma^N(1+\varepsilon_0)^{nN}\right]^{2N/[\gamma\delta_2(1+\varepsilon_0)^n]} \\ &\le z_n C^{2nN^2/[\gamma\delta_2(1+\varepsilon_0)^n]}(1+\gamma^N)^{2N/[\gamma\delta_2(1+\varepsilon_0)^n]} \\ &\qquad\cdot(1+\varepsilon_0)^{2nN^2/[\gamma\delta_2(1+\varepsilon_0)^n]} \\ &\le z_n C^{2nN^2/[\gamma\delta_2(1+\varepsilon_0)^n]}(1+\gamma)^{2nN^2/[\gamma\delta_2(1+\varepsilon_0)^n]} \\ &\qquad\cdot(1+\varepsilon_0)^{2nN^2/[\gamma\delta_2(1+\varepsilon_0)^n]} \\ &= C_*^{n(1+\varepsilon_0)^{-n}} z_n \end{aligned}$$

for all $n \in \mathbf{N}$, where $C_* = [C(1+\gamma)(1+\varepsilon_0)]^{2N^2/(\gamma\delta_2)}$ and $\delta_2 = \min_{\overline{\Omega}}(N-p(x))$. Iterating the previous inequality, we then obtain

$$z_{n+1} \le C_*^{\sum_{k=1}^n k(1+\varepsilon_0)^{-k}} z_1$$

for all $n \in \mathbf{N}$. It's easy to conclude that

$$\sum_{k=1}^{\infty} k(1+\varepsilon_0)^{-k} < \infty.$$

Combining the above two inequalities and letting $n \to \infty$, there exists a positive constant $C(N,p,q,\gamma)$, such that

$$\sup_{B_1(0)} v \le C\left(\int_{B_2(0)} v^{\gamma_1}\,dx\right)^{1/\gamma_1},$$

and consequently

$$\sup_{B_R(x_0)} u \le C\left(\frac{1}{R^N}\int_{B_{2R}(x_0)} u^{\gamma_1}\,dx\right)^{1/\gamma_1}. \tag{3.21}$$

Subcase II-2. $\gamma < 0$. Let $a_n$, $\beta_n$ and $l_n$ be the same notations in Subcase II-1 with the symbol $\gamma_1$ replaced by $\gamma$ in the expressions. Note that in this case (3.18) still holds. Moreover, since $\gamma < 0$, estimate (3.19) can be easily proved by the expression of $\beta_n$, and consequently, (3.20) holds. The rest of the proof is similar to that of subcase II-1, except that the directions of the inequalities are opposite. Therefore, we obtain

$$\inf_{B_R(x_0)} u \geq C\left(\frac{1}{R^N}\int_{B_{2R}(x_0)} u^\gamma dx\right)^{1/\gamma}, \quad \gamma < 0. \tag{3.22}$$

Finally, assuming in addition that $B_{8R}(x_0) \subseteq \Omega$ with some fixed $x_0 \in \Omega$ and combining (3.14) with (3.21) and (3.22), we have

$$\sup_{B_R(x_0)} u \leq C \inf_{B_R(x_0)} u.$$

The proof is complete. □

Proposition 3.1 together with Lemma 3.1 implies the following corollary.

**Corollary 3.1.** *Suppose that all the conditions in Proposition* 3.1 *hold. Then two positive constants $R_0$ and $C$ exist such that, for any $R \leq R_0$, $x \in \Omega$ and $B_{4R}(x) \subseteq \Omega$, it follows that*

$$u(x) \leq CR^{-p(x)/[q(x)-p(x)+1]}.$$

*Proof.* Let $R_0$ be the smaller one of that in Lemma 3.1 and Proposition 3.1. Take arbitrary $x \in \Omega$, such that $B_{4R}(x) \subseteq \Omega$ with $R \leq R_0$. By Proposition 3.1, one obtains

$$u(x) \leq C\left(\frac{1}{R^N}\int_{B_{2R}(x)} u^\gamma dy\right)^{1/\gamma}, \quad \gamma > 0.$$

Choosing $\gamma$ which is close enough to 0 and using Lemma 3.1, we conclude that

$$\left(\frac{1}{R^N}\int_{B_{2R}(x)} u^\gamma dy\right)^{1/\gamma} \leq CR^{-p_2/(q_1-p_2+1)},$$

where $p_2 = \max_{\overline{B_{4R}(x)}} p(y)$ and $q_1 = \min_{\overline{B_{4R}(x)}} q(y)$. Combining the previous two inequalities, we have

$$u(x) \leq CR^{-p_2/(q_1-p_2+1)} = CR^{-p(x)/[q(x)-p(x)+1]+\varepsilon(x)},$$

where $\varepsilon(x) = p(x)/[q(x)-p(x)+1] - p_2/(q_1-p_2+1)$. By simple calculations, we conclude that

$$\varepsilon(x) = \frac{p(x)-p_2}{q(x)-p(x)+1} + \frac{p_2[(q_1-q(x))+(p(x)-p_2)]}{(q(x)-p(x)+1)(q_1-p_2+1)}.$$

Using condition (H1), one can easily conclude that

$$\begin{aligned} R^{\varepsilon(x)} &\leq R^{-C(\operatorname{osc}\{p;B_{4R}\}+\operatorname{osc}\{q;B_{4R}\})} \\ &\leq R^{-C(\|\nabla p\|+\|q\|_{C^{\alpha_0}})R} \\ &= e^{-C(\|\nabla p\|+\|q\|_{C^{\alpha_0}})R\ln R} \\ &\leq C, \end{aligned}$$

and consequently,

$$u(x) \leq CR^{-p(x)/[q(x)-p(x)+1]}.$$

The proof is complete. $\square$

**4. The $L^\infty$ estimate.** In this section, we focus on obtaining the $L^\infty$ estimate on positive solutions of problem (1.1). Combining the results obtained in the previous section with the global $C^{1,\alpha}$ estimates on bounded weak solutions, together with a Liouville type result in [**24**], we can derive the a priori estimates by the blow-up argument.

We first state the following assumptions on functions $f$ and $g$, and the domain $\Omega$:

(H2) For any $(x,z,\xi) \in \Omega \times \mathbf{R}^+ \times \mathbf{R}^N$,

$$\begin{gathered} f(x,z,\xi) = z^{q(x)} + g(x,z,\xi), \quad g(x,z,\xi) \geq 0, \\ K_1' z^{\kappa(x)} - K_2'\big(|\xi|^{\lambda(x)}+1\big) \leq g(x,z,\xi) \leq K_2'\big(z^{\kappa(x)}+|\xi|^{\lambda(x)}+1\big), \end{gathered}$$

where $K_1'$ and $K_2'$ are positive constants, $p(x)$, $q(x)$ and $\lambda(x)$ satisfy (H1) in Section 3, and $\kappa(x) \in C(\overline{\Omega})$ satisfies

$$0 \le \kappa(x) < q(x),$$

for all $x \in \overline{\Omega}$.

(H3) Let $r_0$ and $c_0$ be positive constants. For any $x \in \partial\Omega$, $r \ge r_0$ and a function $h : \mathbf{R}^{N-1} \to \mathbf{R}$ exist with $h(0) = 0$, $\nabla h(0) = 0$ and $\|h\|_{C^{1,\alpha_0}} \le c_0$, such that $\Omega_r(x) := \Omega \cap B_r(x)$ can be represented as

$$\left\{ y \in \mathbf{R}^N \;\middle|\; h(\widehat{y}) < y^N < \sqrt{r^2 - |\widehat{y}|^2} \right\},$$

where $\widehat{y} = (y^1, \dots, y^{N-1}) \in \mathbf{R}^{N-1}$ for any $y = (y^1, \dots, y^N) \in \mathbf{R}^N$ and $y^i$, $1 \le i \le N$, is the rectangular coordinate under some basis which may be different from the original basis $e_1, e_2, \dots, e_N$.

In fact, we perhaps encounter the case that $\Omega$ satisfies assumption (H3) only on the part of the boundary, that is, instead of (H3), we give the following assumption:

(H3$'$) There is a subset $\Sigma \subset \partial\Omega$, such that the statements in (H3) hold true for all $x \in \Sigma$.

In order to gain the $L^\infty$ estimate on positive solutions of problem (1.1), we can firstly consider the global $C^{1,\alpha}$ estimates on bounded weak solutions of a class of elliptic equations satisfying some structure conditions. Assume that

(A1) $p : \mathbf{R}^N \to \mathbf{R}$ is a bounded Hölder continuous function, that is, positive constants $L_0$ and $\alpha_0$ exist such that

$$\begin{gathered} |p(x_1) - p(x_2)| \le L_0 |x_1 - x_2|^{\alpha_0}, \\ 1 < p_- \le p(x) \le p_+ < \infty, \\ x_1, x_2, x \in \mathbf{R}^N, \end{gathered}$$

where $p_-$ and $p_+$ are positive constants.

(A2) Let $A : \overline{\Omega} \times [-M, M] \times \mathbf{R}^N \to \mathbf{R}^N$ and $B : \Omega \times [-M, M] \times \mathbf{R}^N \to \mathbf{R}$. For any $(x, u) \in \overline{\Omega} \times [-M, M]$, $A(x, u, \cdot) \in C^1(\mathbf{R}^N \setminus \{0\}; \mathbf{R}^N)$, and for any $x, x_1, x_2 \in \overline{\Omega}$, $u, u_1, u_2 \in [-M, M]$, $\eta \in \mathbf{R}^N \setminus \{0\}$ and $\xi \in \mathbf{R}^N$, the following conditions are satisfied:

$$\begin{gathered} A(x, u, 0) = 0, \\ \xi^T A_\eta(x, u, \eta) \xi \ge \lambda |\eta|^{p(x)-2} |\xi|^2, \\ |A_\eta(x, u, \eta)| \le \Lambda |\eta|^{p(x)-2}, \end{gathered}$$

$$
\begin{aligned}
&|A(x_1,u_1,\eta)-A(x_2,u_2,\eta)| \\
&\qquad \le \Lambda\left(|x_1-x_2|^{\alpha_0}+|u_1-u_2|^{\alpha_0}\right)\left(|\eta|^{p(x_1)-1}+|\eta|^{p(x_2)-1}\right),
\end{aligned}
$$

where $\lambda$, $\Lambda$ and $\alpha_0$ are positive constants,

$$A_\eta(x,u,\eta) := \left(\frac{\partial A_i}{\partial x^j}(x,u,\eta)\right)_{N\times N},$$

and $|E| := (\sum_{ij} |e_{ij}|^2)^{1/2}$ for any matrix $E=(e_{ij})_{N\times N}$.

(A3) Two positive constants $\rho_0$ and $\theta_0 \in (0,1)$ exist such that, for any ball $B_\rho$ with center on $\partial\Omega$ and radius $\rho \le \rho_0$, the following holds

$$|\Omega_\rho| \le (1-\theta_0)|B_\rho|,$$

where $|E|$ denotes the Lebesgue measure of $E$.

We obtain the following two propositions. Their proofs are very lengthy, but the methods we used are classical; hence, we omit them here and give the proofs in Section 7.

**Proposition 4.1.** *Let $\Omega$ be a domain in $\mathbf{R}^N$ and $u$ a bounded weak solution with* $\max_\Omega |u(x)| \le M$, *of the problem*

$$\begin{cases} -\mathrm{div}\, A(x,u,\nabla u) = B(x,u,\nabla u) & x\in\Omega, \\ u(x)=0 & x\in\partial\Omega. \end{cases}$$

*Assume that* (H3), (A1) *and* (A2) *hold true. Then positive constants $\alpha \in (0,1)$, $R_0$ and $C$ exist depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda$, $\Lambda$, $L_0$, $\alpha_0$, $r_0$ and $c_0$, such that*

$$\mathrm{osc}\,\{\nabla u; \Omega_R(x)\} \le CR^\alpha, \quad x\in\overline{\Omega},\ R\le R_0$$

*and*

$$|\nabla u(x)| \le C, \quad x\in\overline{\Omega},$$

*where $p_-$, $p_+$ $L_0$ and $\alpha_0$ are the constants in* (A1), *and $\lambda$, $\Lambda$, $r_0$ and $c_0$ are the constants stated in* (A2).

**Proposition 4.2.** *$\Omega$ and $u$ are defined as in Proposition* 4.1. *Assume* (H3′), (A1), (A2) *and* (A3) *hold true. Then positive constants*

*$\alpha \in (0,1)$, $R_0$ and $C$ exist depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda$, $\Lambda$, $L_0$, $\alpha_0$, $r_0$, $c_0$ and $\theta_0$ ($\theta_0$ is the constant stated in* (A3)*), such that*

$$\operatorname{osc}\{\nabla u; \Omega_R(x)\} \le CR^\alpha, \quad x \in \Omega_{2R_0} \cup (\Sigma_{3R_0} \cap \overline{\Omega}), \quad R \le R_0$$

*and*

$$|\nabla u(x)| \le C, \quad x \in \Omega_{2R_0} \cup (\Sigma_{3R_0} \cap \overline{\Omega}),$$

*where $\Omega_{2R_0} = \{x \in \Omega \mid d(x, \partial\Omega) \ge 2R_0\}$ and $\Sigma_{3R_0} = \{x \in \mathbf{R}^N \mid d(x, \Sigma) < 3R_0\}$.*

We also need the following Liouville-type result, which is a special case of Theorem 1.1 in [**24**].

**Lemma 4.1** [**24**]. *Let $p$ and $q$ be two positive constants satisfying $p \in (1, N)$ and $q \in (p-1, p^*-1)$, where $p^* = Np/(N-p)$ is the Sobolev critical exponent. Then the problem*

$$\begin{cases} -\Delta_p u = u^q & x \in H, \\ u(x) = 0 & x \in \partial H \end{cases}$$

*has no positive solution, where $H$ is a half space of $\mathbf{R}^N$.*

Now we can state and prove the a priori estimates on positive solutions for problem (1.1), namely, we can prove the following result.

**Lemma 4.2.** *Suppose* (H1)–(H3) *hold true. Then, for any $C^1$ positive solution $u$ of problem* (1.1)*, we have*

$$\|u\| \le C,$$

*where $C$ is a positive constant and $\|\cdot\|$ denotes the uniform norm.*

*Proof.* Suppose, by contradiction, that a sequence $\{u_n\}$ exists such that $u_n$ is a $C^1$ positive solution of problem (1.1) and $\|u_n\| \to \infty$. Take $x_n \in \Omega$ such that $u_n(x_n) = \|u_n\| = s_n$. Denote $d_n = d(x_n, \partial\Omega)$ and define

$$\widetilde{u}_n(x) := u_n(x) + 1, \quad \text{for all } x \in \Omega.$$

Then $\widetilde{u}_n \geq 1$ and this satisfies

$$-\Delta_{p(x)}\widetilde{u}_n = (\widetilde{u}_n - 1)^{q(x)} + g(x, \widetilde{u}_n - 1, \nabla\widetilde{u}_n), \quad x \in \Omega.$$

With the aid of (H1) and (H2), one can easily conclude that positive constants $\widetilde{K}_1$ and $\widetilde{K}_2$ exist, such that

$$\widetilde{K}_1\widetilde{u}_n^{q(x)} - \widetilde{K}_2\big(|\nabla\widetilde{u}_n|^{\lambda(x)} + 1\big) \leq -\Delta_{p(x)}\widetilde{u}_n \leq \widetilde{K}_2\big(\widetilde{u}_n^{q(x)} + |\nabla\widetilde{u}_n|^{\lambda(x)} + 1\big).$$

Applying Corollary 3.1 to the above inequality, a positive constant $C$ exists such that

$$s_n \leq \widetilde{u}_n(x_n) \leq Cd_n^{-p(x_n)/[q(x_n)-p(x_n)+1]}, \tag{4.1}$$

and consequently, it follows that $d_n \to 0$ as $n \to \infty$. Let

$$v_n(x) = s_n^{-1}u_n(y), \qquad y = \delta_n x + x_n, \quad x \in \Omega_n,$$

where $\delta_n = s_n^{-[q(x_n)-p(x_n)+1]/p(x_n)}$ and

$$\Omega_n = \{x \in \mathbf{R}^N \mid \delta_n x + x_n \in \Omega\}.$$

Obviously $\delta_n \to 0$ as $n \to \infty$. Define $p_n(x) := p(\delta_n x + x_n)$ for any $x \in \Omega_n$. $q_n(x)$, $\lambda_n(x)$ and $\kappa_n(x)$ are similarly defined. Then $p(x_n) = p_n(0)$, $q(x_n) = q_n(0)$, $\lambda(x_n) = \lambda_n(0)$ and $\kappa(x_n) = \kappa_n(0)$. By simple calculations, we conclude that $v_n$ satisfies

$$\begin{cases} -\Delta_{p_n(x)}v_n = s_n^{l_n(x)}v_n^{q_n(x)} + g_n(x, v_n, \nabla v_n) & x \in \Omega_n, \\ v_n(x) = 0 & x \in \partial\Omega_n, \end{cases} \tag{4.2}$$

where

$$l_n(x) = \frac{p_n(x)}{p_n(0)}[q_n(x) - q_n(0)] - \frac{q_n(x) + 1}{p_n(0)}[p_n(x) - p_n(0)], \quad x \in \Omega_n,$$

and

$$g_n(x, z, \xi) = \delta_n^{p_n(x)}s_n^{1-p_n(x)}g(\delta_n x + x_n, s_n z, s_n\delta_n^{-1}\xi)$$

$$
\begin{aligned}
&+ \delta_n(\ln s_n - \ln \delta_n)|\xi|^{p_n(x)-2}\xi \cdot \nabla p(\delta_n x + x_n)\\
&= g_{n,1}(x,z,\xi) + g_{n,2}(x,z,\xi)
\end{aligned}
$$

for any $(x,z,\xi) \in \Omega_n \times \mathbf{R}^+ \times \mathbf{R}^N$.

Next, we do the a priori estimates on $v_n$. We first estimate $g_{n,1}$ and $g_{n,2}$. Recalling that $p(x) \in C^1(\overline{\Omega})$ and $\delta_n = s_n^{-[q_n(0)-p_n(0)+1]/p_n(0)}$, it follows that a positive constant $C$ exists such that

$$
\begin{aligned}
|g_{n,2}(x,z,\xi)| &= \frac{q_n(0)+1}{p_n(0)} s_n^{-[q_n(0)-p_n(0)+1]/p_n(0)} (\ln s_n)\\
&\qquad \cdot |\nabla p(\delta_n x + x_n)||\xi|^{p_n(x)-1}\\
&\le C s_n^{1-[(q_n(0)+1)/p_n(0)]} (\ln s_n)|\xi|^{p_n(x)-1}
\end{aligned}
\tag{4.3}
$$

for any $(x,z,\xi) \in \Omega_n \times \mathbf{R}^+ \times \mathbf{R}^N$. Since $p(x)$ and $q(x)$ are continuous and $q(x) > p(x) - 1$ on $\overline{\Omega}$, a $\varepsilon_0(p,q) > 0$ exists such that $q(x) + 1 \ge (1+\varepsilon_0)p(x)$ for any $x \in \Omega$. Recalling that $s_n \to \infty$ as $n \to \infty$, a positive constant $C$ exists such that

$$
s_n^{1-[(q_n(0)+1)/p_n(0)]} \ln s_n = s_n^{1-[(q(x_n)+1)/p(x_n)]} \ln s_n \le s_n^{-\varepsilon_0} \ln s_n. \tag{4.4}
$$

Combining the previous two inequalities, we can estimate that

$$
|g_{n,2}(x,z,\xi)| \le C|\xi|^{p_n(x)-1}
$$

for any $(x,z,\xi) \in \Omega_n \times \mathbf{R}^+ \times \mathbf{R}^N$. Now we estimate $g_{n,1}(x,z,\xi)$. Since $g(x,z,\xi)$ satisfies (H2), a $C > 0$ exists such that

$$
\begin{aligned}
|g_{n,1}(x,z,\xi)| &= \delta_n^{p_n(x)} s_n^{1-p_n(x)} |g(\delta_n x + x_n, s_n z, s_n \delta_n^{-1}\xi)|\\
&\le C\big(\delta_n^{p_n(x)} s_n^{\kappa_n(x)-p_n(x)+1} z^{q_n(x)} + \delta_n^{p_n(x)} s_n^{1-p_n(x)}\\
&\qquad + \delta_n^{p_n(x)-\lambda_n(x)} s_n^{\lambda_n(x)-p_n(x)+1} |\xi|^{\lambda_n(x)}\big)\\
&\le C\delta_n^{p_n(x)} s_n^{\kappa_n(x)-p_n(x)+1}(z^{q_n(x)}+1)\\
&\quad + C\delta_n^{p_n(x)-\lambda_n(x)} s_n^{\lambda_n(x)-p_n(x)+1}|\xi|^{\lambda_n(x)}\\
&= C\alpha_1(x)(z^{q_n(x)}+1) + C\alpha_2(x)|\xi|^{\lambda_n(x)}
\end{aligned}
\tag{4.5}
$$

for any $(x,z,\xi) \in \Omega_n \times \mathbf{R}^+ \times \mathbf{R}^N$. Denote $\widehat{\delta}_n = \max\{\delta_n, d_n\}$ and

$$
\begin{aligned}
\Omega_n' = \{x \in \mathbf{R}^N \mid \text{ there exists a } y \in B_{\sqrt{\widehat{\delta}_n}}(x_n) \cap \Omega,&\\
\text{such that } y = \delta_n x + x_n\},&
\end{aligned}
$$

$$\Omega_n'' = \{x \in \mathbf{R}^N \mid \text{ there exists a } y \in B_{1/2\sqrt{\hat{\delta}_n}}(x_n) \cap \Omega, \\ \text{such that } y = \delta_n x + x_n\}.$$

Obviously $\Omega_n'' \subseteq \Omega_n'$. Noticing that $d_n, \delta_n \to 0$ as $n \to \infty$, it follows that $\widehat{\delta}_n \to 0$ as $n \to \infty$. To estimate $g_{n,1}$, we only need to estimate $\alpha_1(x)$ and $\alpha_2(x)$. By condition (H1), one obtains

$$\varepsilon_1 = \min_{x\in\overline{\Omega}}(q(x) - \kappa(x)) > 0, \qquad \varepsilon_2 = \min_{\overline{\Omega}}\left(\frac{p(x)q(x)}{q(x)+1} - \lambda(x)\right) > 0.$$

Recalling that $\delta_n = s_n^{-[q_n(0)-p_n(0)+1]/p_n(0)}$, and by using condition (H1), it follows that, for any $x \in \Omega_n'$,

$$\begin{aligned}
\alpha_1(x) &= \delta_n^{p_n(x)} s_n^{\kappa_n(x)-p_n(x)+1} \\
&= s_n^{\kappa_n(x)+1-[(q_n(0)+1)/p_n(0)]p_n(x)} \\
&= s_n^{\kappa_n(x)-q_n(0)-[(q_n(0)+1)/p_n(0)](p_n(x)-p_n(0))} \\
&\le s_n^{-\varepsilon_1+q_n(x)-q_n(0)-[(q_n(0)+1)/p_n(0)](p_n(x)-p_n(0))} \\
&= s_n^{-\varepsilon_1+q(\delta_n x+x_n)-q(x_n)-[(q(x_n)+1)/p(x_n)](p(\delta_n x+x_n)-p(x_n))} \\
&\le s_n^{-\varepsilon_1+|\delta_n x|^{\alpha_0}+C|\delta_n x|} \\
&\le s_n^{-\varepsilon_1+C\hat{\delta}_n^{(\alpha_0/2)}},
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_2(x) \\
&= \delta_n^{p_n(x)-\lambda_n(x)} s_n^{\lambda_n(x)-p_n(x)+1} \\
&= s_n^{1+[(q_n(0)+1)/p_n(0)](\lambda_n(x)-p_n(x))} \\
&\le s_n^{[(p_n(x)q_n(x))/((q_n(x)+1))-p_n(x)][(q_n(0)+1)/p_n(0)]+1-[\varepsilon_2(q_n(0)+1)/p_n(0)]} \\
&= s_n^{[p_n(0)(q_n(x)+1)-p_n(x)(q_n(0)+1)]/[p_n(0)(q_n(x)+1)]-[\varepsilon_2(q_n(0)+1)/p_n(0)]} \\
&= s_n^{[(q_n(x)-q_n(0))/(q_n(x)+1)]-[(p_n(x)-p_n(0))(q_n(0)+1)]/[p_n(0)(q_n(x)+1)]-[\varepsilon_2(q_n(0)+1)/p_n(0)]} \\
&\le s_n^{C(|\delta_n x|^{\alpha_0/2}+|\delta_n x|-\varepsilon_2)} \\
&\le s_n^{C(\hat{\delta}_n^{\alpha_0/2}-\varepsilon_2)}.
\end{aligned}$$

Recalling that $\widehat{\delta}_n \to 0$, it follows from the above two inequalities and (4.1) that

$$\lim_{n\to\infty} \sup_{x\in\Omega_n'} \alpha_1(x) = \lim_{n\to\infty} \sup_{\Omega_n'} \alpha_{12}(x) = 0. \tag{4.6}$$

Putting (4.6) into (4.5), we can estimate that

$$|g_{n,1}(x,z,\xi)| \le C(|z|^{q_n(x)} + |\xi|^{\lambda_n(x)} + 1)$$

for any $(x,z,\xi) \in \Omega_n' \times \mathbf{R}^+ \times \mathbf{R}^N$. Therefore, by applying the Young inequality and the fact that $p_n(x) - 1 \le \lambda_n(x) \le p_n(x)$ on $\Omega_n'$, we have

$$\begin{aligned} |g_n(x,z,\xi)| &\le |g_{n,1}(x,z,\xi)| + |g_{n,2}(x,z,\xi)| \\ &\le C(|z|^{q_n(x)} + |\xi|^{p_n(x)-1} + |\xi|^{\lambda_n(x)} + 1) \\ &\le C(|z|^{q_n(x)} + |\xi|^{\lambda_n(x)} + 1) \\ &\le C(|z|^{q_n(x)} + |\xi|^{p_n(x)} + 1) \end{aligned} \tag{4.7}$$

for any $(x,z,\xi) \in \Omega_n' \times \mathbf{R}^+ \times \mathbf{R}^N$. We now estimate the term $s_n^{l_n(x)}$. For any $x \in \Omega_n'$, one has

$$\begin{aligned} |l_n(x)| &\le \left|\frac{p_n(x)}{p_n(0)}\right| |q_n(x) - q_n(0)| + \left|\frac{q_n(x)+1}{p_n(0)}\right| |p_n(x) - p_n(0)| \\ &\le C(|q_n(x) - q_n(0)| + |p_n(x) - p_n(0)|) \\ &= C(|q(\delta_n x + x_n) - q(x_n)| + |p(\delta_n x + x_n) - p(x_n)|) \\ &\le C(\widehat{\delta}_n^{\alpha_0/2} + \widehat{\delta}_n) \le C_1 \widehat{\delta}_n^{\alpha_0/2}, \end{aligned} \tag{4.8}$$

and consequently,

$$-C_1 \widehat{\delta}_n^{\alpha_0/2} \le l_n(x) \le C_1 \widehat{\delta}_n^{\alpha_0/2}.$$

By (4.1) and the definitions of $\delta_n$ and $\widehat{\delta}_n$, two positive constants $C_2$ and $\sigma$ exist such that

$$s_n \le C_2 \widehat{\delta}_n^{-\sigma}.$$

Combining the previous two inequalities, then

$$C_2^{-C_1 \hat{\delta}_n^{\alpha_0/2}} \hat{\delta}_n^{C_1 \sigma \hat{\delta}_n^{\alpha_0/2}} \le s_n^{l_n(x)} \le C_2^{C_1 \hat{\delta}_n^{\alpha_0/2}} \hat{\delta}_n^{-C_1 \sigma \hat{\delta}_n^{\alpha_0/2}}, \quad x \in \Omega_n'.$$

Recalling that $\widehat{\delta}_n \to 0$, the foregoing inequality implies that

$$\lim_{n\to\infty} C_2^{-C_1 \hat{\delta}_n^{\alpha_0/2}} \hat{\delta}_n^{C_1 \sigma \hat{\delta}_n^{\alpha_0/2}} = \lim_{n\to\infty} C_2^{C_1 \hat{\delta}_n^{\alpha_0/2}} \hat{\delta}_n^{-C_1 \sigma \hat{\delta}_n^{\alpha_0/2}} = 1.$$

Consequently, it follows from the previous two inequalities that

$$\lim_{n\to\infty} \sup_{x\in\Omega_n'} |s_n^{l_n(x)} - 1| = 0. \tag{4.9}$$

Combining (4.7) with (4.9) and noticing that $v_n \le 1$, we conclude that

$$\begin{aligned} |s_n^{l_n(x)} v_n^{q_n(x)} + g_n(x, v_n, \nabla v_n)| \\ &\le C(v_n^{q_n(x)} + |\nabla v_n|^{p_n(x)} + 1) \\ &\le C(|\nabla v_n|^{p_n(x)} + 1), \quad x \in \Omega_n'. \end{aligned} \tag{4.10}$$

Recalling that $\hat{\delta}_n \to 0$, by the definition of $\Omega_n'$, we claim that $\Omega_n'$ satisfies assumption (H3$'$) with $\Omega$ replaced by $\Omega_n'$ and $\Sigma$ by $\Sigma_n := \partial\Omega_n'' \cap \partial\Omega_n$, and here the constants $\alpha_0$, $r_0$ and $c_0$ are all independent of $n$. For this purpose, we take arbitrary $y_0 \in \Sigma_n$ and $\rho \le r_0$; then, by the definition of $\Omega_n$ and $\Omega_n''$, we have $x_0 = \delta_n y_0 + x_n \in \partial\Omega$ and $|\delta_n y_0| \le \sqrt{\widehat{\delta}_n}/2$. Moreover, the following holds:

$$B_\rho(y_0) \cap \Omega_n' = B_\rho(y_0) \cap \Omega_n, \tag{4.11}$$

for $n$, which is large enough. Obviously $B_\rho(y_0) \cap \Omega_n' \subseteq B_\rho(y_0) \cap \Omega_n$. In order to verify (4.11), we only need to verify that $B_\rho(y_0) \cap \Omega_n \subseteq B_\rho(y_0) \cap \Omega_n'$. Taking arbitrary $y \in B_\rho(y_0) \cap \Omega_n$ and denoting $x = \delta_n y + x_n$, then by the definition of $\Omega_n$ and $\Omega_n'$, it suffices to verify that $|x - x_n| < \sqrt{\widehat{\delta}_n}$ or $\delta_n |y| < \sqrt{\widehat{\delta}_n}$. Noticing that for large $n$ we have $\delta_n r_0 \le \sqrt{\widehat{\delta}_n}/4$, and recalling that $|\delta_n y_0| \le \sqrt{\widehat{\delta}_n}/2$, it follows that

$$\delta_n |y| \le \delta_n |y_0| + \delta_n |y - y_0| \le \frac{\sqrt{\widehat{\delta}_n}}{2} + \delta_n r_0 \le \frac{3\sqrt{\widehat{\delta}_n}}{4} < \sqrt{\widehat{\delta}_n},$$

and consequently, (4.11) holds for large $n$. On account of (4.11), to verify $\Omega_n'$ satisfies assumption (H3$'$) on $\Sigma_n$, we only need to verify that $\Omega_n$ satisfies assumption (H3). In fact, by (H3), a Hermite matrix $K$ and a function $h \in C^{1,\alpha_0}(\mathbf{R}^{N-1}; \mathbf{R})$ exist, with $h(0) = 0$, $\nabla h(0) = 0$ and $\|h\|_{C^{1,\alpha_0}} \le c_0$, such that

$$\begin{aligned} T\big(\Omega \cap B_r(x_0)\big) &= V \\ &:= \Big\{ z \in \mathbf{R}^N \mid h(\widehat{z}) < z^N < \sqrt{r^2 - |\widehat{z}|^2} \Big\}, \quad 0 < r \le r_0, \end{aligned}$$

where $T$ is given by

$$z = T(x) := K\left(x - x_0\right), \quad x \in \Omega \cap B_r(x_0).$$

Define a mapping $L : \Omega \cap B_r(x_0) \to L(\Omega \cap B_r(x_0))$, such that

$$y = Lx = \delta_n^{-1}(x - x_n), \quad x \in \Omega \cap B_r(x_0).$$

Obviously, $L$ is bijective and, by the definition of $\Omega_n$, we conclude

$$L\big(\Omega \cap B_r(x_0)\big) = \Omega_n \cap B_{\delta_n^{-1}r}(y_0).$$

Define a mapping $\widetilde{T} : \Omega_n \cap B_{\delta_n^{-1}r}(y_0) \to \widetilde{T}(\Omega_n \cap B_{\delta_n^{-1}r}(y_0))$, such that

$$y' = \widetilde{T}y := K(y - y_0).$$

Then $\widetilde{T} = \delta_n^{-1} T \circ L^{-1}$, and consequently $T(B_{\delta_n^{-1}r}(y_0) \cap \Omega_n) = \delta_n^{-1} V$, namely,

$$\begin{aligned}\widetilde{T}\big(B_{\delta_n^{-1}r}(y_0) \cap \Omega_n\big) \\ &= \Big\{ y' \in \mathbf{R}^N \mid \delta_n^{-1} h(\delta_n \widehat{y'}) < y'^N \\ &\qquad < \sqrt{(\delta_n^{-1}r)^2 - |\widehat{y'}|^2} \Big\}, \quad 0 < r \le r_0.\end{aligned}$$

or

$$\widetilde{T}\big(B_r(y_0) \cap \Omega_n\big) = \left\{ y' \in \mathbf{R}^N \mid \delta_n^{-1} h(\delta_n \widehat{y'}) < y'^N < \sqrt{r^2 - |\widehat{y'}|^2} \right\}, \quad 0 < r \le \delta_n^{-1} r_0.$$

Recalling that $K$ is a Hermite matrix and $\delta_n \to 0$ as $n \to \infty$, from the above formula and the definition of $\widetilde{T}$, we can see that $\Omega_n \cap B_r(y_0)$ can be represented as

$$\left\{ y' \in \mathbf{R}^N \mid \delta_n^{-1} h(\delta_n \widehat{y'}) < {y'}^N < \sqrt{r^2 - |\widehat{y'}|^2} \right\}, \quad 0 < r \le r_0.$$

Thus, the remainder to be verified is that, for large $n$,

$$\|h_n(y)\|_{C^{1,\alpha_0}} = \|\delta_n^{-1} h(\delta_n y)\|_{C^{1,\alpha_0}} \le c_0,$$

which is clear from the properties of $h$.

On account of the previous statements and (4.10), we can use Proposition 4.2 to conclude that constants $C$, $\alpha \in (0,1)$ and $R_0$ exist such that, for $n$ which is large enough, the following hold:

$$|\nabla v_n(x)| \le C, \quad x \in \overline{\Omega''_n},$$

and

$$|\nabla v_n(x) - \nabla v(y)| \le C|x-y|^\alpha, \qquad x,y \in \overline{\Omega''_n}, \quad |x-y| \le R_0.$$

Finally, by the limitation process, we can obtain the a priori estimates on $u$. Take $\widetilde{x}_n \in \partial\Omega$ such that $|\widetilde{x}_n - x_n| = d_n$. By the mean value theorem, we obtain

$$\begin{aligned}1 &= u_n(x_n) - u_n(\widetilde{x}_n) = v_n(0) - v_n(\delta_n^{-1}(\widetilde{x}_n - x_n)) \\ &\le \delta_n^{-1}\|\nabla v_n\||\widetilde{x}_n - x_n| \le C\delta_n^{-1} d_n.\end{aligned}$$

Combining (4.1) with the previous inequality, two positive constants $C_3$ and $C_4$ exist such that

$$C_3 \le \delta_n^{-1} d_n \le C_4.$$

Using the same argument as that of subcase I-2 in the proof of Theorem 1.2 in [**24**], $\varepsilon > 0$, subsequences of $\{\Omega''_n\}$ and subsequences of $\{v_n\}$ exist, which are denoted by $\{\Omega''_n\}$ and $\{v_n\}$, respectively, and $v \in C^{1,\alpha/2}(\mathbf{R}^N_\varepsilon)$, such that

$$\lim_{n\to\infty} \Omega''_n = \mathbf{R}^N_\varepsilon := \{(y_1, y_2, \ldots, y_N) \in \mathbf{R}^N \mid y_N > -\varepsilon\}.$$

With some appropriate rotation,

$$v(y) = 0, \quad y \in \partial\mathbf{R}^N_\varepsilon, \quad v(0) = 1,$$

and

$$\lim_{n\to\infty} v_n(y) = v(y) \tag{4.12}$$

uniformly on any compact subset of $\mathbf{R}^N_\varepsilon$ in $C^{1,\alpha/2}$-topology. Recalling that $|\nabla v_n(x)| \le C$, for any $x \in \Omega'_n$ and combining (4.3) with (4.4), together with (4.5), one obtains

$$\begin{aligned}|g_n(x, v_n, \nabla v_n)| &\le |g_{n,1}(x, v_n, \nabla v_n)| + |g_{n,2}(x, v_n, \nabla v_n)| \\ &\le C(s_n^{-\varepsilon_0} \ln s_n + \alpha_1(x) + \alpha_2(x)).\end{aligned}$$

Since $s_n \to \infty$ and

$$\lim_{n\to\infty} \sup_{\Omega'_n} \alpha_1(x) = \lim_{n\to\infty} \sup_{\Omega'_n} \alpha_2(x) = 0,$$

it follows from the previous inequality that

$$\lim_{n\to\infty} \sup_{x\in\Omega'_n} |g_n(x, v_n(x), \nabla v_n(x))| = 0. \tag{4.13}$$

Note that $d_n \to 0$. A subsequence of $\{x_n\}$ exists, still denoted by $\{x_n\}$, and $x_0 \in \partial\Omega$, such that $x_n \to x_0$. Combining (4.9)–(4.13), and recalling that $v_n$ is the solution of problem (4.2), we can see that $v$ satisfies

$$\begin{cases} -\Delta_{p(x_0)} v = v^{q(x_0)} & x \in \mathbf{R}^N_\varepsilon, \\ v(x) = 0 & x \in \partial\mathbf{R}^N_\varepsilon. \end{cases}$$

By the strong maximum principle, $v > 0$ in $\mathbf{R}^N_\varepsilon$. Thus, $v$ is a positive solution for the above problem, which is a contradiction to Lemma 4.1. This contradiction provides the a priori estimates on $u$. The proof is complete. □

**5. The existence.** In this section, we will prove the existence of positive solutions for problem (1.1) based on the a priori estimates obtained in Section 4 and the Krasnoselskii fixed point theorem on the cone raised in [**14**], which also can be found in [**4**].

As preparation, we need a nonexistence result for the following problem with large $\mu > 0$.

$$\begin{cases} -\Delta_{p(x)} u = f(x, u, \nabla u) + \mu & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{5.1}$$

**Lemma 5.1.** *Problem* (5.1) *has no positive solution for* $\mu \geq \mu_0$ *with some suitable* $\mu_0 > 0$.

*Proof.* Let $u$ be a positive solution for problem (5.1) with $\mu \geq 2$. Recalling that $f(x, u, \nabla u) = u^{q(x)} + g(x, u, \nabla u)$ and $g(x, u, \nabla u) \geq 0$, it follows that

$$-\Delta_{p(x)} u \geq u^{q(x)} + \mu, \quad x \in \Omega.$$

Define $\widetilde{u}(x) := u(x) + 1$ for any $x \in \overline{\Omega}$. Note that $(u+1)^{q(x)} \leq 2^{q(x)}(u^{q(x)}+1) \leq M(u^{q(x)}+1)$, where $M = \max_{\overline{\Omega}} 2^{q(x)}$. Then $\widetilde{u}$ satisfies

$$-\Delta_{p(x)} \widetilde{u} \geq M^{-1}\widetilde{u}^{q(x)} + \mu - 1 \geq M^{-1}\widetilde{u}^{q(x)} + 1, \quad x \in \Omega.$$

Taking a fixed point $x_0 \in \Omega$ and $0 < R_0 \leq 1$, such that $B_{2R_0} := B_{2R_0}(x_0) \subseteq \Omega$ and recalling that $p(x)$ and $q(x)$ are continuous on $\overline{\Omega}$ and $p(x) - 1 < q(x)$, we can let $R_0 = R_0(p, q)$, which is small enough, be such that $q_1 > p_2 - 1 + \varepsilon_0$ for some small positive constant $\varepsilon_0 = \varepsilon_0(p, q)$, where $p_2 = \max_{\overline{B_{2R_0}}} p(x)$ and $q_1 = \min_{\overline{B_{2R_0}}} q(x)$. Recalling that $p(x) < N$ on $\overline{\Omega}$, it follows that

$$\begin{aligned}
\frac{(q_1+1)(p_2-1)}{q_1 - p_2 + 1} &= p_2 - 1 + \frac{p_2(p_2-1)}{q_1 - p_2 + 1} \\
&\leq p_2 - 1 + \frac{p_2(p_2-1)}{\varepsilon_0} \\
&\leq \frac{(N-1)(N+\varepsilon_0)}{\varepsilon_0} \\
&\leq \frac{N^2-1}{\varepsilon_0}.
\end{aligned}$$

By Lemma 3.1, a $C_0$ exists such that

$$\begin{aligned}
\int_{B_{R_0}(x_0)} |\nabla u|^{p_2 - 1} dx &= \int_{B_{R_0}(x_0)} |\nabla \widetilde{u}|^{p_2-1} dx \\
&\leq C_0 R_0^{N - [(q_1+1)(p_2-1)]/(q_1 - p_2 + 1)} \\
&\leq C_0 R_0^{N - [(N^2-1)/\varepsilon_0]}.
\end{aligned}$$

Let $e(x)$ be the unique solution of the problem

$$\begin{cases} -\Delta_{p(x)} e = 1 & x \in \Omega, \\ e(x) = 0 & x \in \partial\Omega. \end{cases}$$

Then $e \in C^1(\overline{\Omega})$ and, for any $k > 0$, $ke(x)$ satisfies

$$-\Delta_{p(x)}(ke) = k^{p(x)-1} - k^{p(x)-1} \ln k (\nabla p(x) \cdot \nabla e) |\nabla e|^{p(x)-2}.$$

Taking $C > 0$ such that $\max_{x \in \overline{\Omega}} |\nabla p(x)| |\nabla e|^{p(x)-1} \leq C$, it follows from the previous equation that

$$-\Delta_{p(x)}(ke) \leq k^{p(x)-1} + C k^{p(x)-1} |\ln k|.$$

Choosing $k > 0$, which is small enough, such that for any $x \in \overline{\Omega}$,

$$k^{p(x)-1} + C k^{p(x)-1} |\ln k| \leq 2 \leq \mu.$$

By the comparison theorem, one has

$$ke \leq u, \quad x \in \Omega.$$

Let $\eta$ be a standard cut-off function on $B_{R_0}$. Taking $\phi = (\eta e)^{p(x)}/u^{p(x)-1}$ as a test function, then we obtain

$$\begin{aligned}\int_{B_{R_0}(x_0)} &|\nabla u|^{p(x)-2} \nabla u \nabla \left( \frac{(\eta e)^{p(x)}}{u^{p(x)-1}} \right) dx \\ &= \int_{B_{R_0}(x_0)} \left( u^{q(x)} + g(x, u, \nabla u) + \mu \right) \frac{(\eta e)^{p(x)}}{u^{p(x)-1}} \, dx.\end{aligned}$$

Recalling $g(x, u, \nabla u) \geq 0$ and considering $\ln(\eta e)$ as $0$ on the points where $\eta(x)e(x)=0$, it follows from the previous inequality that

$$
\begin{aligned}
(5.2)\quad & \int_{B_{R_0}(x_0)} \frac{u^{q(x)}+\mu}{u^{p(x)-1}}(\eta e)^p dx \\
& \leq \int_{B_{R_0}(x_0)} |\nabla u|^{p(x)-2}\nabla u \nabla\left(\frac{(\eta e)^{p(x)}}{u^{p(x)-1}}\right) dx \\
& = \int_{B_{R_0}(x_0)} \Bigg[ p(x)\left(\frac{\eta e}{u}\right)^{p(x)-1} |\nabla u|^{p(x)-2}\nabla u\nabla(\eta e) \\
& \qquad - \big(p(x)-1\big)\left(\frac{\eta e}{u}\right)^{p(x)} |\nabla u|^{p(x)} \\
& \qquad + \frac{(\eta e)^{p(x)}}{u^{p(x)-1}}\big(\ln(\eta e)-\ln u\big)|\nabla u|^{p(x)-2}\nabla u\nabla p(x)\Bigg] dx \\
& \leq \int_{B_{R_0}(x_0)} \Bigg[ p(x)\left(\frac{\eta e}{u}\right)^{p(x)-1} |\nabla u|^{p(x)-1}|\nabla(\eta e)| \\
& \qquad - \big(p(x)-1\big)\left(\frac{\eta e}{u}\right)^{p(x)} |\nabla u|^{p(x)} \\
& \qquad + \frac{(\eta e)^{p(x)}}{u^{p(x)-1}}\big(|\ln(\eta e)|+|\ln u|\big)|\nabla u|^{p(x)-1}|\nabla p(x)|\Bigg] dx.
\end{aligned}
$$

In fact, since $p(x)>1$ on $\overline{\Omega}$, it's reasonable for us to deal with $\ln(\eta e)$ like this. By the Young inequality, we have

$$
\begin{aligned}
(5.3)\quad & p(x)\left(\frac{\eta e}{u}\right)^{p(x)-1} |\nabla u|^{p(x)-1}|\nabla(\eta e)| \\
& \qquad \leq (p(x)-1)|\nabla u|^{p(x)}\left(\frac{\eta e}{u}\right)^{p(x)} + |\nabla(\eta e)|^{p(x)}.
\end{aligned}
$$

Putting (5.3) into (5.2), we conclude

$$
\begin{aligned}
(5.4)\quad & \int_{B_{R_0}(x_0)} \frac{u^{q(x)}+\mu}{u^{p(x)-1}}(\eta e)^{p(x)} dx \\
& \leq \int_{B_{R_0}(x_0)} \Bigg( |\nabla(\eta e)|^{p(x)} + \frac{(\eta e)^{p(x)}}{u^{p(x)-1}}\big(|\ln(\eta e)|+|\ln u|\big) \\
& \qquad \cdot |\nabla u|^{p(x)-1}|\nabla p(x)|\Bigg) dx.
\end{aligned}
$$

Recalling that $ke \le u$, it follows that a positive constant $C$ exists, such that

$$\left|\frac{(\eta e)^{p(x)}}{u^{p(x)-1}}\ln(\eta e)\right| \le \left(\frac{e}{u}\right)^{p(x)-1}|(\eta e)\ln(\eta e)| \le k^{p(x)-1}|(\eta e)\ln(\eta e)| \le C, \quad \text{if } \eta e \ne 0, \tag{5.5}$$

and

$$\left|\frac{(\eta e)^{p(x)}}{u^{p(x)-1}}\ln u\right| \le \left(\frac{e}{u}\right)^{p(x)-1}\sup_{u\in(0,1]}|e\ln u| + e^{p(x)}\sup_{u\in[1,\infty)}\frac{\ln u}{u^{p(x)-1}} \le k^{-p(x)}\sup_{u\in(0,1]}|u\ln u| + e^{p(x)}\sup_{u\in[1,\infty)}\frac{\ln u}{u^{p(x)-1}} \le C. \tag{5.6}$$

Denote

$$l(\mu) = \min_{\substack{x\in\overline{B_{R_0}}\\ t>0}}\frac{t^{q(x)}+\mu}{t^{p(x)-1}}.$$

Then $l(\mu) \to \infty$ as $\mu \to \infty$. Putting (5.5) and (5.6) into (5.4), we obtain

$$l(\mu)\int_{B_{R_0}(x_0)}(\eta e)^{p(x)}dx \le \int_{B_{R_0}(x_0)}|\nabla(\eta e)|^{p(x)}dx + C\int_{B_{R_0}(x_0)}|\nabla u|^{p(x)-1}dx.$$

It follows from the Young inequality that

$$\int_{B_{R_0}(x_0)}|\nabla u|^{p(x)-1}dx \le CR_0^N + \int_{B_{R_0}(x_0)}|\nabla u|^{p_2-1}dx \le CR_0^N + C_0R_0^{N-[(N^2-1)/\varepsilon_0]}.$$

Combining the foregoing two inequalities, we can see that $l(\mu)$ is bounded. Recalling that $l(\mu) \to \infty$ as $\mu \to \infty$, a suitable positive constant $\mu_0$ exists such that $\mu < \mu_0$. The proof is complete. ◻

The following Krasnoselskii fixed point theorem on the cone is raised in [**14**], see also in [**4**].

**Lemma 5.2.** *Let $\mathscr{C}$ be a cone in a Banach space and $\mathscr{K}\colon \mathscr{C} \to \mathscr{C}$ a compact operator, such that $\mathscr{K}(0) = 0$. Assume that an $r > 0$ exists, verifying*:

(A) $u \neq t\mathscr{K}(u)$ *for all* $\|u\| = r$, $t \in [0,1]$.

*Assume also that a compact homotopy* $\mathscr{H}\colon [0,1]\cdot\mathscr{C} \to \mathscr{C}$ *and* $R > r$ *exist such that*:

(B1) $\mathscr{K}(u) = \mathscr{H}(0,u)$ *for all* $u \in \mathscr{C}$.

(B2) $\mathscr{H}(t,u) \neq u$ *for any* $\|u\| = R$, $t \in [0,1]$.

(B3) $\mathscr{H}(1,u) \neq u$ *for any* $\|u\| \leq R$.

*Let* $D = \{u \in \mathscr{C} : r < \|u\| < R\}$. *Then,* $\mathscr{K}$ *has a fixed point in* $D$.

Now, we can state and prove our main result.

**Theorem 5.1.** *Suppose that* (H1)–(H3) *hold true with* $|\xi|^{\lambda(x)} + 1$ *replaced by* $|\xi|^{\lambda(x)}$ *in* (H2). *Denote by* $p_-$ *and* $p_+$ *the minimum and maximum of* $p(x)$ *on* $\overline{\Omega}$, *respectively. The minimum and maximum of* $q(x)$, $\lambda(x)$ *and* $\kappa(x)$ *are denoted by similar symbols. Assume that* $\lambda_- > p_+ - 1$, $\kappa_- > p_+ - 1$ *and* $q_- > p_+ - 1$. *Then, at least one positive solution for problem* (1.1) *exists.*

*Proof.* We use Lemma 5.2 to prove our result. Denote

$$\mathscr{C} = \{u \in C^{1,\alpha}(\overline{\Omega}) \mid u(x) \geq 0 \text{ on } \overline{\Omega}\}.$$

Then $\mathscr{C}$ is a cone in $C^{1,\alpha}(\overline{\Omega})$. Define a mapping $\mathscr{K} : \mathscr{C} \to \mathscr{C}$, such that for any $u \in \mathscr{C}$, $\mathscr{K}(u)$ denotes the unique solution of the following problem

$$\begin{cases} -\Delta_{p(x)}\mathscr{K}(u) = f(x,u,\nabla u) & x \in \Omega, \\ \mathscr{K}(u)(x) = 0 & x \in \partial\Omega. \end{cases}$$

By the strong maximum principle in [**11**] and the $C^{1,\alpha}$ estimates in [**5**], the definition of $\mathscr{K}$ is reasonable, and in addition, $\mathscr{K}$ is compact. Note that $f(x,0,0) \equiv 0$ in $\Omega$. Thus, $\mathscr{K}(0) = 0$.

We now verify the conditions stated in Lemma 5.2. We first verify item (A). Let $0 < r < 1$ be small enough. Suppose $\|u\|_{C^{1,\alpha}} = r$ and $u = t\mathscr{K}(u)$ for some $t \in [0,1]$. Obviously, $t \neq 0$. By the definition of

$\mathscr{K}(u)$, it follows that

$$-\Delta_{p(x)} \frac{u}{t} = f(x, u, \nabla u), \quad x \in \Omega.$$

Taking $u$ as a test function for the above equation, then we have

$$\int_\Omega t^{1-p(x)} |\nabla u|^{p(x)} dx = \int_\Omega f(x, u, \nabla u) u \, dx.$$

On one hand, recalling that $t \in (0, 1]$, it follows that

$$\int_\Omega t^{1-p(x)} |\nabla u|^{p(x)} dx \ge \int_\Omega |\nabla u|^{p(x)} dx.$$

On the other hand, by condition (H2), a $C > 0$ exists such that

$$\int_\Omega f(x, u, \nabla u) u \, dx \le C \int_\Omega \left( u^{q(x)+1} + u^{\kappa(x)+1} + |\nabla u|^{\lambda(x)} u \right) dx.$$

Combining the previous three inequalities, we have

$$\int_\Omega |\nabla u|^{p(x)} dx \le C \int_\Omega \left( u^{q(x)+1} + u^{\kappa(x)+1} + |\nabla u|^{\lambda(x)} u \right) dx.$$

Recalling that $0 < r < 1$, it follows that $|u(x)| < 1$ and $|\nabla u(x)| < 1$ on $\overline{\Omega}$. Consequently, we obtain

$$\int_\Omega |\nabla u|^{p_+} dx \le \int_\Omega |\nabla u|^{p(x)} dx,$$

and

$$\int_\Omega \left( u^{q(x)+1} + u^{\kappa(x)+1} + |\nabla u|^{\lambda(x)} u \right) \le \int_\Omega \left( u^{q_-+1} + u^{\kappa_-+1} + |\nabla u|^{\lambda_-} u \right) dx.$$

Combining the previous three inequalities, it follows that

$$\int_\Omega |\nabla u|^{p_+} dx \le \int_\Omega \left( u^{q_-+1} + u^{\kappa_-+1} + |\nabla u|^{\lambda_-} u \right) dx.$$

Denote $a = (\int_\Omega |\nabla u|^{p_+} dx)^{1/p_+}$. Combining the Hölder inequality with the Sobolev embedding theorem, we deduce

$$\begin{aligned}
&\int_\Omega \big(u^{q_-+1} + u^{\kappa_-+1} + |\nabla u|^{\lambda_1} u\big)\, dx \\
&\qquad \le Ca^{q_-+1} + Ca^{\kappa_-+1} + Ca^{\lambda_-} \left( \int_\Omega u^{p_+/(p_+-\lambda_-)} dx \right)^{(p_+-\lambda_-)/p_+} \\
&\qquad \le C\big(a^{q_-+1} + a^{\kappa_-+1} + a^{\lambda_-+1}\big).
\end{aligned}$$

Therefore, it follows from the previous two inequalities that

$$a^{p_+} \le C\big(a^{q_-+1} + a^{\kappa_-+1} + a^{\lambda_-+1}\big),$$

or

$$a^{q_--p_++1} + a^{\kappa_--p_++1} + a^{\lambda_--p_++1} \ge C_0$$

for some constant $C_0 > 0$. Note that $q_- > p_+ - 1$, $\kappa_- > p_+ - 1$ and $\lambda_- > p_+ - 1$. It follows from the above inequality that a constant $\varepsilon_0 > 0$ exists such that

$$a = \left( \int_\Omega |\nabla u|^{p_+} dx \right)^{1/p_+} \ge \varepsilon_0.$$

Consequently, $0 < r_0 < 1$ exists such that

$$\max_{x\in\overline{\Omega}} |\nabla u(x)| \ge \left( \frac{1}{|\Omega|} \int_\Omega |\nabla u|^{p_+} dx \right)^{1/p_+} \ge r_0,$$

and hence,

$$\|u\|_{C^{1,\alpha}} \ge r_0.$$

So, if we take $r = r_0/2$, then $u \ne t\mathscr{K}(u)$ for any $\|u\|_{C^{1,\alpha}} = r$ and $t \in [0,1]$. Item (A) is verified.

Next, we verify (B1)–(B3). Let $\mu_0$ be the same constant in Lemma 5.1 and the homotopy $\mathscr{H}$: $[0,1]\times\mathscr{C} \to \mathscr{C}$. For any $(t,u) \in [0,1]\times\mathscr{C}$, $\mathscr{H}(t,u)$ denotes the unique solution of the problem

$$\begin{cases} -\Delta_{p(x)}\mathscr{H}(t,u) = f(x,u,\nabla u) + \mu_0 t & x \in \Omega, \\ \mathscr{H}(t,u)(x) = 0 & x \in \partial\Omega. \end{cases} \tag{5.7}$$

By the strong maximum principle in [**11**] and the $C^{1,\alpha}$ estimates in [**5**], the definition of $\mathscr{H}$ is reasonable, and $\mathscr{H}$ is compact. Obviously, $\mathscr{K}(u) = \mathscr{H}(0,u)$ for any $u \in \mathscr{C}$. Thus, (B1) is verified. By condition (H2), two positive constants $\widetilde{K}_1$ and $\widetilde{K}_2$ exist such that

$$\begin{aligned}\widetilde{K}_1 u^{q(x)} - \widetilde{K}_2\big(|\nabla u|^{\lambda(x)} + 1\big) &\le f(x,u,\nabla u) + \mu_0 t\\ &\le \widetilde{K}_2\big(u^{q(x)} + |\nabla u|^{\lambda(x)} + 1\big).\end{aligned}$$

Thus, we can apply Lemma 4.2 to problem (5.7) with $\mathscr{H}(t,u)$ replaced by $u$, and consequently, a positive constant $C$ exists, such that $\|u\| \le C$ for any fixed point of $\mathscr{H}(t,u)$, where $\|\cdot\|$ stands for the uniform norm. Then, by the $C^{1,\alpha}$ estimate in [**5**], a constant $R > 0$ exists such that $\|u\|_{C^{1,\alpha}} < R$ for any fixed point of $\mathscr{H}(t,u)$. Thus, (B2) is verified, while (B3) is the direct corollary of Lemma 5.1.

By Lemma 5.2, a fixed point $u$ for $\mathscr{K}(u)$ in $\mathscr{C}$ satisfies $r \le \|u\|_{C^{1,\alpha}} \le R$. By the definition of $\mathscr{K}$, $u$ is a solution of problem (1.1). Utilizing the strong maximum principle in [**11**], we know that $u$ is a positive solution of problem (1.1). The proof is complete. □

## APPENDIX

**6. Global $C^\alpha$ estimates.** The appendices are employed to prove Propositions 4.1 and 4.2 stated in Section 4, in other words, we do the global $C^{1,\alpha}$ estimates on the bounded weak solutions for elliptic equations of the form

$$\begin{cases} -\operatorname{div} A(x,u,\nabla u) = B(x,u,\nabla u) & x \in \Omega,\\ u(x) = 0 & x \in \partial\Omega.\end{cases} \tag{6.1}$$

Obviously, the global $C^{1,\alpha}$ estimates are based on the global $C^\alpha$ estimates. In this section, we concentrate on doing the global $C^\alpha$ estimates, while the global $C^{1,\alpha}$ estimates and the proof of Propositions 4.1 and 4.2 are given in Section 7.

Since the weak solution we considered here and in Section 7 is bounded, without loss of generality, we can suppose that

$$\max_{x\in\Omega} |u(x)| \le M, \quad M > 0.$$

Assume that

(A2′) $A : \Omega \times [-M, M] \times \mathbf{R}^N \to \mathbf{R}^N$ and $B : \Omega \times [-M, M] \to \mathbf{R}$. $A(x, u, \eta)$ and $B(x, u, \eta)$ are measurable in $x$ and continuous in $(u, \eta)$. Positive constants $\lambda^*$ and $\Lambda^*$ exist such that

$$A(x, u, \eta)\eta \geq \lambda^*|\eta|^{p(x)}, \qquad |A(x, u, \eta)| \leq \Lambda^*|\eta|^{p(x)-1}$$

and

$$|B(x, u, \eta)| \leq \Lambda^*(1 + |\eta|^{p(x)}),$$

for any $(x, u, \eta) \in \Omega \times [-M, M] \times \mathbf{R}^N$.

Throughout this appendix, we always suppose (A1), (A2′) and (A3) hold true.

The global $C^\alpha$ estimates are based on the Hölder continuity of functions in the class $\mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$, which was introduced in [**10**]. $\mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$ is the natural generalization of class $\mathscr{B}_p(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$, which was introduced in [**15**]. The approach we used here is similar to that in [**10, 15**]. In fact, the interior $C^\alpha$ estimates can be deduced from the results stated in [**10**]. However, the boundary estimates were not considered there.

**Definition 6.1** [**10**]. Let $M, \gamma, \gamma_1$ and $\delta$ be positive constants with $\delta \leq 2$. We will say that a function $u(x)$ belongs to class $\mathscr{B}_{p(x)}(\Omega, M, \gamma, \gamma_1, \delta)$ if $u \in W^{1,p(x)}(\Omega)$, $\max_\Omega |u(x)| \leq M$, and the functions $w(x) = \pm u(x)$ satisfy the inequality

(6.2)
$$\int_{B_{k,r}} |\nabla w|^{p(x)} dx \leq \gamma \int_{B_{k,\rho}} \left|\frac{w(x) - k}{\rho - r}\right|^{p(x)} dx + \gamma_1 |B_{k,\rho}|, \quad 0 < r < \rho$$

for arbitrary $B_\rho \subseteq \Omega$ and such that $k$

$$k \geq \max_{B_\rho} w(x) - \delta M, \tag{6.3}$$

where $B_{k,\rho} := \{x \in B_\rho \mid w(x) > k\}$.

**Definition 6.2** [**10**]. We will say that function $u$ belongs to $\mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$ if $u \in \mathscr{B}_{p(x)}(\Omega, M, \gamma, \gamma_1, \delta)$ and, in addition, the following holds:

$$\int_{\Omega_{k,r}} |\nabla w|^{p(x)} dx \le \gamma \int_{\Omega_{k,\rho}} \left| \frac{w(x)-k}{\rho - r} \right|^{p(x)} dx + \gamma_1 |\Omega_{k,\rho}|, \quad 0 < r < \rho$$

for arbitrary ball $B_\rho$ with center on $\partial\Omega$ and $k$ such that

$$k \ge \max\left\{ \max_{\Omega_\rho} w(x) - \delta M, \max_{S_\rho} w \right\},$$

where $\Omega_\rho := B_\rho \cap \Omega$, $S_\rho := \partial\Omega \cap B_\rho$ and $\Omega_{k,\rho} := \{x \in \Omega_\rho \mid w(x) > k\}$.

The following lemma is taken from [**10**], which states the interior estimates on functions in the class $\mathscr{B}_{p(x)}(\Omega, M, \gamma, \gamma_1, \delta)$.

**Lemma 6.1.** *Let $\Omega$ be a domain in $\mathbf{R}^N$, $B_R$ and $B_{R/4}$ concentric balls contained in $\Omega$. Then a positive constant $R_0 = R_0(M, L_0, \alpha_0)$ and an integer $s = s(N, p_+, \gamma) \ge 2$ exist such that, for any function $u \in \mathscr{B}_{p(x)}(\Omega, M, \gamma, \gamma_1, \delta)$, at least one of the following two inequalities holds*:

$$\operatorname{osc}\{u; B_R\} \le \tau^{-1} 2^s \frac{\gamma + \gamma_1 + 1}{\gamma} R,$$

$$\operatorname{osc}\{u; B_{R/4}\} \le (1 - \tau 2^{-s}) \operatorname{osc}\{u; B_R\},$$

*for any $R \le R_0$, where $\tau = \min\{(1/2), (\delta/2)\}$.*

Similarly to Lemma 6.1, we can obtain global estimates on functions in the class $\mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$. In fact, we have the following lemma.

**Lemma 6.2.** *Let $\Omega$ be a domain in $\mathbf{R}^N$ and satisfy* (A3). *Let $u$ be a function of class $\mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$. Suppose that, for any ball $B_\rho$ with center on $\partial\Omega$ and $\rho \le \rho_0$, the following holds*:

$$\operatorname{osc}\{u; S_\rho\} \le K\rho^\varepsilon, \quad \varepsilon > 0. \tag{6.4}$$

*Let $B_R$ and $B_{R/4}$ be two concentric balls with center on $\partial\Omega$ and $R \le \rho_0$. Then a positive constant $R_0 = R_0(M, L_0, \alpha_0) \le \rho_0$ and an integer $s = s(N, p_+, \gamma, \theta_0) \ge 2$ exist, such that at least one of the following two inequalities hold*:

$$\begin{aligned} &\operatorname{osc}\{u; \Omega_R\} \le \max\left\{ 2K, \tau^{-1} 2^s \rho_0^{1-\varepsilon} \frac{\gamma + \gamma_1 + 1}{\gamma} \right\} R^\varepsilon, \\ &\operatorname{osc}\{u; \Omega_{R/4}\} \le (1 - \tau 2^{-s}) \operatorname{osc}\{u; \Omega_R\}, \end{aligned} \tag{6.5}$$

*for any* $R \le R_0$*, where* $\tau = \min\{(1/2), (\delta/2)\}$.

The proof of Lemma 6.2 is based on the following Lemmas 6.3 and 6.4.

**Lemma 6.3.** *Let* $\Omega \subseteq \mathbf{R}^N$ *be a domain,* $w(x) \in W^{1,p(x)}(\Omega)$ *a bounded measurable function with* $\max_{x\in\Omega} |w(x)| \le M$*,* $M \ge 1$*.* $B_R$*,* $B_{R/2}$ *and* $B_{R/4}$ *are concentric balls contained in* $\Omega$*. Suppose that, for any balls* $B_r$ *and* $B_\rho$*, which have the common center with* $B_R$*, and* $R/4 \le r < \rho \le R$*, the following inequality holds*:

(6.6)
$$\int_{B_{k,r}\setminus B_{l,r}} |\nabla w|^{p(x)} dx \le \gamma \int_{B_{k,\rho}} \left|\frac{w-k}{\rho - r}\right|^{p(x)} dx + \gamma_1 |B_{k,\rho}|, \quad l \ge k \ge k',$$

*where* $k' \ge -M$ *is a fixed constant, which satisfies* $|B_{k',R/2}| \le (1-\delta_0)|B_{R/2}|$*, and* $\gamma \ge 1, \gamma_1$ *and* $\delta_0 < 1$ *are positive constants. Denote* $\omega = \max_{B_R} w(x) - k'$*. Then a positive constant* $R_0 = R_0(M, L_0, \alpha_0)$ *and an integer* $s = s(N, p_+, \gamma, \delta_0) \ge 2$ *exist, such that*

$$\omega \le 2^s \max\left\{ \max_{B_R} w(x) - \max_{B_{R/4}} w(x), \frac{\gamma+\gamma_1+1}{\gamma} R\right\},$$

*for any* $R \le R_0$*.*

*Proof.* Denote $p_-^* = \min_{B_R} p(x)$ and $p_+^* = \max_{B_R} p(x)$. By (A1), a positive constant $R_0 = (M, L_0, \alpha_0)$ exists such that

$$R^{-(p_+^* - p_-^*)} \le 2, \qquad (2M)^{p_+^* - p_-^*} \le 2, \tag{6.7}$$

provided $R \le R_0$. We complete the proof in the following three steps.

Step 1. A positive constant $\theta = \theta(N, p_+, \gamma) < 1$ exists, such that, for any $k^0 \ge k'$, if

$$|B_{k^0, R/2}| \le \theta R^N, \tag{6.8}$$

then at least one of the following inequalities holds

$$\max_{B_{R/4}} w(x) \le \frac{1}{2}\left(\max_{B_R} w(x) + k^0\right),$$
$$\max_{B_R} w(x) \le k^0 + \frac{\gamma+\gamma_1+1}{\gamma} R.$$

Denote $H = \max_{B_R} w(x) - k^0$. Obviously $H \le 2M$. We assume that

$$\max_{B_R} w(x) > k^0 + \frac{\gamma + \gamma_1 + 1}{\gamma} R,$$

namely, $H > [(\gamma + \gamma_1 + 1)/\gamma]R$. Set

$$\rho_j = \frac{R}{4} + \frac{R}{2^{j+2}}, \qquad k_j = k^0 + \frac{H}{2} - \frac{H}{2^{j+1}},$$

$$y_j = R^{-N}|B_{k_j,\rho_j}|, \qquad D_{j+1} = B_{k_j,\rho_{j+1}} \setminus B_{k_{j+1},\rho_{j+1}}, \quad j = 0, 1, \ldots .$$

Obviously $k_j \ge k^0 \ge k'$. Taking $l = k_{j+1}$, $k = k_j$, $r = \rho_{j+1}$ and $\rho = \rho_j$ in (6.6), then one obtains

$$\int_{D_{j+1}} |\nabla w|^{p(x)} dx \le \gamma \int_{B_{k_j,\rho_j}} \left(\frac{2^{j+3}}{R}\right)^{p(x)} |w - k_j|^{p(x)} dx + \gamma_1 |B_{k_j,\rho_j}|, \\ j = 0, 1, \ldots .$$

It follows from the Young inequality and the inequality above that

$$\begin{aligned} \int_{D_{j+1}} |\nabla w|^{p_-^*} dx &\le (\gamma + \gamma_1 + 1)|B_{k_j,\rho_j}| \\ &\quad + \gamma 2^{(j+3)p_+^*} R^{-p_+^*} \int_{B_{k_j,\rho_j}} |w - k_j|^{p_+^*} dx \\ &\le (\gamma + \gamma_1 + 1)|B_{k_j,\rho_j}| + \gamma 2^{(j+3)p_+^*} R^{-p_+^*} H^{p_+^*} |B_{k_j,\rho_j}|. \end{aligned}$$

Recalling that $H > [(\gamma + \gamma_1 + 1)/\gamma]R$ and $p_+^* > 1$, one has $\gamma + \gamma_1 + 1 < \gamma H^{p_+^*} R^{-p_+^*}$, and consequently, it follows from the previous inequality, that

$$\begin{aligned} \int_{D_{j+1}} |\nabla w|^{p_-^*} dx &\le 2^{(j+4)p_+^*} \gamma R^{-p_+^*} H^{p_+^*} |B_{k_j,\rho_j}| \\ &= 2^{(j+4)p_+^*} \gamma R^{N - p_+^*} H^{p_+^*} y_j. \end{aligned}$$

Applying the Hölder inequality to the left side of the above inequality and recalling (6.7) and $\gamma \ge 1$, one obtains

(6.9)
$$\begin{aligned} \int_{D_{j+1}} |\nabla w| \, dx &\le \left( \int_{D_{j+1}} |\nabla w|^{p_-^*} dx \right)^{1/p_-^*} |D_{j+1}|^{1 - 1/p_-^*} \\ &\le (y_j R^N)^{1 - 1/p_-^*} 2^{(j+4)p_+^*/p_-^*} \gamma^{1/p_-^*} R^{(N - p_+^*)/p_-^*} H^{p_+^*/p_-^*} y_j^{1/p_-^*} \\ &= R^{N - p_+^*/p_-^*} 2^{(j+4)p_+^*/p_-^*} \gamma^{1/p_-^*} H^{p_+^*/p_-^*} y_j \\ &\le R^{N-1} 2^{(j+5)p_+} \gamma H^{p_+^*/p_-^*} y_j, \quad j = 0, 1, 2, \ldots . \end{aligned}$$

Recalling that $k_j \geq k^0$, $R/4 < \rho_{j+1} \leq R/2$, $j = 0, 1, \ldots$, it follows from (6.8) and Lemma 2.2 that

$$\begin{aligned}(6.10)\quad &\int_{D_{j+1}} |\nabla w|\,dx \\ &\geq (k_{j+1} - k_j)|B_{k_{j+1},\rho_{j+1}}|^{1-1/N}|B_{\rho_{j+1}} \setminus B_{k_j,\rho_{j+1}}|\beta(N)^{-1}\rho_{j+1}^{-N} \\ &\geq \frac{H}{2^{j+2}}(R^N y_{j+1})^{1-1/N}(4^{-N}\sigma_N - \theta)\beta(N)^{-1} \\ &= \frac{4^{-N}\sigma_N - \theta}{2^{j+2}\beta(N)} H R^{N-1} y_{j+1}^{1-1/N},\end{aligned}$$

where $\sigma_N$ is the volume of the unit ball in $\mathbf{R}^N$. Combining (6.7) and (6.9) with (6.10), and recalling $H \leq 2M$, one has

$$y_{j+1} \leq c b^j y_j^{1+\varepsilon}, \quad j = 0, 1, \ldots,$$

where

$$c = \left(\frac{\beta(N)2^{6(p_+ + 1)}\gamma}{4^{-N}\sigma_N - \theta}\right)^{N/(N-1)}, \qquad b = 2^{N(p_+ + 1)/(N-1)}, \qquad \varepsilon = \frac{1}{N-1}.$$

By Lemma 2.3, if

$$\begin{aligned} y_0 = R^{-N}|B_{k^0,R/2}| &\leq c^{-1/\varepsilon} b^{-1/\varepsilon^2} \\ &= \left(\frac{4^{-N}\sigma_N - \theta}{\beta(N)2^{6(p_+ + 1)}\gamma}\right)^{N-1} 2^{-N(N-1)(p_+ + 1)},\end{aligned}$$

namely,

$$|B_{k^0,R/2}| \leq c^{-1/\varepsilon} b^{-1/\varepsilon^2} = \left(\frac{4^{-N}\sigma_N - \theta}{\beta(N)2^{6(p_+ + 1)}\gamma}\right)^{N-1} 2^{-N(N-1)(p_+ + 1)} R^N,$$

then

$$y_j \longrightarrow 0 \quad \text{as } j \to \infty.$$

Take

$$\theta = \min\left\{\frac{1}{2}4^{-N}\sigma_N, \left(\frac{4^{-N}\sigma_N}{2\beta(N)2^{4(p_+ + 1)}}\right)^{N-1} 2^{-N(N^2-1)}\right\}.$$

Then $y_j \to 0$ as $j \to \infty$ provided $|B_{k^0,R/2}| \leq \theta R^N$. By the definition of $y_j$, we conclude that

$$|B_{k^0+H/2,R/4}| = \lim_{j\to\infty} |B_{k_j,\rho_j}| = \lim_{j\to\infty} R^N y_j = 0,$$

and therefore

$$\max_{B_{R/4}} w(x) \leq k^0 + \frac{H}{2},$$

namely,

$$\max_{B_{R/4}} w(x) \leq \frac{1}{2}\Big( \max_{B_R} w(x) + k^0 \Big).$$

Thus, we complete Step 1.

Step 2. For any $\theta > 0$, there exists an integer $s = s(N, p_-, p_+, \theta, \gamma, \delta_0) \geq 2$, such that if $\omega > 2^s[(\gamma + \gamma_1 + 1)/\gamma]R$, then (6.8) holds for

$$k^0 = \max_{B_R} w(x) - 2^{-(s-1)}\omega.$$

Let $s \geq 2$ be an integer, which will be determined later. Suppose that $\omega > 2^s[(\gamma + \gamma_1 + 1)/\gamma]R$. Denote

$$k_j = \max_{B_R} w(x) - 2^{-j}\omega, \quad D_j = B_{k_j,R/2} \setminus B_{k_{j+1},R/2}, \quad j = 0, 1, \ldots . \tag{6.11}$$

Taking $r = R/2$, $\rho = R$, $k = k_j$ and $l = k_{j+1}$ in (6.6), $j = 0, 1, \ldots, s-2$, then we conclude

$$\int_{D_j} |\nabla w|^{p(x)} dx \leq \gamma \int_{B_{k_j,R}} \left(\frac{2}{R}\right)^{p(x)} |w - k_j|^{p(x)} dx + \gamma_1 |B_{k_j,R}|.$$

Combining the above inequality with the Young inequality and recalling that $\omega > 2^s[(\gamma + \gamma_1 + 1)/\gamma]R$ and $p_+^* > 1$, we have

$$\begin{aligned} \int_{D_j} |\nabla w|^{p_-^*} dx &\leq (\gamma + \gamma_1 + 1)|B_{k_j,R}| \\ &\quad + \gamma\left(\frac{2}{R}\right)^{p_+^*} \int_{B_{k_j,R}} |w - k_j|^{p_+^*} dx \\ &\leq (\gamma + \gamma_1 + 1)|B_{k_j,R}| + 2^{(1-j)p_+^*}\gamma(\omega R^{-1})^{p_+^*}|B_{k_j,R}| \\ &\leq \gamma 2^{(2-j)p_+^*}(\omega R^{-1})^{p_+^*}|B_{k_j,R}|, \quad j = 0, 1, \ldots, s-2. \end{aligned} \tag{6.12}$$

By the Hölder inequality, it follows from Lemma 2.2 that

$$\begin{aligned}(6.13)\quad &(k_{j+1}-k_j)|B_{k_{j+1},R/2}|^{1-1/N}\\ &\le \frac{\beta(N)(R/2)^N}{|B_{R/2}\setminus B_{k_j,R/2}|}\int_{B_{k_j,R/2}\setminus B_{k_{j+1},R/2}}|\nabla w|\,dx\\ &\le \frac{\beta(N)}{\delta_0\sigma_N}\int_{D_j}|\nabla w|\,dx\\ &\le \frac{\beta(N)}{\delta_0\sigma_N}\left(\int_{D_j}|\nabla w|^{p_-^*}dx\right)^{1/p_-^*}|D_j|^{1-1/p_-^*},\quad j=0,1,\cdots,s-2.\end{aligned}$$

Putting (6.12) into (6.13) and recalling that $|B_{k_{j+1},R/2}|\ge|B_{k_{s-1},R/2}|$, $j=0,1,\dots,s-2$ and $\omega\le 2M$, $M\ge1$, $\gamma\ge1$, $p_->1$ and $R^{-(p_+^*-p_-^*)}\le 2$, we have

$$\begin{aligned}&|B_{k_{s-1},R/2}|^{1-1/N}\\ \le&\frac{\beta(N)}{\delta_0\sigma_N}\gamma^{1/p_-^*}2^{(1+2p_+^*/p_-^*)-j(p_+^*-p_-^*)/p_-^*}\omega^{1/p_-^*-1}R^{(N-p_+^*)/p_-^*}|D_j|^{1-1/p_-^*}\\ \le&\frac{2^{3+2p_+}\beta(N)\gamma}{\delta_0\sigma_N}R^{(N-p_-^*)/p_-^*}|D_j|^{1-1/p_-^*}.\end{aligned}$$

Summing up the previous inequalities with $j=0,1,\dots,s-2$, and noticing that

$$\begin{aligned}\sum_{j=0}^{s-2}|D_j|^{1-1/p_-^*}&\le\left(\sum_{j=0}^{s-2}|D_j|\right)^{1-1/p_-^*}(s-1)^{1/p_-^*}\\ &=|B_{k^0,R/2}\setminus B_{k_{s-1},R/2}|^{1-1/p_-^*}(s-1)^{1/p_-^*}\\ &\le(\sigma_NR^N)^{1-1/p_-^*}(s-1)^{1/p_-},\end{aligned}$$

then we conclude that

$$|B_{k_{s-1},R/2}|\le\left(\frac{8\cdot4^{p_+}\beta(N)\gamma}{(s-1)^{1-1/p_-}\delta_0\sigma_N^{1/p_+}}\right)^{N/(N-1)}R^N.$$

By the aid of the above estimates, we can choose the integer $s$ with

$$s\ge\left(\frac{8\cdot4^{P_+}\beta(N)\gamma}{\theta^{(N-1)/N}\delta_0\sigma_N^{1/p_+}}\right)^{p_-/(p_--1)}+1,$$

such that

$$|B_{k_{s-1},R/2}| \leq \theta R^N.$$

Hence, we complete Step 2.

Step 3. Let $\theta$ and $s$ be the constants stated in Step 1 and Step 2. Denote $k^0 = \max_{B_R} w - 2^{-(s-1)}\omega$. From Step 2, we know that at least one of the following inequalities holds

$$\omega \leq 2^s \frac{\gamma + \gamma_1 + 1}{\gamma} R, \qquad |B_{k^0,R/2}| \leq \theta R^N.$$

If the first one holds, then the conclusion is valid. Otherwise, by Step 1, at least one of the following inequalities holds

$$\max_{B_{R/4}} w(x) \leq \frac{1}{2}\Big(\max_{B_R} w(x) + k^0\Big), \qquad \max_{B_R} w(x) \leq k^0 + \frac{\gamma + \gamma_1 + 1}{\gamma} R,$$

from which the conclusion follows immediately. The proof is complete. $\square$

**Lemma 6.4.** *Let $\Omega$ be a domain and satisfy* (A3). *Let $w(x) \in W^{1,p(x)}(\Omega)$ be a bounded measurable function with $\max_\Omega |w(x)| \leq M$, $M \geq 1$. $B_R$, $B_{R/2}$ and $B_{R/4}$ are concentric balls with center on $\partial\Omega$. Suppose that for any balls $B_r$ and $B_\rho$ which have the common center with $B_R$, and $R/4 \leq r < \rho \leq R$, the following inequality holds*:

$$\int_{\Omega_{k,r}\setminus\Omega_{l,r}} |\nabla w|^{p(x)} dx \leq \gamma \int_{\Omega_{k,\rho}} \Big|\frac{w-k}{\rho - r}\Big|^{p(x)} dx + \gamma_1 |\Omega_{k,\rho}|, \quad l \geq k \geq k', \tag{6.14}$$

*where $k'$ is a fixed constant, which satisfies $\max_{S_R} w(x) \leq k' \leq \max_{\Omega_R} w(x)$, $\gamma \geq 1$ and $\gamma_1$ are both positive constants. Denote $\omega = \max_{B_R} w(x) - k'$. Then a positive constant $R_0 = R_0(M, L_0, \alpha_0) \leq \rho_0$ and an integer exist $s = s(N, p_-, p_+, \gamma, \theta_0) \geq 2$, such that*

$$\omega \leq 2^s \max\left\{\max_{B_R} w(x) - \max_{B_{R/4}} w(x), \frac{\gamma + \gamma_1 + 1}{\gamma} R\right\},$$

*for any $R \leq R_0$.*

*Proof.* Define

$$\widehat{w}(x)=\begin{cases}\max\{w(x),k'\} & x\in\Omega_R,\\ k' & x\in B_R\setminus\Omega_R.\end{cases}$$

Noticing that $k'\ge \max_{S_R} w$, one has $\widehat{w}\in W^{1,p(x)}(B_R)$. By the aid of (6.14), for any $k\ge k'$, it follows

$$\begin{aligned}\int_{B_{k,r}\setminus B_{l,r}}|\nabla\widehat{w}|^{p(x)}dx &= \int_{\Omega_{k,r}\setminus\Omega_{l,r}}|\nabla w|^{p(x)}dx\\ &\le \gamma\int_{\Omega_{k,\rho}}\left|\frac{w-k}{\rho-r}\right|^{p(x)}dx+\gamma_1|\Omega_{k,\rho}|\\ &\le \gamma\int_{B_{k,\rho}}\left|\frac{\widehat{w}-k}{\rho-r}\right|^{p(x)}dx+\gamma_1|B_{k,r}|,\end{aligned}\tag{6.15}$$

where $B_{k,\rho}:=\{x\in B_\rho\mid \widehat{w}(x)>k\}$. On account of (A3), the following holds

$$|B_{k',R/2}|=|\Omega_{k',R/2}|\le(1-\theta_0)|B_{R/2}|.\tag{6.16}$$

Combining (6.15) with (6.16), we infer that $\widehat{w}$ satisfies all the conditions in Lemma 6.3, and consequently, it follows from Lemma 6.3 that a positive constant $R_0=R_0(M,L_0,\alpha_0)\le\rho_0$ and an integer $s=s(N,p_-,p_+,\gamma,\theta_0)\ge 2$ exist such that

$$\max_{B_R}\widehat{w}(x)-k'\le 2^s\max\left\{\max_{B_R}\widehat{w}(x)-\max_{B_{R/4}}\widehat{w}(x),\frac{\gamma+\gamma_1+1}{\gamma}R\right\},$$

for any $R\le R_0$. Recalling that $\max_{S_R}w(x)\le k'\le\max_{\Omega_R}w(x)$, the above inequality implies that at least one of the following inequalities holds

$$\max_{B_R}w(x)-k'\le 2^s\frac{\gamma+\gamma_1+1}{\gamma}R,\tag{6.17}$$

$$\max_{B_R}w(x)-k'\le 2^s\Big(\max_{\Omega_R}w(x)-\max\Big\{\max_{\Omega_{R/4}}w(x),k'\Big\}\Big).\tag{6.18}$$

If (6.17) is valid, then the proof is completed. We now assume that (6.18) holds. If $\max_{\Omega_{R/4}} w \geq k'$, then it follows from (6.18) that

$$\max_{\Omega_R} w(x) - k' \leq 2^s \Big( \max_{\Omega_R} w(x) - \max_{\Omega_{R/4}} w(x) \Big). \tag{6.19}$$

If $\max_{\Omega_{R/4}} w < k'$, then obviously we have

$$\max_{\Omega_R} w(x) - k' \leq \max_{\Omega_R} w(x) - \max_{\Omega_{R/4}} w(x) \leq 2^s \Big( \max_{\Omega_R} w(x) - \max_{\Omega_{R/4}} w(x) \Big),$$

which implies that (6.19) still holds. Combining (6.17) with (6.19), we complete the proof. □

Now, we can give the proof of Lemma 6.2 as follows:

*Proof of Lemma* 6.2. Suppose $B_R$ and $B_{R/4}$ are two concentric balls with centers on $\partial\Omega$ and $R \leq \rho_0$. Set $\tau = \min\{1/2, \delta/2\}$. If $\operatorname{osc}\{u; \Omega_R\} \leq KR^{\varepsilon}$, then (6.5) holds. If $\operatorname{osc}\{u; \Omega_R\} > KR^{\varepsilon}$, then $\operatorname{osc}\{u; \Omega_R\} > \operatorname{osc}\{\mathrm{u}; \mathrm{S_R}\}$, and thus at least one the following two inequalities holds:

$$\max_{S_R} u(x) < \max_{\Omega_R} u(x) - \frac{1}{2}\operatorname{osc}\{u; \Omega_R\},$$
$$\max_{S_R}(-u(x)) < \max_{\Omega_R}(-u(x)) - \frac{1}{2}\operatorname{osc}\{u; \Omega_R\}.$$

Let $w$ be $u$ or $-u$, such that

$$\max_{S_R} w(x) < \max_{\Omega_R} w(x) - \frac{1}{2}\operatorname{osc}\{w; \Omega_R\},$$

and consequently

$$\max_{\Omega_R} w(x) - \tau\operatorname{osc}\{w; \Omega_R\} > \max_{S_R} w(x).$$

Set $k' = \max_{\Omega_R} w(x) - \tau\operatorname{osc}\{w; \Omega_R\}$. Recalling that $\tau = \min\{1/2, \delta/2\}$, this yields

$$\max_{\Omega_R} w(x) \geq k' \geq \max\Big\{ \max_{S_R} w(x), \max_{\Omega_R} w(x) - \delta M \Big\}.$$

By the definition of $\mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$, $w$ satisfies all the conditions in Lemma 6.4. By Lemma 6.4, a positive constant $R_0 = R_0(M, L_0, \alpha_0) \le \rho_0$ and an integer $s = s(N, p_-, p_+, \gamma, \theta_0) \ge 2$ exist such that

$$\max_{\Omega_R} w(x) - k' \le 2^s \max\left\{ \max_{\Omega_R} w(x) - \max_{\Omega_{R/4}} w(x), \frac{\gamma+\gamma_1+1}{\gamma} R \right\},$$

for any $R \le R_0$, which implies that at least one of the following two inequalities holds (recalling that $k' = \max_{\Omega_R} w(x) - \tau \operatorname{osc}\{w; \Omega_R\}$)

$$\operatorname{osc}\{w; \Omega_R\} \le \tau^{-1} 2^s \Big( \max_{\Omega_R} w(x) - \max_{\Omega_{R/4}} w(x) \Big), \tag{6.20}$$

$$\operatorname{osc}\{w; \Omega_R\} \le \tau^{-1} 2^s \frac{\gamma+\gamma_1+1}{\gamma} R \le \tau^{-1} 2^s \rho_0^{1-\varepsilon} \frac{\gamma+\gamma_1+1}{\gamma} R. \tag{6.21}$$

If (6.20) is valid, then it follows that

$$\begin{aligned}\tau^{-1} 2^s \operatorname{osc}\{w; \Omega_{R/4}\} \le{}& (\tau^{-1} 2^s - 1) \operatorname{osc}\{w; \Omega_R\} \\ &+ \tau^{-1} 2^s \Big( \min_{\Omega_R} w(x) - \min_{\Omega_{R/4}} w(x) \Big),\end{aligned}$$

and consequently, noticing that $\min_{\Omega_R} w(x) \le \min_{\Omega_{R/4}} w(x)$, we conclude that

$$\operatorname{osc}\{u; \Omega_{R/4}\} \le (1 - \tau 2^{-s}) \operatorname{osc}\{u; \Omega_R\},$$

which together with (6.20) and (6.21) implies the conclusion of Lemma 6.2. ◻

Combining Lemma 6.1 with Lemma 6.2, we have the following proposition, which states the Hölder continuity of functions in the class $\mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$.

**Proposition 6.1.** *Let $\Omega$ be a domain and satisfy* (A3). *Suppose that $u \in \mathscr{B}_{p(x)}(\overline{\Omega}, M, \gamma, \gamma_1, \delta)$ and for any ball $B_R$ with center on $\partial\Omega$ and $R \le \rho_0$, the following holds*:

$$\operatorname{osc}\{u; S_R\} \le K R^{\varepsilon}, \quad \varepsilon > 0.$$

*Then a positive constant* $R_0 = R_0(M, L_0, \alpha_0) \le \rho_0$ *and an integer* $s = s(N, P_-, p_+, \gamma, \theta_0)$ *exist such that, for any* $R \le R_0/24$ *and* $x_0 \in \overline{\Omega}$, *the following holds*:

$$\operatorname{osc}\{u; \Omega_R(x_0)\} \le c R_0^{-\alpha} R^{\alpha},$$

*where* $\alpha = \min\{\varepsilon, -\log_{24}(1 - \tau 2^{-s})\}$, $\tau = \min\{1/2, \delta/2\}$, $c = (24)^{\alpha} \max\{c_* R_0^{\varepsilon}, 2M\}$ *and*

$$c_* = 5^{\varepsilon} \max\left\{2K, 4\tau^{-1} 2^s \rho_0^{1-\varepsilon} \frac{\gamma + \gamma_1 + 1}{\gamma}\right\}.$$

*Proof.* Let $R_0$ and $s$ be the larger of those in Lemmas 6.1 and 6.2, respectively. Take arbitrary $x_0 \in \overline{\Omega}$ and $R \le R_0/24$. We first conclude that at least one of the following two inequalities holds

$$\operatorname{osc}\{u; \Omega_R(x_0)\} \le 5^{\varepsilon} \max\left\{2K, 4\tau^{-1} 2^s \rho_0^{1-\varepsilon} \frac{\gamma + \gamma_1 + 1}{\gamma}\right\} R^{\varepsilon}, \tag{6.22}$$

$$\operatorname{osc}\{u; \Omega_R(x_0)\} \le (1 - \tau 2^{-s}) \operatorname{osc}\{u; \Omega_{24R}(x_0)\}. \tag{6.23}$$

In fact, set $d = d(x_0, \partial\Omega)$. If $d \ge 4R$, then $B_{4R}(x_0) \subseteq \Omega$, and it follows from Lemma 6.1 that at least one of the following inequalities holds:

$$\operatorname{osc}\{u; B_{4R}(x_0)\} \le 4\tau^{-1} 2^s \frac{\gamma + \gamma_1 + 1}{\gamma} R \le 4\tau^{-1} 2^s \rho_0^{1-\varepsilon} \frac{\gamma + \gamma_1 + 1}{\gamma} R^{\varepsilon}, \tag{6.24}$$

$$\operatorname{osc}\{u; B_R(x_0)\} \le (1 - \tau 2^{-s}) \operatorname{osc}\{u; B_{4R}(x_0)\}. \tag{6.25}$$

If $d < 4R$, taking $y_0 \in \partial\Omega$ with $|x_0 - y_0| = d$, then $\Omega_R(x_0) \subseteq \Omega_{R+d}(y_0)$ and $\Omega_{4(R+d)}(x_0) \subseteq \Omega_{(4R+5d)}(y_0) \subseteq \Omega_{24R}(x_0)$. By using Lemma 6.2, at least one of the following inequalities holds:

$$\operatorname{osc}\{u; \Omega_{R+d}(y_0)\} \le \max\left\{2K, \tau^{-1} 2^s \rho_0^{1-\varepsilon} \frac{\gamma + \gamma_1 + 1}{\gamma}\right\} (R + d)^{\varepsilon},$$

$$\operatorname{osc}\,\{u; \Omega_{R+d}(y_0)\} \le (1 - \tau 2^{-s}) \operatorname{osc}\,\left\{u; \Omega_{4(R+d)}(y_0)\right\},$$

and consequently, at least one of the following inequalities holds:

(6.26)
$$\operatorname{osc}\{u;\Omega_R(x_0)\} \le 5^\varepsilon \max\left\{2K, \tau^{-1}2^s\rho_0^{1-\varepsilon}\frac{\gamma+\gamma_1+1}{\gamma}\right\}R^\varepsilon,$$

(6.27)
$$\operatorname{osc}\{u;\Omega_R(x_0)\} \le (1-\tau 2^{-s}) \operatorname{osc}\{u;\Omega_{24R}(x_0)\}.$$

Combining (6.24)–(6.27), we obtain (6.22) and (6.23). Set

$$c_* = 5^\varepsilon \max\left\{2K, 4\tau^{-1}2^s\rho_0^{1-\varepsilon}\frac{\gamma+\gamma_1+1}{\gamma}\right\}.$$

Then, by Lemma 2.4, the following holds:

$$\operatorname{osc}\{u;\Omega_R(x_0)\} \le cR_0^{-\alpha}R^\alpha,$$

where $\alpha = \min\{\varepsilon, -\log_{24}(1-\tau 2^{-s})\}$ and $c = (24)^\alpha \max\{c_* R_0^\varepsilon, 2M\}$. The proof is complete. □

By applying Proposition 6.1, we can obtain the global $C^\alpha$ estimates on the bounded weak solutions for problem (6.1).

**Proposition 6.2.** *Let $\Omega$ be a domain in $\mathbf{R}^N$ and satisfy* (A3). *Then positive constants $R^* = R^*(M, L_0, \alpha_0) \le \rho_0$, $\alpha^* = \alpha^*(N, M, p_-, p_+, \lambda^*, \Lambda^*, \theta_0)$ and $c^* = c^*(N, M, p_-, p_+, \lambda^*, \Lambda^*, \theta_0)$ exist such that, for any bounded weak solution $u$ of* (6.1) *and $R \le R^*$, the following holds*:

$$\operatorname{osc}\{u;\Omega_R\} \le c^* R^{\alpha^*}.$$

*Proof.* By Theorem 4.2 in [**9**], positive constants $\gamma$, $\gamma_1$ and $\delta$ exist depending only upon $\lambda^*, \Lambda^*, \Lambda, p_-, p_+$ and $M$, such that $u \in \mathscr{B}_{p(x)}(\bar\Omega, M, \gamma, \gamma_1, \delta)$. By using Proposition 6.1 for $u$, we can obtain the conclusion. The proof is complete. □

**7. Global $C^{1,\alpha}$ estimates.** In this appendix, we give the proof of Propositions 4.1 and 4.2, in another words, we establish the global $C^{1,\alpha}$ estimates on bounded weak solutions of problem (6.1). The interior $C^{1,\alpha}$ estimates can be deduced from the results stated in [**5**]; therefore we only need to consider boundary $C^{1,\alpha}$ estimates.

As was mentioned in Section 6, here we always suppose that $u$ is a bounded weak solution of problem (6.1). Let the domain $\Omega$ be satisfied by condition (H3). And, throughout this appendix, we always suppose that (A1)–(A3) hold true. As we can see from (A2), a direct calculation shows that (A2$'$) also holds with some positive constants $\lambda^*$ and $\Lambda^*$; thus, without loss of generality, we always suppose that (A2$'$) holds throughout this appendix.

The following lemma is taken from [**21**].

**Lemma 7.1** [**21**]. *Suppose that $A(x, z, \eta)$ satisfies assumptions* (A2). *Then we have*

$$(7.1)\quad (A(x,u,\eta) - A(x,u,\eta'))(\eta-\eta') \geq \begin{cases} \lambda_0|\eta-\eta'|^{p(x)} & p(x) \geq 2, \\ \lambda_0(|\eta|^2+|\eta'|^2)^{[p(x)-2]/2}|\eta-\eta'|^2 & p(x) < 2, \end{cases}$$

*where $\lambda_0$ is a constant depending only upon $N$, $p_-$, $p_+$, $\lambda$ and $\Lambda$.*

As the first step of proving the boundary $C^{1,\alpha}$ estimates, we translate problem (6.1) into a new problem, which is defined on hemisphere $B_r^+(0)$ and is equipped with a structure similar to (A2) by using condition (H3).

For this purpose, we take arbitrary $x_0 \in \partial\Omega$, and without loss of generality, via translation transformation, we can assume that $x_0 = 0$. On account of (H3), positive constants $r_0$, $c_0$, $\alpha_0 \in (0,1)$ and function $h \in C^{1,\alpha_0}(\mathbf{R}^{N-1})$ exist with $h(0) = 0$, $\nabla h(0) = 0$ and $\|h\|_{C^{1,\alpha_0}} \leq c_0$, such that $\Omega_{r_0}(0) := \Omega \cap B_{r_0}(x_0) = \Omega \cap B_{r_0}(0)$ can be represented as

$$\left\{ y \in \mathbf{R}^N \mid h(\widehat{y}) < y^N < \sqrt{r_0^2 - |\widehat{y}|^2} \right\},$$

under some rectangular coordinates systems in $\mathbf{R}^N$ centered at 0 in a basis $f_1, \dots, f_N$, which may be different from the original basis $e_1, \dots, e_N$, where $y^i$, $i = 1, \dots, N$ are the coordinates corresponding to $f_1, \dots, f_N$. Noticing that (H3) still holds if we replace $r_0$ by any positive constant $r_0' \leq r_0$, recalling that $\nabla h(0) = 0$, so without loss of generality, we can assume that $r_0$ is small enough such that

$$(7.2)\qquad |\nabla h(y^1, \dots, y^{N-1})| < \frac{1}{2}, \qquad (y^1, \dots, y^{N-1}) \in B_{r_0}(0),$$

where $B_{r_0}$ is a ball in $\mathbf{R}^{N-1}$. For any point $P \in \mathbf{R}^N$, we denote by $(x^1, \dots, x^N)$ and $(y^1, \dots, y^N)$ the coordinates of point $P$ in the rectangular coordinate systems centered at 0 in the bases $e_1, \dots, e_N$ and $f_1, \dots, f_N$, respectively. Then the following holds:

$$(y^1, \dots, y^N)^T = K(x^1, \dots, x^N)^T, \tag{7.3}$$

where $K = (f_i e_j)_{N\times N}$. It's easy to see that $K$ is a Hermite matrix, namely

$$KK^T = K^T K = I, \tag{7.4}$$

where $I$ is the unit matrix of order $N \times N$. We denote $U_{r_0} = K(\Omega_{r_0}(0))$, then

$$U_{r_0} = \Big\{(y^1, \dots, y^N) \mid h(y^1, \dots, y^{N-1}) < y^N < \sqrt{r_0^2 - ((y^1)^2 + \cdots + (y^{N-1})^2)}\Big\}.$$

Define a mapping $\Phi_0 : U_{r_0} \to \Phi_0(U_{r_0})$, $y \mapsto z = \Phi_0(y)$ as follows

$$z^i = y^i, \quad i = 1, \dots, N-1, \qquad z^N = y^N - h(y^1, \cdots, y^{N-1}). \tag{7.5}$$

Denote $V_0 := \Phi_0(U_{r_0})$, then

$$V_0 = \Big\{(z^1, \dots, z^N) \mid 0 < z^N < \sqrt{r_0^2 - ((z^1)^2 + \cdots + (z^{N-1})^2)} - h(z^1, \dots, z^{N-1})\Big\}. \tag{7.6}$$

Obviously $\Phi_0$ is a reversible mapping, and its inverse mapping is denoted by $\Psi_0$. Then for any $z \in V_0$, $y = \Psi_0(z)$ can be represented as

$$y^i = z^i, \quad i = 1, \dots, N-1, \qquad y^N = z^N + h(z^1, \dots, z^{N-1}). \tag{7.7}$$

Define a mapping $\Phi : \Omega_{r_0} \to V_0$ such that

$$\Phi(x) = \Phi_0(Kx), \quad x \in \Omega_{r_0}, \tag{7.8}$$

and its inverse mapping is denoted by $\Psi$. Then by (7.4), one obtains

$$\Psi : V_0 \longrightarrow \Omega_{r_0}, \qquad \Psi(z) = K^T(\Psi_0(z)), \quad z \in V_0. \tag{7.9}$$

For any $(z, v, \eta) \in V_0 \times [-M, M] \times \mathbf{R}^N$, define

$$\widetilde{A}(z, v, \eta) = \Phi'(\Psi(z))A(\Psi(z), v, \Phi'(\Psi(z))^T\eta), \tag{7.10}$$

$$\widetilde{B}(z, v, \eta) = B(\Psi(z), v, \Phi'(\Psi(z))^T\eta). \tag{7.11}$$

On account of the assumptions on $A(x, u, \eta)$ and $B(x, u, \eta)$, we can infer that $\widetilde{A}$ and $\widetilde{B}$ have the same continuity as $A$ and $B$, respectively. For any $(z, v) \in V_0 \times [-M, M]$ and $\eta \in \mathbf{R}^N \setminus \{0\}$, we denote

$$\widetilde{A}_\eta(z, v, \eta) := \left(\frac{\partial \widetilde{A}_i}{\partial \eta^j}(z, v, \eta)\right)_{N\times N}.$$

By (A2), we can see that $\widetilde{A}(z, v, \eta)$ and $\widetilde{B}(z, v, \eta)$ satisfy assumption (A2) with the constants $\lambda$ and $\Lambda$ being replaced by some other constants. In fact, we have the following lemma.

**Lemma 7.2.** *Let mappings $K$, $\Phi_0$, $\Psi_0$, $\Phi$ and $\Psi$ be defined by* (7.3), (7.5), (7.7)–(7.9), *respectively. Assume $\widetilde{A}(z, v, \eta)$ and $\widetilde{B}(z, v, \eta)$ are given by* (7.10) *and* (7.11), *respectively. Define $\widetilde{p}(z) = p(\Psi(z))$ for any $z \in V_0$. Then positive constants $\lambda_1$, $\Lambda_1$ and $L_1$ exist depending only upon $\lambda$, $\Lambda$, $\lambda^*$, $\Lambda^*$, $c_0$, $\alpha_0$, $L_0$ and $p_+$, such that for any $z_1$, $z_2$, $z \in V_0$, $v_1, v_2, v \in [-M, M]$, $\eta \in \mathbf{R}^N \setminus \{0\}$ and $\xi \in \mathbf{R}^N$, the following hold:*

$$\xi^T \widetilde{A}_\eta(z, v, \eta)\xi \ge \lambda_1 |\eta|^{\tilde{p}(z)-2}|\xi|^2, \tag{7.12}$$

$$\widetilde{A}(z, v, \eta)\eta \ge \lambda_1 |\eta|^{\tilde{p}(z)}, \tag{7.13}$$

$$|\widetilde{A}_\eta(z, \eta)| \le \Lambda_1 |\eta|^{\tilde{p}(z)-2}, \tag{7.14}$$

$$|\widetilde{A}(z, v, \eta)| \le \Lambda_1 |\eta|^{\tilde{p}(z)-1}, \tag{7.15}$$

$$|\widetilde{B}(z, v, \eta)| \le \Lambda_1 (1 + |\eta|^{\tilde{p}(z)}), \tag{7.16}$$

(7.17)
$$|\widetilde{A}(z_1, v_1, \eta) - \widetilde{A}(z_2, v_2, \eta)| \le \Lambda_1(|z_1 - z_2|^{\alpha_0} + |v_1 - v_2|^{\alpha_0}) \cdot (|\eta|^{\tilde{p}(z_1)-1} + |\eta|^{\tilde{p}(z_2)-1}),$$

(7.18)
$$|\widetilde{p}(z_1) - \widetilde{p}(z_2)| \le L_1 |z_1 - z_2|^{\alpha_0}, \quad 1 < p_- \le \widetilde{p}(z) \le p_+ < \infty,$$

*where $\eta$ can be equal to* 0 *in* (7.13) *and* (7.15)–(7.17).

*Proof.* A direct calculation shows that

$$\widetilde{A}_\eta(z,v,\eta) = \Phi'(\Psi(z))A_\eta(\Psi(z),v,\Phi'(\Psi(z))^T\eta)\Phi'(\Psi(z))^T, \tag{7.19}$$

$$\Phi'(\Psi(z)) = \Phi_0'(K\Psi(z))K. \tag{7.20}$$

By the aid of the definition of $\Phi_0$, it follows that

$$\Phi_0'(y) = \begin{pmatrix} I_{N-1} & 0 \\ -\nabla h & 1 \end{pmatrix},$$
$$\nabla h = \left(\frac{\partial h(y^1,\dots,y^{N-1})}{\partial y^1},\dots,\frac{\partial h(y^1,\dots,y^{N-1})}{\partial y^{N-1}}\right),$$

for any $y=(y^1,\dots,y^N)\in U_{r_0}$. For any $y\in U_{r_0}$ and $\eta\in\mathbf{R}^N$, one has

$$\Phi_0'(y)\eta = \eta-(0,\dots,0,\nabla h\widehat{\eta})^T, \qquad \Phi_0'(y)^T\eta = \eta-\eta^N(\nabla h,0)^T,$$

where $\widehat{\eta}=(\eta^1,\dots,\eta^{N-1})$, and consequently it follows from (7.2) and (7.20) that

$$\frac{1}{2}|\eta| \le |\Phi'(\Psi(z))\eta| \le \frac{3}{2}|\eta|, \qquad \frac{1}{2}|\eta| \le |\Phi'(\Psi(z))^T\eta| \le \frac{3}{2}|\eta|, \tag{7.21}$$

for any $z\in V_0$. Here we used the fact that $|K\eta|=|\eta|$ for any $\eta\in\mathbf{R}^N$. For any $\xi\in\mathbf{R}^N$, $(z,v)\in V_0\times[-M,M]$ and $\eta\in\mathbf{R}^N\setminus\{0\}$, it follows from (A2), (A2′), (7.19) and (7.21) that

$$\begin{aligned}\xi^T\widetilde{A}_\eta(z,v,\eta)\xi &= (\Phi'(\Psi(z))^T\xi)^T A_\eta(\Psi(z),v,\Phi'(\Psi(z))^T\eta)(\Phi'(\Psi(z))^T\xi)\\ &\ge \lambda|\Phi'(\Psi(z))^T\eta|^{p(\Psi(z))-2}|\Phi'(\Psi(z))^T\xi|^2\\ &\ge' \lambda\left|\frac{1}{2}\eta\right|^{p(\Psi(z))-2}\left|\frac{1}{2}\xi\right|^2 \ge 2^{-p_+}\lambda|\eta|^{\tilde{p}(z)-2}|\xi|^2,\end{aligned}$$

and

$$\begin{aligned}\widetilde{A}(z,v,\eta)\eta &= A(\Psi(z),v,\Phi'(\Psi(z))^T\eta)(\Phi'(\Psi(z))^T\eta)\\ &\ge \lambda^*|\Phi'(\Psi(z))^T\eta|^{p(\Psi(z))} \ge 2^{-p_+}\lambda^*|\eta|^{\tilde{p}(z)}.\end{aligned}$$

And thus (7.12) and (7.13) hold, provided $\lambda_1\le 2^{-p_+}\min\{\lambda,\lambda^*\}$.

For any $(z,v) \in V_0 \times [-M,M]$ and $\eta \in \mathbf{R}^N \setminus \{0\}$, by (A2), (7.19) and (7.21), one has

$$\begin{aligned}|\widetilde{A}_\eta(z,v,\eta)| &= |\Phi'(\Psi(z))A_\eta(\Psi(z),v,\Phi'(\Psi(z))^T\eta)\Phi'(\Psi(z))^T| \\ &\le \frac{3}{2}|A_\eta(\Psi(z),v,\Phi'(\Psi(z))^T\eta)\Phi'(\Psi(z))^T| \\ &= \frac{3}{2}|\Phi'(\Psi(z))A_\eta(\Psi(z),v,\Phi'\Psi(z))^T\eta)^T| \\ &\le \frac{9}{4}|A_\eta(\Psi(z),v,\Phi'(\Psi(z))^T\eta)| \\ &\le \frac{9}{4}\Lambda\left|\frac{3}{2}\eta\right|^{p(\Psi(z))-2} \le \left(\frac{3}{2}\right)^{p_+}\Lambda|\eta|^{\tilde{p}(z)-2},\end{aligned}$$

which implies (7.14) provided $\Lambda_1 \ge (3/2)^{p_+}\Lambda$.

For any $(z,v,\eta) \in V_0 \times [-M,M] \times \mathbf{R}^N$, combining (A2$'$) with (7.21), then we have

$$\begin{aligned}|\widetilde{B}(z,v,\eta)| &= |B(\Psi(z),v,\Phi'(\Psi(z))^T\eta)| \\ &\le \Lambda\left(1 + \left|\frac{3}{2}\eta\right|^{p(\Psi(z))}\right) \\ &\le \left(\frac{3}{2}\right)^{p_+}\Lambda\left(1 + |\eta|^{\tilde{p}(z)}\right),\end{aligned}$$

and

$$\begin{aligned}|\widetilde{A}(z,v,\eta)| &= |\Phi'(\Psi(z))A(\Psi(z),v,\Phi'(\Psi(z))^T\eta)| \\ &\le \frac{3}{2}|A(\Psi(z),v,\Phi'(\Psi(z))^T\eta) \\ &\le \frac{3}{2}\Lambda^*\left|\frac{3}{2}\eta\right|^{p(\Psi(z))-1} \\ &\le \left(\frac{3}{2}\right)^{p_+}\Lambda^*|\eta|^{\tilde{p}(z)-1}.\end{aligned}$$

Therefore, (7.15) and (7.16) hold provided $\Lambda_1 \ge \max\{(3/2)^{p_+}\Lambda, (3/2)^{p_+}\Lambda^*\}$.

Let $z_1, z_2 \in V_0$, $v_1, v_2 \in [-M,M]$ and $\eta \in \mathbf{R}^N$. We denote $x_i = \Psi(z_i)$

and $y_i = Kx_i$, $i = 1, 2$. Recalling that $\|h\|_{C^{1,\alpha_0}} \le c_0$, we have

$$\begin{aligned}|\Phi'(x_1) - \Phi'(x_2)| &= |(\Phi_0'(y_1) - \Phi_0'(y_2))K| \\ &= |(\Phi_0'(y_1) - \Phi_0'(y_2))| \\ &= |\nabla h(\widehat{y_1}) - \nabla h(\widehat{y_2})| \\ &\le c_0|\widehat{y_1} - \widehat{y_2}|^{\alpha_0} \\ &\le c_0|y_1 - y_2|^{\alpha_0} \\ &= c_0|x_1 - x_2|^{\alpha_0},\end{aligned}$$

and

$$\begin{aligned}|x_1 - x_2| &= |\Psi(z_1) - \Psi(z_2)| \\ &= |K^T(\Psi_0(z_1) - \Psi_0(z_2))| \\ &= |\Psi_0(z_1) - \Psi_0(z_2)| \\ &\le |z_1 - z_2| + |h(\widehat{z_1}) - h(\widehat{z_2})| \\ &\le (c_0 + 1)|z_1 - z_2|,\end{aligned} \tag{7.22}$$

where $\widehat{\eta} = (\eta^1, \dots, \eta^{N-1})^T$ for any $\eta \in \mathbf{R}^N$, and consequently, by (A1), (A2), (7.21) and the mean value theorem, we conclude

$$\begin{aligned}|\widetilde{p}(z_1) - \widetilde{p}(z_2)| &= |p(\Psi(z_1)) - p(\Psi(z_2))| \\ &\le L_0|\Psi(z_1) - \Psi(z_2)|^{\alpha_0} \\ &\le L_0(c_0 + 1)^{\alpha_0}|z_1 - z_2|^{\alpha_0},\end{aligned}$$

and

$$\begin{aligned}&|\widetilde{A}(z_1,v_1,\eta) - \widetilde{A}(z_2,v_2,\eta)| \\ &\quad= |\Phi'(x_1)A(x_1,v_1,\Phi'(x_1)^T\eta) - \Phi'(x_2)A(x_2,v_2,\Phi'(x_2)^T\eta)| \\ &\quad\le \big|\Phi'(x_1)\big(A(x_1,v_1,\Phi'(x_1)^T\eta) - A(x_1,v_1,\Phi'(x_2)^T\eta)\big)\big| \\ &\qquad + \big|\Phi'(x_1)\big(A(x_1,v_1,\Phi'(x_2)^T\eta) - A(x_2,v_2,\Phi'(x_2)^T\eta)\big)\big| \\ &\qquad + \big|\big(\Phi'(x_1) - \Phi'(x_2)\big)A(x_2,v_2,\Phi'(x_2)^T\eta)| \\ &\quad\le \frac{3}{2}c_0|x_1 - x_2|^{\alpha_0}\big|A_\eta\big(x_1,v_1,(\theta\Phi'(x_1) + (1-\theta)\Phi'(x_2))\eta\big)\big||\eta| \\ &\qquad + \frac{3}{2}\Lambda(|x_1 - x_2|^{\alpha_0} + |v_1 - v_2|^{\alpha_0}) \\ &\qquad \cdot \left(\left|\frac{3}{2}\eta\right|^{p(x_1)-2} + \left|\frac{3}{2}\eta\right|^{p(x_2)-1}\right)\left|\frac{3}{2}\eta\right|\end{aligned}$$

$$
\begin{aligned}
&+ c_0\Lambda|x_1-x_2|^{\alpha_0}\left|\frac{3}{2}\eta\right|^{p(x_2)-2}|\eta| \\
&\le 3\left(\frac{3}{2}\right)^{p_+}(c_0+1)\Lambda(|x_1-x_2|^{\alpha_0}+|v_1-v_2|^{\alpha_0}) \\
&\quad\cdot\left(|\eta|^{p(x_1)-2}+|\eta|^{p(x_2)-1}\right)|\eta| \\
&\le 3\left(\frac{3}{2}\right)^{p_+}(c_0+1)^{1+\alpha_0}\Lambda(|z_1-z_2|^{\alpha_0}+|v_1-v_2|^{\alpha_0}) \\
&\quad\cdot\left(|\eta|^{\tilde{p}(z_1)-2}+|\eta|^{\tilde{p}(z_2)-1}\right)|\eta|.
\end{aligned}
$$

Thus (7.17) and (7.18) are obtained if we take $\Lambda_1 \ge 3(3/2)^{p_+}(c_0+1)^{1+\alpha_0}$ and $L_1 = L_0(1+c_0)^{\alpha_0}$.

Combining the previous proof and taking

$$
\begin{aligned}
L_1 &= L_0(1+c_0)^{\alpha_0}, \\
\lambda_1 &= 2^{-p_+}\min\{\lambda,\lambda^*\}, \\
\Lambda_1 &= \left(\frac{3}{2}\right)^{p_+}\max\{\Lambda,\Lambda^*,3(c_0+1)^{1+\alpha_0}\},
\end{aligned}
$$

then (7.12)–(7.18) hold, and the proof is complete. $\square$

**Lemma 7.3.** *Let $V_0$ be given by* (7.6). *Then a positive constant $r_1 \le 1$ exists, such that*

$$B^+_{r_1}(0) := \left\{(z^1,\dots,z^N) \mid \sum_{i=1}^N (z^i)^2 < r_1^2, z^N > 0\right\} \subseteq V_0.$$

*Proof.* For any $z=(z^1,\dots,z^N)^T \in \mathbf{R}^N$, we denote $\widehat{z}=(z^1,\dots,z^{N-1})^T \in \mathbf{R}^{N-1}$. For any $z \in \mathbf{R}^N$ with $|\widehat{z}| < 2/\sqrt{13}r_0$, recalling that $h(0)=0$, we infer from (7.2) that

$$\sqrt{r_0^2-|\widehat{z}|^2}-h(\widehat{z}) > \frac{3}{\sqrt{13}}r_0 - |h(\widehat{z})-h(0)| \ge \frac{3}{\sqrt{13}}r_0 - \frac{1}{2}|\widehat{z}| > \frac{2}{\sqrt{13}}r_0.$$

For any $z \in B^+_{2/\sqrt{13}r_0}(0)$, it is obvious that $|\widehat{z}| < 2/\sqrt{13}r_0$ and $z^N < 2/\sqrt{13}r_0$, and consequently, it follows from the above inequality that

$$0 < z^N < \frac{2}{\sqrt{13}}r_0 < \sqrt{r_0^2-|\widehat{z}|^2}-h(\widehat{z}),$$

which implies that (recalling (7.6))

$$B^+_{2/\sqrt{13}r_0}(0) \subseteq \left\{z \in \mathbf{R}^N \mid 0 < z^N < \sqrt{r_0^2 - |\widehat{z}|^2} - h(\widehat{z}), |\widehat{z}| < \frac{2}{\sqrt{13}} r_0 \right\} \subseteq V_0.$$

Let $r_1 = 2/\sqrt{13}r_0$. The proof is complete. □

For any bounded weak solution $u$ of problem (6.1), we define a new function

$$v(z) = u(\Psi(z)), \quad z \in V_0. \tag{7.23}$$

Then, it's easy to see that $v$ is a bounded weak solution for the problem

$$-\mathrm{div}\widetilde{A}(z, v, \nabla v) = \widetilde{B}(z, v, \nabla v), \quad z \in V_0. \tag{7.24}$$

Moreover, by Proposition 6.2, it follows from (7.22) that

$$|v(z_1) - v(z_2)| \leq c^*|\Psi(z_1) - \Psi(z_2)|^{\alpha^*} \leq c^*(1 + c_0)^{\alpha^*}|z_1 - z_2|^{\alpha^*}, \tag{7.25}$$

for any $z_1, z_2 \in V_0$, such that $|z_1 - z_2| \leq R^*/(c_0 + 1)$.

By the definition of $v$, we can see that one can firstly obtain the estimate on $v$ to derive the boundary $C^{1,\alpha}$ estimates on $u$. For this purpose, we use a similar argument used in [**5, 16**]. Let $r_1$ be the constant in Lemma 7.3. For any $z_0 \in \overline{B^+_{r_1/2}(0)}$ and $0 < R < r_1/2$, select $z_0^* \in \overline{B_R(z_0) \cap B^+_{r_1}(0)}$, such that

$$\widetilde{p}(z_0^*) = p_+(z_0; R) = \max\left\{\widetilde{p}(z) \mid z \in \overline{B_R(z_0) \cap B^+_{r_1}(0)}\right\},$$

and define

$$\overline{A}(\eta) = \widetilde{A}(z_0^*, v(z_0^*), \eta), \quad \eta \in \mathbf{R}^N.$$

We introduce two auxiliary functions $w_1$ and $w_2$ as follows: if $B_{2R}(z_0) \subseteq B^+_{r_1}(0)$, we consider the boundary value problem

$$\begin{cases} -\mathrm{div}\,\overline{A}(\nabla w_1) = 0 & z \in B_R(z_0), \\ w_1(z) = v(z) & z \in \partial B_R(z_0), \end{cases} \tag{7.26}$$

and if $z_0 \in B^0_{r_1/2}(0) := B_{r_1/2}(0) \cap \{z \in \mathbf{R}^N \mid z^N = 0\}$, we consider the boundary value problem

$$\begin{cases} -\mathrm{div}\,\overline{A}(\nabla w_2) = 0 & z \in B^+_R(z_0), \\ w_2(z) = v(z) & z \in \partial B^+_R(z_0). \end{cases} \tag{7.27}$$

Then we have the following two lemmas, which state the properties of $w_1$ and $w_2$. As we will see later, these properties will be frequently mentioned to study the properties of $v$.

**Lemma 7.4.** *There is a unique solution* $w_1 \in W^{1,p_+(z_0;R)}(B_R(z_0)) \cap L^\infty(B_R(z_0))$ *of problem* (7.26), *such that*

(7.28)
$$\sup_{B_{R/2}(z_0)} |\nabla w_1|^{p_+(z_0;R)} \le CR^{-N} \int_{B_R(z_0)} |\nabla w_1|^{p_+(z_0;R)} dz,$$

(7.29)
$$\operatorname{osc}\{\nabla w_1; B_r(z_0)\} \le C\left(\frac{r}{\rho}\right)^\sigma \operatorname{osc}\{\nabla w_1; B_\rho(z_0)\}, \ 0 < r < \rho \le R,$$

(7.30)
$$\int_{B_R(z_0)} |\nabla w_1|^{p_+(z_0;R)} dz \le C \int_{B_R(z_0)} (1 + |\nabla v|^{p_+(z_0;R)})\, dz,$$

(7.31)
$$\sup_{B_R(z_0)} |w_1 - v| \le \operatorname{osc}\{v; B_R(z_0)\},$$

*where* $\sigma \in (0,1)$ *and* $C$ *are positive constants depending only upon* $N$, $\lambda_1$, $\Lambda_1$ *and* $p_+$.

*Proof.* The existence and uniqueness of the solution of problem (7.26) can be obtained by using the standard argument on strongly monotonic functionals. The strongly monotonic functional considered here is given by $T : W_0^{1,p_+(z_0;R)}(B_R(z_0)) \to W^{-1,(p_+(z_0;R))'}(B_R(z_0))$,

$$T(w)(\eta) := \int_{B_{R(x_0)}} \overline{A}(\nabla w + \nabla v)\nabla\eta\, dx, \quad \eta, w \in W_0^{1,p_+(z_0;R)}(B_R(z_0)),$$

where $(p_+(z_0;R))' = p_+(z_0;R)/[p_+(z_0;R) - 1]$. Let $w_1$ be the unique weak solution of problem (7.26). Then, by the weak comparison principle, we infer that $-\|v\| \le w_1(x) \le \|v\|$ for any $x \in B_R(z_0)$, and thus $w_1 \in L^\infty(B_R(z_0))$. Set

$$\overline{A}_\varepsilon(\eta) = \overline{A}(\eta) + \varepsilon\eta, \quad \eta \in \mathbf{R}^N.$$

Let $v_\varepsilon$ be the unique weak solution of the problem

$$\begin{cases} -\operatorname{div} \overline{A}_\varepsilon(\nabla w_\varepsilon) = 0 & x \in B_R(z_0), \\ w_\varepsilon(x) = v(x) & x \in \partial B_R(z_0). \end{cases}$$

By using the global $C^{1,\alpha}$ estimates in [**16**], positive constants $\alpha \in (0,1)$ and $C$ exist such that

$$\|w_\varepsilon\|_{C^{1,\alpha}(\overline{B_R(z_0)})} \le C.$$

By utilizing the Arzela-Ascoli theorem, a function $w_0 \in C^{1,\alpha/2}(\overline{B_R(z_0)})$ and a subsequence of $\{v_\varepsilon\}$ exist such that

$$w_\varepsilon \longrightarrow w_0, \quad \text{in } C^{1,\alpha/2}(\overline{B_R(z_0)}).$$

One can easily conclude that $w_0$ is a weak solution of problem (7.26). Thus, it follows from the uniqueness of solutions for problem (7.26) that $w_1 = w_0$.

Set

$$F_\varepsilon(t) = \lambda_1 t^{p_+(z_0;R)-2} + \varepsilon, \quad t > 0.$$

Then, for any $0 < t \le s$,

$$F_\varepsilon(t) \ge \min\{4^{2-p_+}, 1\} F_\varepsilon(4t), \quad F_\varepsilon(t)t \le F_\varepsilon(s)s, \quad F_\varepsilon(t) \ge \varepsilon. \tag{7.32}$$

Let $\overline{\Lambda} = \Lambda_1/\lambda_1$. By (7.12), (7.14) and (7.15), the following hold:

$$\begin{gathered} \xi^T \overline{A}_{\varepsilon\eta}(\eta)\xi \ge F_\varepsilon(|\eta|)|\xi|^2, \qquad |\overline{A}_{\varepsilon\eta}(\eta)| \le \overline{\Lambda} F_\varepsilon(|\eta|), \\ |\overline{A}_\varepsilon(\eta)| \le \overline{\Lambda} F_\varepsilon(|\eta|)|\eta|, \end{gathered} \tag{7.33}$$

for all $\eta \in \mathbf{R}^N \setminus \{0\}$ and $\xi \in \mathbf{R}^N$. Combining (7.32) with (7.33), we can apply Lemma 1 in [**16**] to obtain that positive constants $\sigma \in (0,1)$ and $C$ exist depending only upon $N$, $p_+$ and $\overline{\Lambda}$, such that

$$\operatorname{osc}\{\nabla w_\varepsilon; B_r(z_0)\} \le C\left(\frac{r}{\rho}\right)^\sigma \operatorname{osc}\{\nabla w_\varepsilon; B_\rho(z_0)\}, \ 0 < r < \rho \le R,$$

$$\sup_{B_{R/2}(z_0)} |\nabla w_\varepsilon|^2 F_\varepsilon(|\nabla w_\varepsilon|) \le C R^{-N} \int_{B_R(z_0)} F_\varepsilon(|\nabla w_\varepsilon|)|\nabla w_\varepsilon|^2 dx.$$

Letting $\varepsilon \to 0$ in the above two inequalities, and recalling that $w_\varepsilon \to w_1$ in $C^{1,\alpha/2}(\overline{B_R(z_0)})$, we obtain (7.28) and (7.29). Taking $w_1 - v$ as a test function, by simply calculating, we deduce (7.30), while (7.31) can easily be deduced by the weak maximum principle for (7.26). Thus, the proof is complete. $\square$

*Remark* 7.1. Lemma 7.4 has been essentially proved in [**16**], where the constant $C$ also depends upon $p_+(z_0;R)$. Here we show that the constant $C$ can be taken independent of $p_+(z_0;R)$, but of $p_+$ the super bound of $p(x)$. This fact is necessary for us to obtain the interior or global $C^{1,\alpha}$ estimates.

**Lemma 7.5.** *Problem* (7.27) *has a unique solution* $w_2 \in W^{1,p_+(z_0;R)}(B_R^+(z_0)) \cap L^\infty(B_R^+(z_0))$ *such that*

(7.34)
$$\operatorname{osc}\left\{\nabla w_2; B_r^+(z_0)\right\} \le C\left(\frac{r}{R}\right)^\sigma \sup_{B_R^+(z_0)} |\nabla w_2|, \quad 0 < r \le R,$$

(7.35)
$$\sup_{B_{R/2}^+(z_0)} |\nabla w_2| \le C\left(\frac{1}{R^N}\int_{B_R^+(z_0)} |\nabla w_2|^{p_+(z_0;R)} dz\right)^{1/p_+(z_0;R)},$$

(7.36)
$$\int_{B_R^+(z_0)} |\nabla w_2|^{p_+(z_0;R)} dz \le C\int_{B_R^+(z_0)} (1 + |\nabla v|^{p_+(z_0;R)})\, dz,$$

(7.37)
$$\sup_{B_R^+(z_0)} |w_2 - v| \le \operatorname{osc}\left\{v; B_R^+(z_0)\right\},$$

*where* $C$ *and* $\sigma \in (0,1)$ *are positive constants depending only upon* $N$*,* $\Lambda_1$*,* $\lambda_1$ *and* $p_+$*.*

*Proof.* The proof of the existence and uniqueness of the solution of problem (7.27) is similar to that in Lemma 7.4, thus we omit it here. For $\varepsilon > 0$, set

$$\overline{A}_\varepsilon(\eta) = \overline{A}(\eta) + \varepsilon\eta, \quad \eta \in \mathbf{R}^N.$$

Let $w_\varepsilon$ be the unique solution of the boundary value problem

$$\begin{cases} -\operatorname{div} \overline{A}_\varepsilon(\nabla w_\varepsilon) = 0 & z \in B_R^+(z_0), \\ w_\varepsilon(z) = v(z) & z \in \partial B_R^+(z_0). \end{cases}$$

Using the same argument in Lemma 7.4, it follows from the global $C^{1,\alpha}$ estimates in [**16**], the uniqueness of the solutions for problem (7.27) and the Arzela-Ascoli theorem that a subsequence of $\{w_\varepsilon\}$ exists such that

$$w_\varepsilon \longrightarrow w_2, \quad \text{in } C^{1,\alpha/2}(\overline{B_R^+(z_0)}).$$

Define a function $F_\varepsilon(t) = \lambda_1 t^{p_+(z_0;R)-2} + \varepsilon$ for any $t > 0$, and set $\overline{\Lambda} = \Lambda_1/\lambda_1$, where $\lambda_1$ and $\Lambda_1$ are the positive constants in Lemma 7.2. By the aid of (7.12)–(7.15), one can verify that

$$\begin{gathered} \xi^T \overline{A}_{\varepsilon\eta}(\eta)\xi \geq F^\varepsilon(|\eta|)|\xi|^2, \qquad |\overline{A}_{\varepsilon\eta}(\eta)| \leq \overline{\Lambda} F_\varepsilon(|\eta|), \\ |\overline{A}_\varepsilon(\eta)| \leq \overline{\Lambda}|\eta| F_\varepsilon(|\eta|), \end{gathered} \tag{7.38}$$

for any $\eta \in \mathbf{R}^N \setminus \{0\}$ and $\xi \in \mathbf{R}^N$. In addition,

$$F_\varepsilon \geq \min\{4^{2-p_+}, 1\} F_\varepsilon(4t), \qquad F_\varepsilon(t)t \leq F_\varepsilon(s)s, \qquad F_\varepsilon(t) \geq \varepsilon, \tag{7.39}$$

for any $0 < t \leq s$. Combining (7.38) with (7.39), we can use Lemma 2 and Lemma 4 in [**16**] to deduce that positive constants $\sigma$ and $C$ exist depending upon $N$, $p_+$ and $\overline{\Lambda}$, such that

$$\sup_{B_{R/2}^+(z_0)} |\nabla w_\varepsilon| \leq C \operatorname{osc}\left\{\frac{w_\varepsilon}{R}; B_{3R/4}^+(z_0)\right\}$$

and

$$\operatorname{osc}\left\{\nabla w_\varepsilon; B_r^+(z_0)\right\} \leq C\left(\frac{r}{R}\right)^\sigma \sup_{B_R^+(z_0)} |\nabla w_\varepsilon|, \quad 0 < r \leq R,$$

Letting $\varepsilon \to 0$ in the above inequalities, then

$$\sup_{B_{R/2}^+(z_0)} |\nabla w_2| \leq C \operatorname{osc}\left\{\frac{w_2}{R}; B_{3R/4}^+(z_0)\right\}, \tag{7.40}$$

and

$$\operatorname{osc}\left\{\nabla w_2; B_r^+(z_0)\right\} \le C\left(\frac{r}{R}\right)^{\sigma} \sup_{B_R^+(z_0)} |\nabla w_2|,$$

and thus (7.34) holds. Recalling that $v(z) = 0$ on $B_R^0(z_0)$ and applying the local maximum principle (see Corollary 1.1 in [**22**]) to $w_2$, a positive constant $C$ exists depending only upon $N$, $\lambda_1$, $\Lambda_1$ and $p_+$, such that

$$\operatorname{osc}\left\{w_2; B_{3R/4}^+(z_0)\right\} \le C\left(\frac{1}{R^N}\int_{B_R^+(z_0)} |w_2|^{p_+(z_0;R)}dx\right)^{1/p_+(z_0;R)},$$

which implies that

$$\operatorname{osc}\left\{w_2; B_{3R/4}^+(z_0)\right\} \le CR\left(\frac{1}{R^N}\int_{B_R^+(z_0)} |\nabla w_2|^{p_+(z_0;R)}dx\right)^{1/p_+(z_0;R)}$$

by Poincaré's inequality. Combining (7.40) with the above inequality, we obtain (7.35). Taking $w_2 - v$ as a test function, by simply calculating, we deduce (7.36), while (7.37) can easily be deduced by the weak comparison principle for (7.27). Thus, the proof is complete. $\square$

To study further the properties of $v$, we also need higher integrability on $v$ stated in the following lemma which will be frequently used later.

**Lemma 7.6.** *Let $v$ be defined by* (7.23). *Then, positive constants $\widehat{R} \le r_1$, $\widehat{c}$ and $\widehat{\delta}$ exist depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $c_0$, $\alpha_0$, $\alpha^*$ and $c^*$, such that*

(i) *For any concentric balls $B_{2R}$ and $B_R$ contained in $B_{r_1}^+(0)$, one has*

(7.41)
$$\left(\frac{1}{R^N}\int_{B_R} |\nabla v|^{\tilde{p}(z)(1+\delta)}dz\right)^{1/(1+\delta)} \le \widehat{c}\left(1 + \frac{1}{R^N}\int_{B_{2R}} |\nabla v|^{\tilde{p}(z)}dz\right)$$

*and*

$$\int_{B_{2R}} |\nabla v|^{\tilde{p}(z)}dz \le 1,$$

*provided $R \in (0, \widehat{R}]$ and $\delta \in (0, \widehat{\delta}]$;*

(ii) *For any* $z^* \in B^0_{r_1}(0) := B_{r_1}(0) \cap \{z \in \mathbf{R}^N | z^N = 0\}$ *and* $R > 0$, *with* $B^+_{2R}(z) \subseteq B^+_{r_1}(0)$, *one obtains*

$$\left(\frac{1}{R^N}\int_{B^+_R(z^*)} |\nabla v|^{\tilde{p}(z)(1+\delta)} dz\right)^{1/(1+\delta)} \leq \widehat{c}\left(1 + \frac{1}{R^N}\int_{B^+_{2R}(z^*)} |\nabla v|^{\tilde{p}(z)} dz\right) \tag{7.42}$$

*and*

$$\int_{B^+_{2R}(z^*)} |\nabla v|^{\tilde{p}(z)} dz \leq 1,$$

*provided* $R \in (0, \widehat{R}]$ *and* $\delta \in (0, \widehat{\delta}]$.

*Proof.* Let $R^*$, $c^*$ and $\alpha^*$ be the constants in Proposition 6.2. We first prove part (i). Let $B_{2R}$ and $B_R$ be concentric balls contained in $B^+_{r_1}(0)$ with center $z^*$ and radius $R \leq R^*/2(1+c_0)$. Then, for any $z_1, z_2 \in B_{2R}$, it follows from (7.22) that $|\Psi(z_1) - \Psi(z_2)| \leq (1+c_0)|z_1 - z_2| \leq R^*$, and consequently by the definition of $v$ and Proposition 6.2, one obtains

$$\begin{aligned} |v(z_1) - v(z_2)| &= |u(\Psi(z_1)) - u(\Psi(z_2))| \\ &\leq c^*|\Psi(z_1) - \Psi(z_2)|^{\alpha^*} \\ &\leq c^*(c_0+1)^{\alpha^*}|z_1 - z_2|^{\alpha^*}, \end{aligned} \tag{7.43}$$

provided $|z_1 - z_2| \leq R^*/(1+c_0)$. Note that $v$ satisfies equation (7.24). Take $\xi \in C^\infty_0(B_{2R})$, such that $0 \leq \xi \leq 1$, $\xi = 1$ on $B_R$ and $|\nabla \xi| \leq 4/R$. Taking $\varphi = \xi^{p_+}(v - k)$ as a test function with $k = [1/|B_{2R}|]\int_{B_{2R}} v\,dx$, then by (7.13), (7.15) and (7.16), one has

$$\begin{aligned} \lambda_1 \int_{B_{2R}} |\nabla v|^{\tilde{p}(z)}\xi^{p_+} dz &\leq \Lambda_1 \int_{B_{2R}} (1 + |\nabla v|^{\tilde{p}(z)})|v - k|\xi^{p_+} dz \\ &\quad + p_+\Lambda_1 \int_{B_{2R}} \xi^{p_+ - 1}|v - k||\nabla v|^{\tilde{p}(z)-1}|\nabla \xi|\, dz. \end{aligned} \tag{7.44}$$

By the Young inequality, for any $0 < \varepsilon < 1$, recalling that $\widetilde{p}(z) > 1$, the following holds:

$$\xi^{p_+-1}|v-k|\nabla v|^{\tilde{p}(z)-1}|\nabla\xi| \leq \varepsilon\xi^{p_+}|\nabla v|^{\tilde{p}(z)} + \varepsilon^{-p_+}\xi^{p_+-\tilde{p}(z)}|v-k|^{\tilde{p}(z)}|\nabla\xi|^{\tilde{p}(z)}.$$

Taking $R \le \min\{[1/(c_0+1)](\lambda_1/4\Lambda_1 c^*)^{1/\alpha^*}, R^*/[2(1+c_0)]\}$, putting the above inequality with $\varepsilon = \lambda_1/(4p_+\Lambda_1)$ into (7.44), recalling that $|\nabla \xi| \le 4/R$ and (7.43), one obtains

$$\frac{\lambda_1}{2}\int_{B_{2R}} |\nabla v|^{\tilde{p}(z)}\xi^{p_+}dz \le \frac{\lambda_1}{4}|B_{2R}| + \lambda_1^{-p_+}(4p_+\Lambda_1)^{1+p_+}\int_{B_{2R}}\left|\frac{v-k}{R}\right|^{\tilde{p}(z)}dz,$$

or

$$\int_{B_{2R}} |\nabla v|^{\tilde{p}(z)}\xi^{p_+}dz \le \left(\frac{1}{2} + 2(4p_+)^{1+p_+}\right)\left(|B_{2R}| + \int_{B_{2R}}\left|\frac{v-k}{R}\right|^{\tilde{p}(z)}dz\right). \tag{7.45}$$

By applying Proposition 6.2 and (7.43), the above inequality implies that a positive constant $R_1 = R_1(N, p_+, c_0, c^*, \alpha^*) \le \min\{[1/(c_0+1)] (\lambda_1/4\Lambda_1 c^*)^{1/\alpha^*}, R^*/[2(1+c_0)]\}$ exists such that

$$\int_{B_{2R}} |\nabla v|^{\tilde{p}(z)}dz \le 1, \quad R \le R_1.$$

Hence, we can use Lemma 2.1 to conclude that positive constants $\sigma$, $R_2$ and $c_2$ exist depending only upon $N$, $L_0$, $\alpha_0$, $c_0$, $p_-$ and $p_+$ such that

$$\frac{1}{R^N}\int_{B_{2R}}\left|\frac{v-k}{R}\right|^{\tilde{p}(z)}dz \le c_2 + c_2\left(\frac{1}{R^N}\int_{B_{2R}}|\nabla v|^{\tilde{p}(z)/(1+\sigma)}dz\right)^{1+\sigma}, \tag{7.46}$$

for any $R \le R_2$. By utilizing (7.45) and (7.46), we deduce the Gehting type inequality

$$\frac{1}{R^N}\int_{B_R}|\nabla v|^{\tilde{p}(z)}dz \le c_3 + c_3\left(\frac{1}{R^N}\int_{B_{2R}}|\nabla v|^{\tilde{p}(z)/(1+\sigma)}dz\right)^{1+\sigma}, \tag{7.47}$$

for any $R \le \min\{R_1, R_2\}$, which implies (7.41) (see [**12**, Chapter 5, Proposition 1.1]).

Now we prove part (ii). Take arbitrary $z^* \in B^0_{r_1}(0)$ and $R \le R^*/(2(1+c_0))$, such that $B^+_{2R}(z^*) \subseteq B^+_{r_1}(0)$. Noticing that $v(z) = 0$ on $B^0_{r_1}(0)$, (7.43) yields $|v(z)| \le c^*(1+c_0)^{\alpha^*}(2R)^{\alpha^*}$ for any $z \in B^+_{2R}(z^*)$.

Taking $\varphi = \xi^{p_+} v$ as a test function and recalling the argument of the proof of part (i), we then have

$$\begin{aligned} \int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)} \xi^{p_+} dz \\ \leq \left(\frac{1}{2} + 2\,(4p_+)^{1+p_+}\right)\left(|B_{2R}^+(z^*)| + \int_{B_{2R}^+(z^*)} \left|\frac{v}{R}\right|^{\tilde{p}(z)} dz\right), \end{aligned} \tag{7.48}$$

and

$$\int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)} dx \leq 1,$$

for any $R \leq R_1$. For convenience, we temporarily denote $p_-^*$ and $p_+^*$ the minimum and the maximum values of $\tilde{p}(z)$ on $\overline{B_{2R}^+(z^*)}$, respectively. Without loss of generality, we suppose that $R_1$ is small enough, such that $\delta := (p_+^* p_-^* - N(p_+^* - p_-^*))/(Np_+^*) > 1 + \sigma$ with some positive constant $\sigma$ depending only upon $N$, $p_-$ and $p_+$. Then it is easy to conclude that

$$\frac{Np_+^*}{N + p_+^*} \leq \frac{\tilde{p}(z)}{1+\delta}, \quad z \in B_{2R}^+(z^*). \tag{7.49}$$

Consequently, it follows from the Young inequality that

$$\begin{aligned} \int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)/(1+\delta)} dz &\leq |B_{2R}^+(z^*)| + \int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)} dz \\ &\leq 1 + 2^N \sigma_N, \end{aligned} \tag{7.50}$$

for all $R \leq \min\{R_1, 1\}$. Combining (7.49) with (7.50), recalling that $\delta = [p_+^* p_-^* - N(p_+^* - p_-^*)]/Np_+^*$, and noticing that $1 + \delta \leq (N + p_+^*)/N$, it follows from the Sobolev embedding theorem, the Young inequality and the Hölder inequality that

(7.51)

$$\begin{aligned} &\frac{1}{R^N} \int_{B_{2R}^+(z^*)} \left|\frac{v}{R}\right|^{\tilde{p}(z)} dz \\ &\quad \leq 2^N \sigma_N + \frac{1}{R^N} \int_{B_{2R}^+(z^*)} \left|\frac{v}{R}\right|^{p_+^*} dz \end{aligned}$$

$$
\begin{aligned}
&\leq 2^N \sigma_N + C\left[\frac{1}{R^N}\int_{B_{2R}^+(z^*)} |\nabla v|^{Np_+^*/(N+p_+^*)} dz\right]^{(N+p_+^*)/N} \\
&\leq 2^N \sigma_N + C\left[\frac{1}{R^N}\int_{B_{2R}^+(z^*)} \left(1+|\nabla v|^{\tilde{p}(z)/(1+\delta)}\right) dz\right]^{(N+p_+^*)/N} \\
&\leq C + CR^{-(N+p_+^*)}\left[\int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)/(1+\delta)} dz\right]^{(N+p_+^*)/N} \\
&\leq C + CR^{-(N+p_+^*)}\left[\int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)/(1+\delta)} dz\right]^{1+\delta} \\
&\quad \cdot (1+2^N\sigma_N)^{(N+p_+^*)/N-(1+\delta)} \\
&\leq C + CR^{-(p_+^*-p_-^*)(N+p_+^*)/p_+^*}\left[\frac{1}{R^N}\int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)/(1+\delta)}\right]^{1+\delta} \\
&\leq C + C\left[\frac{1}{R^N}\int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)/(1+\delta)}\right]^{1+\delta} \\
&\leq C + C\left[\frac{1}{R^N}\int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)/(1+\sigma)}\right]^{1+\sigma},
\end{aligned}
$$

for all $R \leq \{R_1, 1\}$, where $C = C(N, L_0, \alpha_0, c_0, p_-, p_+)$. Combining (7.48) with (7.51), we obtain the Gehting type inequality

$$
\int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)} dz \leq C + C\left(\frac{1}{R^N}\int_{B_{2R}^+(z^*)} |\nabla v|^{\tilde{p}(z)/(1+\sigma)}\right)^{1+\sigma},
$$

for all $R \leq \{R_1, 1\}$, where $C = C(N, L_0, \alpha_0, c_0, p_-, p_+)$, which implies (7.42) (see [**12**, Chapter 5, Proposition 1.1]).

It is easy to see that the constants $\widehat{R}$, $\widehat{c}$ and $\widehat{\delta}$ depend only upon $N$, $M$, $L_0$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $\alpha_0$, $c_0$ and $\alpha^*$. The proof is complete. ◻

Combining Lemma 7.4 with Lemma 7.5, together with Lemma 7.6, we have the following lemma, which states the "gap" between $v$ and $w_1$ or $w_2$.

**Lemma 7.7.** *Let $z^* \in \overline{B_{r_1}^+(0)}$, $w_1$ and $w_2$ be the solutions of problems* (7.26) *and* (7.27) *with $z_0$ being replaced by $z^*$, respectively* (*of course,* *in the meantime,* $\overline{A}$ *is defined by* $\overline{A}(\eta) = A(z^{**}, v(z^{**}), \eta)$*, where* $z^{**}$ *is the point where* $\widetilde{p}(z)$ *gets its maximum value on* $\overline{B_R(z^*) \cap B^+_{r_1}(0)}$*.) We denote* $p^*_+ = \widetilde{p}(z^{**}) = \max\{\widetilde{p}(z); z \in \overline{B_R(z^*) \cap B^+_{r_1}(0)}\}$*. Then two positive constants* $R_0$ *and* $C$ *exist depending only upon* $N$*,* $M$*,* $p_-$*,* $p_+$*,* $\lambda_1$*,* $\Lambda_1$*,* $L_0$*,* $\alpha_0$*,* $c_0$*,* $c^*$ *and* $\alpha^*$*, such that for any* $R \le R_0$*, the following two hold*:

(i) *Let* $B_R$ *and* $B_{2R}$ *be two concentric balls contained in* $B^+_{r_1}(0)$ *with center at* $z^*$*. Then one obtains*

$$\int_{B_R(z^*)} |\nabla v - \nabla w_1|^{p^*_+} dz \le C R^{\alpha^* \alpha_0 / 2} \int_{B_{2R}(z^*)} \left(1 + |\nabla v|^{\tilde{p}(z)}\right) dz; \tag{7.52}$$

(ii) *Let* $z^* \in B^0_{r_1}(0)$ *and* $R > 0$*, such that* $B^+_{2R}(z^*) \subseteq B^+_{r_1}(0)$*. Then*

$$\int_{B^+_R(z^*)} |\nabla v - \nabla w_2|^{p^*_+} dz \le C R^{\alpha^* \alpha_0 / 2} \int_{B^+_{2R}(z^*)} \left(1 + |\nabla v|^{\tilde{p}(z)}\right) dz. \tag{7.53}$$

*Proof.* We only prove part (i), since part (ii) can be proved analogously. Denote

$$I = \int_{B_R(z^*)} \left(\overline{A}(\nabla v) - \overline{A}(\nabla w_1)\right) (\nabla v - \nabla w_1) \, dx.$$

Since $v$ satisfies (7.24) and $w_1$ is the solution of (7.26) with $z_0$ being replaced by $z^*$, we have

$$\begin{aligned} I &= \int_{B_R(z^*)} \overline{A}(\nabla v)(\nabla v - \nabla w_1) \, dz \\ &= \int_{B_R(z^*)} \left(\overline{A}(\nabla v) - \widetilde{A}(z, v, \nabla v)\right) (\nabla v - \nabla w_1) \, dz \\ &\quad + \int_{B_R(z^*)} \widetilde{B}(z, v, \nabla v)(v - w_1) \, dz \\ &:= I_1 + I_2. \end{aligned}$$

Let $\widehat{R}$ and $\widehat{\delta}$ be the constants stated in Lemma 7.6 and let $R^*$, $c^*$ and $\alpha^*$ be the constants stated in Proposition 6.2. Denote $p^*_- =$

$\min_{\overline{B_R(z^*)}} \widetilde{p}(z)$. Using condition (7.18), a positive constant $R_1 = R_1(L_1, \alpha_0, \widehat{\delta})$ exists such that

$$\delta = \frac{p_+^* - p_-^*}{p_-^*} \leq \widehat{\delta}, \quad p_+^* \leq \widetilde{p}(z)(1+\delta), \quad z \in B_R(z^*), \tag{7.54}$$

provided $R \leq R_1$. Set $R_0 = \min\{R^*, \widehat{R}, R_1\}$. Then $\int_{B_{2R}(z^*)} |\nabla v|^{\tilde{p}(z)} dz \leq 1$ (see Lemma 7.6) and $R^{-\delta} \leq C(L_1, \alpha_0)$, provided $R \leq R_0$. By Proposition 6.2, Lemma 7.4 and Lemma 7.6, it follows from (7.17), (7.43), (7.54) and the Young inequality that, for any $R \leq R_0$, one has

$$\begin{aligned} I_1 &\leq C(R^{\alpha_0} + R^{\alpha_0\alpha^*}) \\ &\qquad \int_{B_R(z^*)} (|\nabla v|^{p_+^*-1} + |\nabla v|^{\tilde{p}(z)-1})(|\nabla v| + |\nabla w_1|)\, dz \\ &\leq CR^{\alpha_0\alpha^*} \int_{B_R(z^*)} (1 + |\nabla v|^{p_+^*-1})(|\nabla v| + |\nabla w_1|)\, dx \\ &\leq CR^{\alpha_0\alpha^*} \int_{B_R(z^*)} (1 + |\nabla v|^{p_+^*} + |\nabla w_1|^{p_+^*})\, dz \\ &\leq CR^{\alpha_0\alpha^*} \int_{B_R(z^*)} (1 + |\nabla v|^{p_+^*})\, dz \\ &\leq CR^{\alpha_0\alpha^*} \int_{B_R(z^*)} (1 + |\nabla v|^{\tilde{p}(z)(1+\delta)})\, dz \\ &\leq CR^{\alpha_0\alpha^*} \left[ |B_R(z^*)| + R^N \left( 1 + \frac{1}{R^N} \int_{B_{2R}(z^*)} |\nabla v|^{\tilde{p}(z)} dz \right)^{1+\delta} \right] \\ &\leq CR^{\alpha_0\alpha^*} \left( |B_R(z^*)| + R^{-\delta N} \int_{B_{2R}(z^*)} |\nabla v|^{\tilde{p}(z)} dz \right) \\ &\leq CR^{\alpha_0\alpha^*} \int_{B_{2R}(z^*)} (1 + |\nabla v|^{\tilde{p}(z)})\, dz, \end{aligned} \tag{7.55}$$

where $C$ depends upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$ and $\alpha^*$. Combining (7.16) with Lemma 7.4, together with Proposition 6.2, it follows that

$$\begin{aligned} I_2 &\leq C \operatorname{osc}\{v; B_R(z^*)\} \int_{B_R(z^*)} \left(1 + |\nabla v|^{\tilde{p}(z)}\right)\, dz \\ &\leq CR^{\alpha^*} \int_{B_{2R}(z^*)} \left(1 + |\nabla v|^{\tilde{p}(z)}\right)\, dz, \end{aligned}$$

where $C = C(N, \lambda_1, \Lambda_2, p_+)$. So we obtain that

$$I \le CR^{\alpha^*\alpha_0} \int_{B_{2R}(z^*)} (1 + |\nabla v|^{\tilde{p}(z)})\, dz, \tag{7.56}$$

where $C$ depends upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$ and $\alpha^*$. If $p_+^* \ge 2$, we deduce (7.52) immediately from (7.1) and (7.56). If $p_+^* < 2$, then by using the similar argument of (7.55), it follows from (7.1), (7.56) and Lemma 7.6 that

$$\begin{aligned}
&\int_{B_R(z^*)} |\nabla v - \nabla w_1|^{p_+^*} dz \\
&\quad \le \left[ \int_{B_R(z^*)} (|\nabla v|^2 + |\nabla w_1|^2)^{(p_+^*-2)/2} |\nabla v - \nabla w_1|^2 dz \right]^{1/2} \\
&\qquad \cdot \left[ \int_{B_R(z^*)} (|\nabla v|^2 + |\nabla w_1|^2)^{(2-p_+^*)/2} |\nabla v - \nabla w_1|^{2p_+^*-2} dz \right]^{1/2} \\
&\quad \le C \left( \frac{I}{\lambda_0} \right)^{1/2} \left[ \int_{B_R(z^*)} (|\nabla v|^{p_+^*} + |\nabla w_1|^{p_+^*}) \right]^{1/2} \\
&\quad \le C \left( \frac{I}{\lambda_0} \right)^{1/2} \left[ \int_{B_{2R}(z^*)} (1 + |\nabla v|^{\tilde{p}(z)})\, dz \right]^{1/2} \\
&\quad \le CR^{\alpha^*\alpha_0/2} \int_{B_{2R}(z^*)} \left(1 + |\nabla v|^{\tilde{p}(z)}\right) dz,
\end{aligned}$$

and consequently we also obtain (7.52). It's easy to see that $C$ depends only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$ and $\alpha^*$. □

Using the properties of $w_1$ and $w_2$ (namely, Lemmas 7.4 and 7.5) and the "gap" between $v$ and $w_1$ or $w_2$ (namely, Lemma 7.7), we have the following lemma.

**Lemma 7.8.** *We denote by $p_+^*(z; r)$ the maximum value of $\widetilde{p}$ on $\overline{B_r(z) \cap B_{r_1}^+(0)}$. Then a positive constant $R_0$ exists depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$ and $\alpha^*$, such that for all $R \le R_0$ and $\tau \in (0, N)$, it follows:*

(i) *Let $B_{2R}$, $B_R$ and $B_\rho$ be concentric balls contained in $B^+_{r_1}(0)$ with center at $z^*$. Then*

$$\int_{B_\rho(z^*)} |\nabla v|^{p^*_+(z^*;\rho)} dz \leq C_\tau \left(\frac{\rho}{R}\right)^{N-\tau} \left[\int_{B_{2R}(z^*)} |\nabla v|^{p^*_+(z^*;2R)} dz + R^N\right], \quad 0 < \rho \leq 2R, \tag{7.57}$$

*where $C_\tau$ is a constant depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$ and $\tau$;*

(ii) *Let $z^* \in B^0_{r_1}(0)$ and $R > 0$ be such that $B^+_{2R}(z^*) \subseteq B^+_{r_1}(0)$. Then*

$$\int_{B^+_\rho(z^*)} |\nabla v|^{p^*_+(z^*;\rho)} dz \leq C_\tau \left(\frac{\rho}{R}\right)^{N-\tau} \left[\int_{B^+_{2R}(z^*)} |\nabla v|^{p^*_+(z^*;2R)} dz + R^N\right], \quad 0 < \rho \leq 2R, \tag{7.58}$$

*where $C_\tau$ is a constant depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$ and $\tau$.*

*Proof.* We only need to prove the conclusion for the case $\rho \leq R/2$. Let $B_{2R}$, $B_R$ and $B_\rho$ be concentric balls contained in $B^+_{r_1}(0)$ with $\rho \leq R \leq R_0$, where $R_0$ is the constant stated in Lemma 7.7. Let $w_1$ be the unique solution for problem (7.26) with $z_0$ being replaced by $z^*$ (see Lemma 7.7 for the exact meaning.) Then, by using Lemma 7.4, part (i) of Lemma 7.7 and the Young inequality, we conclude that

$$\begin{aligned} &\int_{B_\rho(z^*)} |\nabla v|^{p^*_+(z^*;\rho)} dz \\ &\qquad \leq C \int_{B_\rho(z^*)} |\nabla v - \nabla w_1|^{p^*_+(z^*;\rho)} dz \\ &\qquad\quad + C \int_{B_\rho(z^*)} |\nabla w_1|^{p^*_+(z^*;\rho)} dz \\ &\qquad \leq CR^N + C \int_{B_R(z^*)} |\nabla v - \nabla w_1|^{p^*_+(z^*;R)} dz \end{aligned} \tag{7.59}$$

$$
\begin{aligned}
&+ C\rho^N \sup_{B_\rho(z^*)} |\nabla w_1|^{p_+^*(z^*;R)} \\
&\le CR^N + CR^{\alpha^*\alpha_0/2} \int_{B_{2R}(z^*)} |\nabla v|^{\tilde p(z)} dz \\
&+ C\left(\frac{\rho}{R}\right)^N \int_{B_R(z^*)} |\nabla v|^{p_+^*(z^*;R)} dz \\
&\le C\left(R^{\alpha^*\alpha_0/2} + \left(\frac{\rho}{R}\right)^N\right) \int_{B_{2R}(z^*)} |\nabla u|^{p_+^*(z^*;2R)} dx \\
&+ CR^N.
\end{aligned}
$$

Applying Lemma 3.2 in [**24**] to (7.59), we obtain (7.57). Hence, we complete the proof of part (i). While the proof of part (ii) is similar to that of part (i), the differences are these: the function $w_1$ is replaced by $w_2$, which is the unique solution of problem (7.27) with $z_0$ being replaced by $z^*$; we use Lemma 7.5 and part (ii) of Lemma 7.7 instead of Lemma 7.4 and part (i) of Lemma 7.7, respectively. The proof is complete. □

By Lemma 7.8, we obtain the following corollary.

**Corollary 7.1.** *We denote by $p_+^*(z;r)$ the maximum value of $\widetilde{p}$ on $\overline{B_r(z) \cap B_{r_1}^+(0)}$. Then a positive constant $R_0 \le r_1/4$ exists depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$ and $\alpha^*$, such that for all $z^* \in \overline{B_{r_1/2}^+(0)}$, $\rho \le R_0$ and $\tau \in (0, N)$, it follows that*

$$\int_{B_\rho(z^*) \cap B_{r_1}^+(0)} |\nabla v|^{p_+^*(z^*;\rho)} dz \le C_\tau \rho^{N-\tau},$$

*where $C_\tau$ is a constant depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$, $r_1$ and $\tau$.*

*Proof.* Take arbitrary $z^* \in \overline{B_{r_1/2}^+(0)}$ and $\rho \le R_0$, where $R_0$ will be determined later. On account of the property of $\widetilde{p}(z)$ and Lemma 7.6, $\widehat{c} > 0$, $\widehat{\delta} > 0$ and $\widehat{R}$ depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $c_0$, $\alpha_0$, $\alpha^*$ and $c^*$, such that for any $\widetilde{z} \in B_{r_1/2}^0(0)$ and $R \le \widehat{R}/2$, it follows that

$$p_+^*(\widetilde{z}; 2R) \le \widetilde{p}(z)(1 + \widehat{\delta}), \quad z \in B_{2R}^+(\widetilde{z}),$$

and, by the Young inequality,

$$\begin{aligned}\int_{B^+_{2R}(\widetilde{z})} |\nabla v|^{p^*_+(\widetilde{z};2R)} dz &\le |B_{2R}| + \int_{B^+_{2R}(\widetilde{z})} |\nabla v|^{\widetilde{p(z)}(1+\widehat{\delta})} dz \\ &\le |B_{\widehat{R}}| + \int_{B^+_{\widehat{R}}(\widetilde{z})} |\nabla v|^{\widetilde{p(z)}(1+\widehat{\delta})} dz \\ &\le \sigma_N \widehat{R}^N + [\widehat{c}(1+\widehat{R}^{-N})]^{1+\hat{\delta}} \widehat{R}^N = c_1.\end{aligned} \tag{7.60}$$

Denote $d = \operatorname{dist}(z^*, B^0_{r_1}(0)) < r_1/2$, where $B^0_{r_1}(0) = B_{r_1}(0) \cap \{z \in \mathbf{R}^N \mid z^N = 0\}$. Take $z_1^* \in B^0_{r_1/2}$, such that $|z_1^* - z_1| = d$.

If $\rho \ge d$, then by Lemma 7.8, for any $\tau \in (0, N)$, two positive constants $R_1 \le \widehat{R}/2$ and $C^*_\tau$ exist such that, for any $\rho \le R_1$, it follows from (7.60) that

$$\begin{aligned}&\int_{B_\rho(z^*) \cap B^+_{r_1/2}(0)} |\nabla v|^{p^*_+(z^*;\rho)} dz \\ &\quad\le \int_{B^+_{\rho+d}(z_1^*)} |\nabla v|^{p^*_+(z^*;\rho)} dz \\ &\quad\le \int_{B^+_{\rho+d}(z_1^*)} \left(1 + |\nabla v|^{p^*_+(z_1^*;\rho+d)}\right) dz \\ &\quad\le C^*_\tau \left(\frac{\rho+d}{R_1}\right)^{N-\tau} \left[\int_{B^+_{2R_1}(z_1^*)} |\nabla v|^{p^*_+(z_1^*;2R_1)} dz + R_1^N\right] \\ &\qquad + \sigma_N (\rho+d)^N \\ &\quad\le \left[C^*_\tau (c_1 + R_1^N) R_1^{\tau-N} 2^{N-\tau} + \sigma_N 2^N R_1^\tau\right] \rho^{N-\tau} \\ &\quad= C_{1\tau} \rho^{N-\tau}.\end{aligned}$$

If $\rho \le d$, then by Lemma 7.8, for any $\tau \in (0, N)$, two positive constants $R_2 \le \widehat{R}/2$ and $C^{**}_\tau$ exist such that, for any $\rho \le R_2$, it follows from the Young inequality and (7.60) that

$$\begin{aligned}&\int_{B_\rho(z^*) \cap B^+_{r_1/2}(0)} |\nabla v|^{p^*_+(z^*;\rho)} dz \\ &\qquad = \int_{B_\rho(z^*)} |\nabla v|^{p^*_+(z^*;\rho)} dz\end{aligned}$$

$$
\begin{aligned}
&\le C_\tau^{**}\left(\frac{\rho}{d}\right)^{N-\tau}\left[\int_{B_d(z^*)}|\nabla v|^{p_+^*(z^*;d)}dz+d^N\right]\\
&\le C_\tau^{**}\left(\frac{\rho}{d}\right)^{N-\tau}\left[\int_{B_{2d}^+(z_1^*)}|\nabla v|^{p_+^*(z_1^*;2d)}dz+d^N\right]\\
&\le C_\tau^{**}\left(\frac{\rho}{d}\right)^{N-\tau}\left[d^N+\left(\frac{2d}{R_2}\right)^{N-\tau}\right.\\
&\qquad\qquad\qquad \left.\cdot\left(\int_{B_{2R_2}^+(z_1^*)}|\nabla v|^{p_+^*(z_1^*;2R_2)}dz+R_2^N\right)\right]\\
&\le C_\tau^{**}\left[d^\tau+2^{N-\tau}R_2^{\tau-N}(c_1+R_2^N)\right]\rho^{N-\tau}\\
&\le C_\tau^{**}\left[\left(\frac{r_1}{2}\right)^\tau+2^{N-\tau}R_2^{\tau-N}(c_1+R_2^N)\right]\rho^{N-\tau}\\
&= C_{2\tau}\rho^{N-\tau}.
\end{aligned}
$$

Taking $R_0=\min\{R_1,R_2\}$ and $C_\tau=\max\{C_{1\tau},C_{2\tau}\}$, then the proof is complete. $\square$

Now we can state and prove the boundary $C^{1,\alpha}$ estimates on $v$, namely, we have the following proposition.

**Proposition 7.1.** *Positive constants $\beta$, $C$ and $R_0$ exist depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$ and $r_1$ such that, for any $z^*\in\overline{B^+_{r_1/2}(0)}$ and $\rho\le R_0$, the following holds*:

$$\int_{B_\rho(z^*)\cap B^+_{r_1}(0)}|\nabla v-\{\nabla v\}_\rho|\,dx\le C\rho^{N+\beta/p_+},$$

*where*

$$\{\nabla v\}_\rho=\frac{1}{|B_\rho(z^*)\cap B^+_{r_1}(0)|}\int_{B_\rho(z^*)\cap B^+_{r_1}(0)}\nabla v\,dz,$$

*and consequently,*

$$\operatorname{osc}\left\{\nabla v;B_\rho(z^*)\cap B^+_{r_1}(0)\right\}\le C\rho^{\beta/p_+},$$

*and*

$$|\nabla v(z)|\le C,\quad z\in\overline{B^+_{r_1/2}(0)}.$$

*Proof.* For any $z \in \overline{B^+_{r_1}(0)}$, we denote by $p^*_+(z;r)$ the maximum value of $\widetilde{p}$ on $\overline{B_r(z) \cap B^+_{r_1}(0)}$. Take arbitrary $z^* \in \overline{B^+_{r_1/2}(0)}$ and $\rho \le R_0$. Denote $d = \operatorname{dist}(z^*, B^0_{r_1}(0))$ and take $z^*_1 \in B^0_{r_1}(0)$ such that $d = |z^* - z^*_1|$, where $B^0_{r_1}(0) := \{z \in \mathbf{R}^N \mid z^N = 0\} \cap B_{r_1}(0)$. We denote

$$\begin{aligned} \delta &= p_- - 1, \\ \varepsilon &= \frac{\alpha^*\alpha_0\sigma(1+\delta)}{4N + 4\sigma(1+\delta) + 2\alpha^*\alpha_0}, \\ \tau &= \frac{\varepsilon\sigma(1+\delta)}{N+\sigma(1+\delta)}, \\ \theta &= \frac{(N+\varepsilon)[N+\sigma(1+\delta)+\alpha^*\alpha_0/2]}{[N+\sigma(1+\delta)](N-\tau+\alpha^*\alpha_0/2)}, \\ \mu &= \frac{N+\sigma(1+\delta)}{N+\sigma(1+\delta)+\alpha^*\alpha_0/2}, \end{aligned}$$

where $\alpha^*$ and $\sigma$ are the positive constants stated in Proposition 6.2 and Lemma 7.4 or Lemma 7.5, respectively. Obviously, we have $0 < \mu < 1$. In addition, one obtains

$$0 < \theta = \frac{(N+(1/2)\alpha^*\alpha_0)[N+\sigma(1+\delta)] - (1/4)\alpha^*\alpha_0\sigma(1+\delta)}{(N+(1/2)\alpha^*\alpha_0)[N+\sigma(1+\delta)] - (1/4)\alpha^*\alpha_0\sigma(1+\delta)[\sigma(1+\delta)/N+\sigma(1+\delta)+\alpha^*\alpha_0]} < 1.$$

Moreover, we have

$$\theta[N+\sigma(1+\delta)] - \theta\mu[\tau+\sigma(1+\delta)] = \theta\mu\left(N + \frac{\alpha^*\alpha_0}{2} - \tau\right) = N+\varepsilon, \tag{7.61}$$

and

$$N+\sigma(1+\delta) - \theta[\tau+\sigma(1+\delta)] > N, \quad \theta\left(N + \frac{\alpha^*\alpha_0}{2} - \tau\right) > N. \tag{7.62}$$

There is no difficulty in verifying (7.61) and the second inequality in (7.62), so we only verify the first inequality in (7.62). For this purpose, we note that the inequality is equivalent to the following

$$\begin{aligned} &\left\{[N+\sigma(1+\delta)]^2 + \varepsilon\left[N+\sigma(1+\delta)+\frac{1}{2}\alpha^*\alpha_0\right] + \frac{1}{2}\alpha^*\alpha_0\right\}\tau \\ &\qquad < \sigma(1+\delta)\left\{\frac{1}{2}\alpha^*\alpha_0\sigma(1+\delta) - \varepsilon\left[N+\sigma(1+\delta)+\frac{1}{2}\alpha^*\alpha_0\right]\right\}. \end{aligned}$$

Note that $\varepsilon[N+\sigma(1+\delta)+(1/2)\alpha^*\alpha_0] = (1/4)\alpha^*\alpha_0\sigma(1+\delta)$ and the above inequality is equivalent to

$$\Big\{[N+\sigma(1+\delta)]^2+\frac{1}{4}\alpha^*\alpha_0[2N+\sigma(1+\delta)]\Big\}\tau<\frac{1}{4}\alpha^*\alpha_0[\sigma(1+\delta)]^2,$$

which can be deduced by

$$4[N+\sigma(1+\delta)]\left[N+\sigma(1+\delta)+\frac{1}{2}\alpha^*\alpha_0\right]\tau\le\alpha^*\alpha_0[\sigma(1+\delta)]^2.$$

Recalling the value of $\varepsilon$, we can take $\tau=[\varepsilon\sigma(1+\delta)]/[N+\sigma(1+\delta)]$, such that the above inequality holds, and consequently, the first inequality in (7.62) holds. We denote

$$N+\beta=\min\{N+\sigma(1+\delta)-\theta[\tau+\sigma(1+\delta)],\theta(N+\alpha^*\alpha_0/2-\tau),N+\varepsilon\},$$

then $0<\beta\le\varepsilon$. Set

$$R=\rho^\theta,\qquad r=R^\mu=\rho^{\theta\mu}.$$

Obviously, a positive constant $R_1$ exists such that

$$2\rho\le R,\qquad \rho+2R\le\frac{r}{2},$$

for any $\rho\le R_1=R_1(N,p_-,p_+,\lambda_1,\Lambda_1,\alpha^*,\alpha_0)$. On account of this, we suppose that $R_0\le R_1$.

We continue the proof in two different cases: $2R\le d$ and $2R>d$.

Case 1. $2R\le d$. We denote by $w_1$ the solution of problem (7.26) with $z_0$ replaced by $z^*$ (see Lemma 7.7 for the exact meaning). By Lemma 7.4, Lemma 7.7 and Corollary 7.1, for any $\tau\in(0,N)$, an $R_2$ exists depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$ and $r_1$, such that

$$\begin{aligned}&\int_{B_\rho(z^*)\cap B^+_{r_1}(0)}|\nabla v-\{\nabla v\}_\rho|^{p^*_+(z^*;R)}dz\\&\qquad=\int_{B_\rho(z^*)}|\nabla v-\{\nabla v\}_\rho|^{p^*_+(z^*;R)}dz\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{B_\rho(z^*)} |\nabla w_1 - \{\nabla w_1\}_\rho|^{p_+^*(z^*;R)} dz\\
&\quad + C\int_{B_\rho(z^*)} |\nabla v - \nabla w_1|^{p_+^*(z^*;R)} dz\\
&\le C\rho^N\left(\frac{\rho}{R}\right)^{\sigma p_+^*(z^*;R)} \left(\text{osc}\left\{\nabla w_1; B_{R/2}(z^*)\right\}\right)^{p_+^*(z^*;R)}\\
&\quad + C\int_{B_R(z^*)} |\nabla v - \nabla w_1|^{p_+^*(z^*;R)} dz\\
&\le C\left(\frac{\rho}{R}\right)^{N+\sigma p_+^*(z^*;R)} \int_{B_R(z^*)} |\nabla v|^{p_+^*(z^*;R)} dz\\
&\quad + CR^{\alpha^*\alpha_0/2}\int_{B_{2R}(z^*)} (1+|\nabla v|^{p_+^*(z^*;2R)}) dz\\
&\le C_\tau\left(\frac{\rho}{R}\right)^{N+\sigma(1+\delta)} R^{N-\tau}\\
&\quad + C_\tau R^{N+\alpha^*\alpha_0/2-\tau}\\
&= C_\tau \rho^{N+\sigma(1+\delta)-\theta[\tau+\sigma(1+\delta)]}\\
&\quad + C_\tau\rho^{\theta(N+\alpha^*\alpha_0/2-\tau)}\\
&\le C_\tau \rho^{N+\beta},
\end{aligned}
$$

for any $\rho \le R_2$, where $C_\tau$ is a positive constant depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$, $r_1$ and $\tau$.

Case 2. $2R > d$. We denote by $w_2$ the solution of problem (7.27) with $R$ and $z_0$ replaced by $r$ and $z_1^*$, respectively. We temporarily denote $\{\nabla v\}_{\rho+d} = 1/|B^+_{\rho+d}(z_1^*)| \int_{B^+_{\rho+d}(z_1^*)} \nabla v\, dz$. Then, it follows from Lemma 7.5, Lemma 7.7 and Corollary 7.1 that, for any $\tau \in (0, N)$, an $R_3$ exists depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$ and $r_1$, such that

$$
\begin{aligned}
&\int_{B_\rho(z^*)\cap B^+_{r_1}(0)} |\nabla v - \{\nabla v\}_\rho|^{p_+^*(z_1^*;r)} dz\\
&\qquad \le \int_{B^+_{\rho+d}(z_1^*)} |\nabla v - \{\nabla v\}_{\rho+d}|^{p_+^*(z_1^*;r)} dz\\
&\qquad \le C\int_{B^+_{\rho+d}(z_1^*)} |\nabla w_2 - \{\nabla w_2\}_{\rho+d}|^{p_+^*(z_1^*;r)} dz
\end{aligned}
$$

$$
\begin{aligned}
&\quad + C\int_{B^+_{\rho+d}(z_1^*)} |\nabla v - \nabla w_2|^{p_+^*(z_1^*;r)} dz \\
&\le C(\rho+d)^N \left(\frac{\rho+d}{r}\right)^{\sigma p_+^*(z_1^*;r)} \\
&\quad \cdot \left(\operatorname{osc}\left\{\nabla w_2; B^+_{r/2}(z_1^*)\right\}\right)^{p_+^*(z_1^*;r)} \\
&\quad + C\int_{B^+_r(z_1^*)} |\nabla v - \nabla w_2|^{p_+^*(z_1^*;r)} dz \\
&\le C\left(\frac{\rho+d}{r}\right)^{N+\sigma p_+^*(z_1^*;r)} \int_{B^+_r(z_1^*)} |\nabla v|^{p_+^*(z_1^*;r)} dz \\
&\quad + C r^{\alpha^*\alpha_0/2} \int_{B^+_{2r}(z_1^*)} \left(1 + |\nabla v|^{p_+^*(z_1^*;2r)}\right) dz \\
&\le C_\tau \left(\frac{R}{r}\right)^{N+\sigma(1+\delta)} r^{N-\tau} + C_\tau r^{N+\alpha^*\alpha_0/2-\tau} \\
&= C_\tau \rho^{\theta[N+\sigma(1+\delta)]-\theta\mu[\tau+\sigma(1+\delta)]} \\
&\quad + C_\tau \rho^{\theta\mu(N+\alpha^*\alpha_0/2-\tau)} \\
&= C_\tau \rho^{N+\varepsilon} \le C_\tau \rho^{N+\beta},
\end{aligned}
$$

for any $\rho \le R_3$, where $C_\tau$ is a positive constant depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$, $\tau$ and $r_1$.

Combining the above two cases, setting $p_+^* = p_+^*(z^*;R)$ or $p_+^* = p_+^*(z_1^*;r)$, it follows from the Hölder inequality that

$$
\begin{aligned}
&\int_{B_\rho(z^*)\cap B^+_{r_1}(0)} |\nabla v - \{\nabla v\}_\rho| dz \\
&\qquad \le C\rho^{(1-1/p_+^*)N} \left(\int_{B_\rho(z^*)\cap B^+_{r_1}(0)} |\nabla v - \{\nabla v\}_\rho|^{p_+^*}\right)^{1/p_+^*} \\
&\qquad \le C\rho^{N+\beta/p_+^*} \le C\rho^{N+\beta/p_+},
\end{aligned}
$$

for all $\rho \le R_0 = \min\{R_1, R_2, R_3\}$ and $C$ depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda_1$, $\Lambda_1$, $L_0$, $\alpha_0$, $c_0$, $c^*$, $\alpha^*$ and $r_1$. The above inequality implies that

$$
\operatorname{osc}\left\{\nabla v; B_\rho(z^*) \cap B^+_{r_1}(0)\right\} \le C\rho^{\beta/p_+},
$$

which together with the interpolation theorem leads to

$$|\nabla v(z)| \leq C, \qquad z \in \overline{B^+_{r_1/2}(0)}.$$

The proof is complete. □

Combining Proposition 7.1 with the interior $C^{1,\alpha}$ estimates, we can obtain the global $C^{1,\alpha}$ estimates, namely, we can give the proof of Proposition 4.1 as follows:

*Proof of Proposition* 4.1. Let $r_1$ be the constant stated in Lemma 7.3. Take arbitrary $x_1$, $x_2 \in \overline{\Omega}$, such that $|x_1 - x_2| \leq (1/9)r_1$. Without loss of generality, we suppose that $d(x_1, \partial\Omega) \geq d(x_2, \partial\Omega)$. Take $x_1^0 \in \partial\Omega$, such that $|x_1 - x_1^0| = d(x_1, \partial\Omega)$. We consider the following two cases: $d(x_1, \partial\Omega) \leq (2/9)r_1$ and $d(x_1, \partial\Omega) > (2/9)r_1$.

Case I. $d(x_1, \partial\Omega) \leq (2/9)r_1$. It follows that

$$|x_1^0 - x_2| \leq |x_1^0 - x_1| + |x_1 - x_2| \leq \frac{1}{3}r_1 \leq \frac{1}{3}r_0.$$

Thus, we have $x_1, x_2 \in B_{r_0}(x_1^0) \cap \Omega$. Let $h$ be the function stated in (H3), and let $\Phi_0$, $\Psi_0$, $\Phi$ and $\Psi$ be defined by (7.5)–(7.9) with 0 replaced by $x_1^0$, respectively. By (7.21), it follows that

$$\begin{aligned} |\Phi(x_1) - \Phi(x_1^0)| &\leq |\Phi'(\xi)(x_1 - x_1^0)| \leq \frac{3}{2}|x_1 - x_1^0| \leq \frac{r_1}{3}, \\ |\Phi(x_2) - \Phi(x_1^0)| &\leq |\Phi'(\zeta)(x_2 - x_1^0)| \leq \frac{3}{2}|x_2 - x_1^0| \leq \frac{r_1}{2}. \end{aligned}$$

Define a function $v(z) = u(\Psi(z))$, for any $z \in V_0$. Denote $z_1 = \Phi(x_1)$, $z_2 = \Phi(x_2)$, $z_0 = \Phi(x_1^0)$, $y_1 = Kx_1$ and $y_2 = Kx_2$. On account of the above two inequalities and Lemma 7.3, we have $z_1, z_2 \in B^+_{r_1/2}(z_0)$. Obviously,

$$u(x) = v(\Phi(x)), \qquad \nabla u(x) = \nabla v(\Phi(x))\Phi'(x).$$

Then, it follows from Proposition 7.1 and (7.21) that

$$|\nabla u(x_i)| = |\nabla v(\Phi(x_i))\Phi'(x_i)| \leq \frac{3}{2}|\nabla v(\Phi(x_i))| \leq C, \quad i = 1, 2,$$

and
$$\begin{aligned}|\nabla u(x_1)-\nabla u(x_2)| &= |\nabla v(z_1)\Phi'(x_1)-\nabla v(z_2)\Phi'(x_2)|\\ &\le |(\nabla v(z_1)-\nabla v(z_2))\Phi'(x_1)|\\ &\quad + |\nabla v(z_2)||\Phi'(x_1)-\Phi'(x_2)|\\ &\le C|\nabla v(z_1)-\nabla v(z_2)|\\ &\quad + C|\nabla h(\widehat{y_1})-\nabla h(\widehat{y_2})|\\ &\le C|z_1-z_2|^{\beta/p_+}+C|y_1-y_2|^{\alpha_0}\\ &\le C|x_1-x_2|^{\alpha_1},\end{aligned}$$
where $\alpha_1=\min\{\alpha_0,\beta/p_+\}$.

Case II. $d(x_1,\partial\Omega)>(2/9)r_1$. Take $x_i^0\in\partial\Omega$, such that $|x_i-x_i^0|=d(x_i,\partial\Omega)$, $i=1,2$. Then one obtains
$$|x_2-x_2^0|\ge|x_1-x_2^0|-|x_1-x_2|\ge d(x_1,\partial\Omega)-|x_1-x_2|\ge\frac{r_1}{9}.$$
Set $x_*=(x_1+x_2)/2$, $d=r_1/18$. Consider the ball $B_d(x_*)$. For arbitrary $x\in B_d(x_*)$, choosing $x^0\in\partial\Omega$, such that $d(x,\partial\Omega)=|x-x^0|$, then one has
$$|x-x^0|\ge|x_1-x^0|-|x-x_1|\ge d(x_1,\partial\Omega)-\frac{r_1}{9}>\frac{r_1}{9}.$$
On account of this, we can use Theorem 1.1 in [**5**] to conclude that positive constants $C$ and $\alpha_2\in(0,1)$ exist depending only upon $N$, $M$, $p_-$, $p_+$, $\lambda$, $\Lambda$, $L_0$, $\alpha_0$ and $r_1$, such that
$$|\nabla u(x_1)-\nabla u(x_2)|\le C|x_1-x_2|^{\alpha_2},$$
and consequently, it follows from the interpolation theorem that
$$|\nabla u(x_1)|\le C.$$
Taking $R_0=r_1/9$, $\alpha=\min\{\alpha_1,\alpha_2\}$, then the conclusion follows by combining Case I with Case II. The proof is complete. □

We can use a similar method to prove Proposition 4.2, and there is no essential difference between the proof of Proposition 4.1 and that of Proposition 4.2. Thus, we omit it here. □

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