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ASYMPTOTIC BEHAVIOR OF ROBIN PROBLEM FOR HEAT EQUATION ON A COATED BODY

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ABSTRACT. We are interested in the problem of protection of an isotropically conducting body from overheating by an anisotropically conducting coating, thin compared to the scale of the body. We assume Newton's cooling law, so that the temperature satisfies the Robin boundary condition on the outer boundary of the coating; we assume that either the whole thermal tensor of the coating is small, or it is small in the directions normal to the body (the case of "optimally aligned coating"). We study the asymptotic behavior of the solution to the heat equation, as the thickness of the coating shrinks. We find that in this singular limit, on the boundary of the body, we effectively have Dirichlet, Robin or a Neumann condition, depending upon the scaling relations among the thermal tensor and the thickness of the coating and the thermal transport coefficient; thus, the scaling relation that leads to the effective Neumann condition ensures good insulation of the body.

1. Introduction. Motivated by the problem of protecting a body from overheating by an anisotropic insulating coating, we studied in [**8**] the heat equation with Dirichlet boundary condition on the outer surface of the coating; in the singular limit as the thickness of the coating approaches zero, we obtained exact scaling relations between the thermal tensor and the thickness of the coating so that the effective

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(asymptotic) condition on the boundary of the body is of Dirichlet, Robin or Neumann type, with the last condition indicating good insulation.

If we assume Newton's law of cooling at the outer boundary of the coating, then the boundary condition should be, before taking the singular limit, of Robin type. The purpose of this paper is to find again how to have good insulation, by obtaining the effective boundary condition in the singular limit.

The mathematical setting is as follows. Let $\Omega \subset \mathbf{R}^n$ be an open bounded domain, divided into three parts: $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, see Figure 1. Here Ω_1 is a fixed open bounded domain with boundary Γ (representing an isotropically conducting body), Ω_2 is a thin layer with uniform thickness δ (representing the anisotropic coating), and $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. Let the thermal tensor $A(x)$ of Ω be given by

$$A(x) = (a_{ij}(x))_{n\times n} = \begin{cases} kI_{n\times n} & x \in \Omega_1, \\ \sigma(\overline{a}_{ij}(x))_{n\times n} & x \in \Omega_2, \end{cases} \tag{1.1}$$

where $k > 0$ is the constant thermal conductivity of Ω_1, $I_{n\times n}$ is the identity matrix, σ is a positive parameter and the matrix $(\overline{a}_{ij}(x))$ is symmetric and positive definite at every $x \in \Omega_2$, with its smallest eigenvalue bounded from below by a positive constant. Let η be the thermal transport coefficient across $\partial\Omega$. Then the temperature function $Q(x,t)$ is the solution of the heat equation

$$\begin{cases} Q_t - \nabla \cdot (A\nabla Q) = 0 & x \in \Omega, \quad t > 0, \\ (\partial Q/\partial \overline{\nu}_A) + \eta(Q - H) = 0 & x \in \partial\Omega, \, t > 0, \\ Q = Q_0(x) & x \in \Omega, \quad t = 0, \end{cases} \tag{1.2}$$

where $(\partial Q/\partial \overline{\nu}_A) = \nabla Q \cdot A \cdot \overline{\nu}$, $\overline{\nu} = (\overline{\nu}_1, \dots, \overline{\nu}_n)$ is the unit outer normal vector field on $\partial\Omega$, Q_0 is the initial temperature distribution, H is some positive constant (temperature), which is large compared to the values of $Q_0(x)$.

To find exact scaling relations among the thermal tensor and thickness of the coating, and the thermal transport coefficient, we assume, *mathematically*, that

$$\sigma = \sigma(\delta), \qquad \eta = \eta(\delta), \quad \sigma \text{ remains bounded as } \delta \to 0,$$

and that the following limits exist

$$\alpha = \lim_{\delta \to 0^+} \frac{\sigma}{\delta} \quad \text{and} \quad \beta = \lim_{\delta \to 0^+} \eta(\delta).$$

The non-technical description of our results is as follows.

1. If $\beta = 0$, then for any fixed finite $T > 0$, $Q \to w$ in $L^2(\Omega_1 \times (0, T))$ as $\delta \to 0^+$, where w is the solution of

$$\begin{cases} w_t - k\Delta w = 0 & x \in \Omega_1,\ t > 0, \\ w = Q_0 & x \in \Omega_1,\ t = 0, \\ (\partial w/\partial \nu) = 0 & x \in \Gamma, \quad t > 0, \end{cases}$$

where ν is the unit outer normal vector field on Γ.

2. If $\beta \in (0, +\infty]$ and $\alpha \in [0, +\infty]$, then for any fixed finite $T > 0$, $Q \to w$ strongly in $L^2(\Omega_1 \times (0, T))$ as $\delta \to 0^+$, where w is the solution of

$$\begin{cases} w_t - k\Delta w = 0 & x \in \Omega_1,\ t > 0, \\ w = Q_0 & x \in \Omega_1,\ t = 0, \\ k(\partial w/\partial \nu) + [(\beta\alpha\nu \cdot \nu_{\overline{A}})/(\beta + \alpha\nu \cdot \nu_{\overline{A}})](w - H) = 0 & x \in \Gamma, \quad t > 0. \end{cases}$$

Here $\nu_{\overline{A}} = (\overline{a}_{ij})\nu$; the boundary condition is understood as

(i) the Dirichlet condition $w = H$ if both α and β are equal to ∞,

(ii) the Robin condition $k(\partial w/\partial \nu) + \alpha\nu \cdot \nu_{\overline{A}}(w - H) = 0$ if $\beta = \infty$ and $\alpha < \infty$,

(iii) $k(\partial w/\partial \nu) + \beta(w - H) = 0$ if $\alpha = \infty$ and $\beta < \infty$. Note that, in the case of $\beta = \infty$, this result agrees to our previous result for the Dirichlet boundary value problem studied in [**8**].

Therefore, if either $\beta = 0$ or $\beta > 0$, but $\alpha = 0$, then on the finite time interval $[0, T]$, the effective boundary condition is of the Neumann type, and hence the body Ω_1 is effectively well-insulated.

In Theorem 2.3, we improve these results by not requiring that the entire thermal tensor be small over Ω_2 in the sense of (1.1), but requiring that the coating be "optimally aligned "and the eigenvalue in the directions normal to the body be small. The notion of "optimally aligned coating" was introduced in [**10**]; it refers to the following situation: for $x \in \Omega_2$, let p be the projection of x on $\partial\Omega_1$. Then the vector $\overline{px}$ is an eigenvector corresponding to the smallest eigenvalue of the thermal tensor $(a_{ij}(x))$. Results of this kind seem to indicate that, when designing an insulator, thermal consideration needs to be taken only in the normal direction, thus leaving room for mechanical considerations in the tangential directions.

The "interior reinforcement problem" for elliptic and parabolic equations was first studied by Sanchez-Palencia [**11**] in a special case where the reinforcing material occupies a lens-shaped domain. Brezis, Caffarelli and Friedman [**1**] studied the elliptic problem with Dirichlet boundary condition in both cases of interior and boundary reinforcement. See also Buttazzo and Kohn [**2**] for the case of rapidly oscillating thickness of the coating. Our results are the counterpart of the elliptic one in [**1**]. However, this prior work did not cover the case of optimally aligned coating, and here we use the method developed in [**8**], which involves only the parabolic $W_2^{1,1}$ a priori estimates, while the methods of [**1, 2**] are based on H^2 estimates and Γ-convergence, respectively. (In this connection, we mention that one of the authors [**7**] has recently improved some of the results in [**1**] by using only H^1 estimates.) Although here we use the framework established in our previous paper [**8**], the adaptation to our current situation is nontrivial; for example, the proof of Theorem 2.2 (ii) involves a new, subtly chosen test function (in (2.24)). The proof of (iii) of the same theorem involves overcoming the difficulty caused by lack of a global $W^{1,1}$ a priori estimate (up to $t = 0$).

The results of the present paper should be compared to those obtained in our previous papers [**10, 12**], where we studied the thermal insulation problem by analyzing Dirichlet and Robin eigenpairs. The basis of these papers is the following: the heat equation can be solved by the eigenexpansion method; for example, the Robin problem (1.2) can be solved by

$$Q(x,t) = H + \sum_{m\geq 1} e^{-\lambda_m t}\phi_m(x)\int_\Omega \phi_m(x')(Q_0(x') - H)\,dx' \tag{1.3}$$

where (λ_m, ϕ_m), $m = 1, 2, \ldots,$ are the eigenvalues and normalized eigenfunctions of the Robin eigenvalue problem

$$\begin{cases} -\nabla\cdot\big(A\nabla u\big) = \lambda u & x \in \Omega, \\ (\partial u/\partial \overline{\nu}_A) + \eta u = 0 & x \in \partial\Omega. \end{cases} \tag{1.4}$$

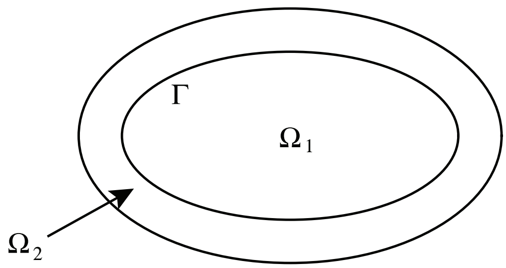

FIGURE 1. $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$.

Thus, $Q(x,t) \to H$ as $t \to \infty$. To protect the body Ω_1 from overheating, we need to slow down the rate of convergence of Q to H. We argue that the following help to slow it down:

(A) As many λ_m as possible should be small, in particular, the first eigenvalue λ_1 should be small;

(B) The first eigenfunction should take large values on the body Ω_1;

(C) The higher eigenfunctions take small absolute values on Ω_1.

In [**12**] we proved that, roughly speaking, λ_1 is small and (B) occurs if either $\alpha = 0$ or $\beta = 0$; all (A)–(C) occur if

$$\frac{\sigma}{\delta^2} \longrightarrow 0 \quad \text{as } \delta \to 0. \tag{1.5}$$

Recall that either $\alpha = 0$ or $\beta = 0$ is the condition that is needed in the current paper to ensure the effective Neumann boundary condition; condition (1.5) is not needed here.

For results on Dirichlet eigenpairs, see [**10**] and the earlier papers by Friedman [**3**] and Panasenko [**9**].

2. Preliminaries and main results. Let $u(x,t) = Q(x,t) - H$, $\varphi(x) = Q_0(x) - H$. Then (1.2) with a given source term is transformed to

$$\begin{cases} u_t - \nabla \cdot (A \nabla u) = f & (x,t) \in Q_T, \\ (\partial u / \partial \overline{\nu}_A) + \eta u = 0 & (x,t) \in S_T, \\ u = \varphi(x) & x \in \Omega,\ t = 0, \end{cases} \tag{2.1}$$

where $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, with T being positive and finite. Let

$$W_2^{1,1}(Q_T) = \left\{u : u, \nabla_x u, u_t \in L^2(Q_T)\right\}$$

be the Hilbert space equipped with the norm

$$\|u\|^2_{W_2^{1,1}(Q_T)} \triangleq \|u\|^2_{L^2(Q_T)} + \|\nabla_x u\|^2_{L^2(Q_T)} + \|u_t\|^2_{L^2(Q_T)}.$$

If $\partial\Omega$ is C^1 smooth, it is well known that

(i) the set of C^∞ smooth functions is dense in $W_2^{1,1}(Q_T)$, and

(ii) $W_2^{1,1}(Q_T)$ is continuously embedded in $C([0,T]; L^2(\Omega))$. We will also need the Hilbert space

$$W_2^{1,0}(\Omega_1 \times (0,T)) \triangleq \left\{u : u, \nabla_x u \in L^2(\Omega_1 \times (0,T))\right\}$$

equipped with the norm

$$\|u\|^2_{W_2^{1,0}(\Omega_1\times(0,T))} \triangleq \|u\|^2_{L^2(\Omega_1\times(0,T))} + \|\nabla_x u\|^2_{L^2(\Omega_1\times(0,T))}.$$

If $\partial\Omega_1$ is C^1 smooth, then the set of C^∞ functions is also dense in $W_2^{1,0}(\Omega_1 \times (0,T))$. Recall that there exists a bounded trace operator from $H^1(\Omega_1)$ to $L^2(\partial\Omega_1)$. Therefore, there exists such an operator from $W_2^{1,0}(\Omega_1 \times (0,T))$ to $L^2(\partial\Omega_1 \times (0,T))$.

The weak solution of (2.1) is a function u satisfying

(i) $u \in W_2^{1,1}(Q_T)$ and $u(\cdot,t) \to \varphi$ in $L^2(\Omega)$ as $t \to 0^+$;

(ii) for any $v \in W_2^{1,1}(Q_T)$, we have

$$\int_0^T \left[\int_\Omega \left(u_t v + a_{ij}(x) u_{x_i} v_{x_j}\right)\, dx + \eta \int_{\partial\Omega} uv\, dS\right] dt = \int_0^T \int_\Omega fv\, dx\, dt.$$

It is easy to see that (i) and (ii) are equivalent to: $u \in W_2^{1,1}(Q_T)$; and for any $v \in W_2^{1,1}(Q_T)$ which is equal to 0 when $t = T$, we have

$$\begin{aligned}\text{(2.2)}\qquad &\int_0^T \left[\int_\Omega \left(-uv_t + a_{ij}(x) u_{x_i} v_{x_j}\right)\, dx + \eta \int_{\partial\Omega} uv\, dS\right] dt \\ &\qquad - \int_\Omega \varphi(x) v(x,0)\, dx = \int_0^T \int_\Omega fv\, dx\, dt.\end{aligned}$$

We can prove, via eigenexpansion that, if $f \in L^2(Q_T)$ and $\varphi \in H^1(\Omega)$, then (2.1) has one and only one weak solution u which is also in $C([0,T];H^1(\Omega))$. The weak solution, in the case when (1.1) holds, satisfies the "transmission condition"on $\Gamma \times (0,T]$ in the weak sense:

$$u_1 = u_2, \qquad k\frac{\partial u_1}{\partial \nu} = \sigma \frac{\partial u_2}{\partial \nu_{\overline{A}}} \tag{2.3}$$

where u_i is the restriction of u on $\Omega_i \times (0,T)$, and the conormal derivative $(\partial u/\partial \nu_{\overline{A}}) = u_{x_i}\overline{a}_{ij}\nu_j$. Furthermore, if $\overline{a}_{ij} \in C^1(\overline{\Omega}_2)$, both $\partial\Omega_i$, $i = 1, 2$, are C^2 smooth, and both f and f_t are bounded on Q_T, then $\nabla_x u$ is Hölder continuous in each $\overline{\Omega}_i \times (0,T]$, and the "transmission condition" holds on $\Gamma \times (0,T]$ in the classical sense (see [**5**, Theorem 7]).

The coating Ω_2 and its thickness are precisely defined as follows: for any small positive δ, define

$$\begin{gathered} F : (p,\tau) \in \Gamma \times [-\delta,\delta] \longrightarrow (x_1,\dots,x_n) \in \mathbf{R}^n, \\ (x_1,\dots,x_n) = F(p,\tau) = p + \tau\nu(p). \end{gathered} \tag{2.4}$$

By [**4**, Lemma 14.16], if $\Gamma \in C^2$, and δ is small enough, then F is a diffeomorphism; moreover, if we take

$$\Omega_2 = F(\Gamma \times (0,\delta)),$$

then $\partial\Omega_2 \in C^2$. In this paper, we assume that Ω_2 is obtained in this way; and we say that the thickness of the thin insulator is δ. For the sake of simplicity, we assume that Γ is connected.

The following $W_2^{1,1}$ a priori estimates are the basis for our asymptotic analysis and are a part of the standard parabolic theory (see [**6**, pages 172–175] for the case of the Dirichlet boundary condition).

Lemma 2.1. *Suppose that* (1.1) *holds*; $\Gamma \in C^2$, $\overline{a}_{ij} \in C^1(\overline{\Omega}_2)$, $f \in L^2(Q_T)$ *and* $\varphi \in H^1(\Omega)$; *and that* $\overline{a}_{ij}$, f *and* φ *are fixed on* Ω *and*

σ is bounded, as Ω_2 shrinks. Then the weak solution u of (2.1) *satisfies*

$$\max_{0\le t\le T}\int_\Omega u^2(x,t)\,dx \le C, \tag{2.5}$$

$$\int_0^T\int_\Omega |u_t|^2 dx\,dt + \max_{0\le t\le T}\Big[k\int_{\Omega_1}|\nabla u|^2 dx + \sigma\int_{\Omega_2}|\nabla u|^2 dx + \eta\int_{\partial\Omega}u^2(p)\,dS_p\Big]$$

(2.6)

$$\le C\Big(k\int_{\Omega_1}|\nabla\varphi|^2 dx + \sigma\int_{\Omega_2}|\nabla\varphi|^2 dx + \eta\int_{\partial\Omega}\varphi^2 dS_p + \int_0^T \mathcal{Q}\int_\Omega f^2 dx\,dt\Big),$$

(2.7)

$$\int_0^T\Big[k\int_{\Omega_1}|\nabla u|^2 dx + \sigma\int_{\Omega_2}|\nabla u|^2 dx + \eta\int_{\partial\Omega}u^2(p)\,dS_p\Big]\,dt \le C,$$

for some constant C independent of σ and δ (but dependent upon T, φ and f).

We are now ready to establish our main results.

Theorem 2.2. *Suppose that all the conditions in Lemma* 2.1 *hold.*

(i) *Assume that*

$$\lim_{\delta\to 0^+}\eta = 0.$$

Then the solution $u(x,t)\to w(x,t)$ strongly in $L^2(\Omega_1\times(0,T))$ as $\delta\to 0^+$, where $w(x,t)$ is the solution of the Neumann problem,

$$\begin{cases} w_t - k\Delta w = f & x\in\Omega_1,\ t>0,\\ w=\varphi(x) & x\in\Omega_1,\ t=0,\\ (\partial w/\partial\nu)=0 & x\in\Gamma,\quad t>0.\end{cases} \tag{2.8}$$

(ii) *Assume that*

$$\lim_{\delta\to 0^+}\eta = \beta\in(0,+\infty),\quad \textit{and}\quad \lim_{\delta\to 0^+}\frac{\sigma}{\delta}=\alpha\in[0,\infty].$$

Then the solution $u(x,t)\to w(x,t)$ strongly in $L^2(\Omega_1\times(0,T))$ as $\delta\to 0^+$, where $w(x,t)$ is the solution of

$$\begin{cases} w_t - k\Delta w = f & x\in\Omega_1,\ t>0,\\ w=\varphi(x) & x\in\Omega_1,\ t=0,\\ k(\partial w/\partial\nu)+[(\beta\alpha\nu\cdot\nu_{\overline{A}})/(\beta+\alpha\nu\cdot\nu_{\overline{A}})]w=0 & x\in\Gamma,\quad t>0.\end{cases} \tag{2.9}$$

If $\alpha = +\infty$, then the boundary condition is understood as $k(\partial w/\partial \nu) + \beta w = 0$.

(iii) *Assume that $\beta = \infty$ and $\alpha \in [0, \infty]$. Then the convergence of u to w holds strongly in $L^2(\Omega_1 \times (0,T))$, and the boundary condition in* (2.9) *is changed to $k(\partial w/\partial \nu) + \alpha \nu \cdot \nu_{\overline{A}} w = 0$ if $\alpha < +\infty$, $w = 0$ if $\alpha = +\infty$.*

Proof of (i). By (2.5) and (2.6), for all small $\delta > 0$,

$$\|u\|_{W_2^{1,1}(\Omega_1 \times (0,T))} \leq C. \tag{2.10}$$

Thus, for some $w \in W_2^{1,1}(\Omega_1 \times (0,T))$, we have that $u(x,t) \to w(x,t)$ strongly in $L^2(\Omega_1 \times (0,T))$, and weakly in $W_2^{1,1}(\Omega_1 \times (0,T))$, after passing to a subsequence of $\delta \to 0^+$. (Here we are only using the assumption that $\beta < \infty$, i.e., η is bounded; thus, the above convergence statement holds in part (ii).)

We shall eventually show that w is a weak solution of (2.8). Since the weak solution of (2.8) is unique, the convergence statement made above holds without passing to a subsequence of $\delta \to 0^+$.

For any $\zeta \in W_2^{1,1}(\Omega_1 \times (0,T))$ that is equal to 0 when $t = T$, extend ζ to $\mathbf{R}^n \times (0,T)$ such that $\zeta \in W_2^{1,1}(\Omega \times (0,T))$ and $\zeta = 0$ when $t = T$. By (2.2), we have

$$\begin{aligned} \int_0^T \Big[-\int_{\Omega_1} u\zeta_t\,dx - \int_{\Omega_2} u\zeta_t\,dx + k\int_{\Omega_1} \nabla u \cdot \nabla \zeta\,dx \\ + \sigma \int_{\Omega_2} \overline{a}_{ij}(x) u_{x_i} \zeta_{x_j}\,dx + \eta \int_{\partial\Omega} u\zeta\,dS \Big]\,dt \\ - \int_{\Omega} \varphi(x)\zeta(x,0)\,dx \\ = \int_0^T \Big[\int_{\Omega_1} f\zeta\,dx + \int_{\Omega_2} f\zeta\,dx \Big]\,dt. \end{aligned} \tag{2.11}$$

From Cauchy's inequality and Lemma 2.1, it follows that, as $\delta \to 0$,

$$\left| \int_0^T \int_{\Omega_2} u\zeta_t \right| \leq \left(\int_0^T \int_{\Omega_2} u^2 \right)^{1/2} \left(\int_0^T \int_{\Omega_2} \zeta_t^2 \right)^{1/2} = o(1),$$

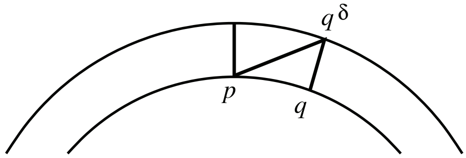

FIGURE 2.

$$\left|\sigma\int_0^T\int_{\Omega_2}\overline{a}_{ij}(x)u_{x_i}\zeta_{x_j}\right|\le\left(\sigma\int_0^T\int_{\Omega_2}\overline{a}_{ij}(x)u_{x_i}u_{x_j}\right)^{1/2}$$
$$\times\left(\sigma\int_0^T\int_{\Omega_2}\overline{a}_{ij}(x)\zeta_{x_i}\zeta_{x_j}\right)^{1/2}=o(1),$$
$$\left|\eta\int_0^T\int_{\partial\Omega}u\zeta\right|\le\left(\eta\int_0^T\int_{\partial\Omega}u^2\right)^{1/2}\left(\eta\int_0^T\int_{\partial\Omega}\zeta^2\right)^{1/2}=o(1),$$
$$\left|\int_0^T\int_{\Omega_2}f\zeta\right|\le\left(\int_0^T\int_{\Omega_2}f^2\right)^{1/2}\left(\int_0^T\int_{\Omega_2}\zeta^2\right)^{1/2}=o(1).$$

Letting $\delta\to 0$ in (2.11), we thus conclude that

$$\int_0^T\left[-\int_{\Omega_1}w\zeta_t\,dx+k\int_{\Omega_1}\nabla w\cdot\nabla\zeta\,dx\right]dt-\int_{\Omega_1}\varphi(x)\zeta(x,0)\,dx$$
$$=\int_0^T\int_{\Omega_1}f\zeta\,dx\,dt.$$

This shows that w is the weak solution of (2.8). $\square$

Proof of (ii). We first assume that α is finite. The proof starts with a carefully chosen test function (to be used in (2.2)). To this end, we introduce a new coordinate system in Ω_2. For any point $p\in\Gamma$, let l be the ray in the conormal direction $\nu_{\overline{A}}$ initiating at p. It intersects $\partial\Omega$ at a point q^δ. Denote by q the projection of q^δ on Γ, and so $q^\delta=q+\delta\nu(q)$ (see Figure 2). Denote by $h(p)$ the distance between p and q^δ. Now reparameterize Ω_2 by

$$(x_1,\dots,x_n)=\overline{F}(p,s)=p+s\frac{\nu_{\overline{A}}}{|\nu_{\overline{A}}|}(p), \tag{2.12}$$

with $s\in(0,h(p))$.

For any $\zeta(x,t) \in C^1(\overline{\Omega}_1 \times [0,T])$ that is equal to zero when $t = T$, in the x variable extend $\zeta(x,t)$ along $\nu_{\overline{A}}$ to Ω_2 by $\zeta((p,s),t) = \zeta(p,t)$ for every $p \in \Gamma$. Define

$$\phi(x) \triangleq \begin{cases} 1 - \frac{\tau}{\delta} & 0 \le \tau \le \delta,\, p \in \Gamma, \\ 1 & \text{otherwise}. \end{cases} \tag{2.13}$$

Here, for ζ, we have used the coordinates (2.12); for ϕ, we have used the coordinates (2.4). It is easy to verify that $\zeta(x,t) \in W_2^{1,1}(\Omega \times (0,T))$, and that

$$\nabla\phi(x) = \frac{\partial \phi}{\partial \tau}\nu(p) = -\frac{\nu(p)}{\delta}. \tag{2.14}$$

By choosing the test function in (2.2) to be $\zeta\phi$, we have

$$\begin{aligned} \int_0^T \Big[-\int_{\Omega_1} u\zeta_t\,dx - \int_{\Omega_2} u\phi\zeta_t\,dx + k\int_{\Omega_1} \nabla u\cdot\nabla\zeta\,dx \\ + \sigma \int_{\Omega_2} \overline{a}_{ij}(x) u_{x_i}(\phi\zeta)_{x_j}\,dx \Big]\,dt \\ - \int_\Omega \varphi(x)\phi(x)\zeta(x,0)\,dx \\ = \int_0^T \Big[\int_{\Omega_1} f\zeta\,dx + \int_{\Omega_2} f\phi\zeta\,dx \Big]\,dt. \end{aligned} \tag{2.15}$$

As in the proof of (i), all the integrals on Ω_2, except that of the $\overline{a}_{ij}$-term, shrink as $\delta \to 0$. To study that exceptional integral, we need the fact

$$\frac{h(p)}{\delta} \longrightarrow \frac{|\nu_{\overline{A}}(p)|}{\nu(p)\cdot\nu_{\overline{A}}(p)} \tag{2.16}$$

uniformly for p on Γ as $\delta \to 0$, which was proved in [**8**].

Let dS_p denote the surface element on Γ at p. It was shown in [**8**] (in the proof of Theorem 2.2 there) that the volume element on Ω_2 at $\overline{F}(p,s)$ is

$$dx_1\cdots dx_n = \left(\frac{\nu\cdot\nu_{\overline{A}}}{|\nu_{\overline{A}}|} + O(1)s\right) dS_p\,ds. \tag{2.17}$$

A similar calculation gives that the surface element on $\partial\Omega$ at q^δ is

$$dS_{q^\delta} = (1 + O(1)\delta)\, dS_p. \tag{2.18}$$

Using (2.14) and (2.17), we compute

$$\begin{aligned}
&\sigma \int_0^T \int_{\Omega_2} \overline{a}_{ij} u_{x_i} \phi_{x_j} \zeta \, dx\, dt \\
&= -\frac{\sigma}{\delta} \int_0^T \int_{\Omega_2} \overline{a}_{ij} u_{x_i} \nu_j \zeta \, dx\, dt \\
&= -\frac{\sigma}{\delta} \int_0^T \int_\Gamma \int_0^{h(p)} [\overline{a}_{ij}(p,s,t) - \overline{a}_{ij}(p,0,t)] \\
&\qquad \times u_{x_i} \nu_j \zeta \left(\frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} + O(s) \right) ds\, dS_p\, dt \\
&\quad - \frac{\sigma}{\delta} \int_0^T \int_\Gamma \int_0^{h(p)} \overline{a}_{ij}(p,0,t) \\
&\qquad \times u_{x_i} \nu_j \zeta \left(\frac{\nu \cdot \nu_{\overline{A}}}{|\nu_{\overline{A}}|} + O(s) \right) ds\, dS_p\, dt.
\end{aligned} \tag{2.19}$$

From Lemma 2.1, (2.16) and (2.17), it follows that

$$\begin{aligned}
&\frac{\sigma}{\delta} \int_0^T \int_\Gamma \int_0^{h(p)} s\, |\zeta \nabla u_2| \, ds\, dS_p\, dt \\
&\le C\sigma \left(\int_0^T \int_{\Omega_2} |\nabla u|^2 \right)^{1/2} \left(\int_0^T \int_{\Omega_2} |\zeta|^2 \right)^{1/2} = o(1).
\end{aligned} \tag{2.20}$$

Set $(\partial u_2/\partial \nu_{\overline{A}}) \triangleq \nabla u_2 \cdot \nu_{\overline{A}}$; then

$$u(p,t) - u(q^\delta, t) = -\int_0^{h(p)} \frac{1}{|\nu_{\overline{A}}|} \cdot \frac{\partial u_2}{\partial \nu_{\overline{A}}}\, ds, \tag{2.21}$$

for $t \in [0,T]$. Observe that $|\overline{a}_{ij}(p,s,t) - \overline{a}_{ij}(p,0,t)| \le Cs$, $\overline{a}_{ij}(p,0,t)\nu_j = \nu_{\overline{A}}$, and $\zeta(p,s,t) = \zeta(p,0,t)$. It follows from (2.19), together with (2.20) and (2.21), that

$$\begin{aligned}
&\sigma \int_0^T \int_{\Omega_2} \overline{a}_{ij} u_{x_i} \phi_{x_j} \zeta \, dx\, dt \\
&= \frac{\sigma}{\delta} \int_0^T \int_\Gamma \nu \cdot \nu_{\overline{A}} u(p,t) \zeta(p,t)\, dS_p\, dt \\
&\quad - \frac{\sigma}{\delta} \int_0^T \int_\Gamma \nu \cdot \nu_{\overline{A}} u(q^\delta, t) \zeta(p,t)\, dS_p\, dt + o(1).
\end{aligned}$$

This estimate, in combination with (2.18) and the fact that $\sigma \to 0$ (implied by the assumption that α is finite), leads to

$$\begin{aligned}(2.22)\quad \sigma\int_0^T\int_{\Omega_2}\overline{a}_{ij}u_{x_i}\phi_{x_j}\zeta\,dx\,dt &= \frac{\sigma}{\delta}\int_0^T\int_\Gamma \nu\cdot\nu_{\overline{A}}u(p,t)\zeta(p,t)\,dS_p\,dt\\ &\quad -\frac{\sigma}{\delta}\int_0^T\int_{\partial\Omega}\nu\cdot\nu_{\overline{A}}u(q^\delta,t)\zeta(q^\delta,t)\,dS_{q^\delta}\,dt+o(1).\end{aligned}$$

Now putting (2.22) back to (2.15), we have

$$\begin{aligned}(2.23)\quad &\int_0^T\Big[-\int_{\Omega_1}u\zeta_t\,dx+k\int_{\Omega_1}\nabla u\cdot\nabla\zeta\,dx\\ &\qquad +\frac{\sigma}{\delta}\int_\Gamma\nu\cdot\nu_{\overline{A}}u(p,t)\zeta(p,t)\,dS_p\\ &\qquad -\frac{\sigma}{\delta}\int_{\partial\Omega}\nu\cdot\nu_{\overline{A}}u(q^\delta,t)\zeta(q^\delta,t)\,dS_{q^\delta}\Big]\,dt\\ &-\int_{\Omega_1}\varphi(x)\zeta(x,0)\,dx=\int_0^T\int_{\Omega_1}f\zeta\,dx\,dt+o(1).\end{aligned}$$

On the other hand, because of $\nu\cdot\nu_{\overline{A}}>0$, we can extend $\nu\cdot\nu_{\overline{A}}$ onto Ω_1 (still denoted by $\nu\cdot\nu_{\overline{A}}$) so that $\nu\cdot\nu_{\overline{A}}>0$ in Ω_1. ($\nu\cdot\nu_{\overline{A}}$ is extended onto Ω_2 the same way as for ζ.) Now in (2.2), use $\nu\cdot\nu_{\overline{A}}\zeta$ as the test function. Similarly to the proof of (i), we get

$$\begin{aligned}(2.24)\quad &\int_0^T\Big[-\int_{\Omega_1}u\zeta_t\nu\cdot\nu_{\overline{A}}\,dx+k\int_{\Omega_1}\nabla u\cdot\nabla(\nu\cdot\nu_{\overline{A}}\zeta)\,dx\\ &\qquad +\eta\int_{\partial\Omega}\nu\cdot\nu_{\overline{A}}u(q^\delta,t)\zeta(q^\delta,t)\,dS_{q^\delta}\Big]\,dt\\ &-\int_{\Omega_1}\varphi(x)\zeta(x,0)\nu\cdot\nu_{\overline{A}}\,dx=\int_0^T\int_{\Omega_1}f\zeta\nu\cdot\nu_{\overline{A}}\,dx\,dt+o(1).\end{aligned}$$

Combining (2.23) with (2.24), we obtain

$$\begin{aligned}\int_0^T \Big[&-\int_{\Omega_1} u\Big(1+\frac{\sigma}{\eta\delta}\nu\cdot\nu_{\overline{A}}\Big)\zeta_t\,dx \\ &+ k\int_{\Omega_1} \nabla u\cdot\nabla\Big(\Big(1+\frac{\sigma}{\eta\delta}\nu\cdot\nu_{\overline{A}}\Big)\zeta\Big)\,dx \\ &+ \frac{\sigma}{\delta}\int_{\Gamma} u(p,t)\nu\cdot\nu_{\overline{A}}\zeta(p,t)\,dS_p\Big]\,dt \\ &-\int_{\Omega_1}\varphi\Big(1+\frac{\sigma}{\eta\delta}\nu\cdot\nu_{\overline{A}}\Big)\zeta(x,0)\,dx \\ &= \int_0^T\int_{\Omega_1} f\Big(1+\frac{\sigma}{\eta\delta}\nu\cdot\nu_{\overline{A}}\Big)\zeta\,dx\,dt + \Big(1+\frac{\sigma}{\eta\delta}\Big)o(1).\end{aligned} \tag{2.25}$$

The first three integrals, including the boundary integral, on the left hand side of the above equation, are bounded linear functionals of $u \in W_2^{1,1}(\Omega_1)$. Thus, by weak convergence of u to w, letting $\delta \to 0^+$, we have that

$$\begin{aligned}\int_0^T \Big[&-\int_{\Omega_1} w\Big(1+\frac{\alpha}{\beta}\nu\cdot\nu_{\overline{A}}\Big)\zeta_t\,dx \\ &+ k\int_{\Omega_1} \nabla w\cdot\nabla\Big(\Big(1+\frac{\alpha}{\beta}\nu\cdot\nu_{\overline{A}}\Big)\zeta\Big)\,dx \\ &+ \alpha\int_{\Gamma} w\nu\cdot\nu_{\overline{A}}\zeta\,dS_p\Big]\,dt \\ &-\int_{\Omega_1}\varphi\Big(1+\frac{\alpha}{\beta}\nu\cdot\nu_{\overline{A}}\Big)\zeta(x,0)\,dx \\ &= \int_0^T\int_{\Omega_1} f\Big(1+\frac{\alpha}{\beta}\nu\cdot\nu_{\overline{A}}\Big)\zeta\,dx\,dt.\end{aligned}$$

Set $\Psi \triangleq (1+(\alpha/\beta)\nu\cdot\nu_{\overline{A}})\zeta$; then $\zeta = [\Psi/(1+(\alpha/\beta)\nu\cdot\nu_{\overline{A}})]$, and

$$\begin{aligned}\int_0^T &\Big[-\int_{\Omega_1} w\Psi_t\,dx + k\int_{\Omega_1}\nabla w\cdot\nabla\Psi\,dx + \int_\Gamma w\frac{\beta\alpha\nu\cdot\nu_{\overline{A}}}{\beta+\alpha\nu\cdot\nu_{\overline{A}}}\Psi\,dS_p\Big]\,dt \\ &-\int_{\Omega_1}\varphi\Psi(x,0)\,dx \\ &= \int_0^T\int_{\Omega_1} f\Psi\,dx\,dt,\end{aligned}$$

which means that w is the weak solution of (2.9). Item (ii) is proved for finite α.

If $\alpha = +\infty$, by (2.2) we have

$$\begin{aligned}
&\int_0^T \Big[-\int_{\Omega_1} u\zeta_t \, dx + k\int_{\Omega_1} \nabla u \nabla \zeta \, dx \\
&\qquad + \eta \int_{\partial\Omega} u(q^\delta, t)\zeta(q^\delta, t)\, dS_{q^\delta} \Big] dt \\
&- \int_{\Omega_1} \varphi\zeta(x,0)\, dx \\
&\qquad = \int_0^T \int_{\Omega_1} f\zeta \, dx\, dt + o(1).
\end{aligned} \tag{2.26}$$

We claim

$$\eta \int_0^T \int_{\partial\Omega} u(q^\delta, t)\zeta(q^\delta, t)\, dS_{q^\delta}\, dt = \eta \int_0^T \int_\Gamma u(p,t)\zeta(p,t)\, dS_p\, dt + o(1). \tag{2.27}$$

In fact,

$$\begin{aligned}
&\left| \eta \int_0^T \int_{\partial\Omega} u(q^\delta, t)\zeta(q^\delta, t)\, dS_{q^\delta}\, dt - \eta \int_0^T \int_\Gamma u(p,t)\zeta(p,t)\, dS_p\, dt \right| \\
&\qquad \le \left| \eta \int_0^T \int_\Gamma \zeta(p,t)[u(p,t) - u(q^\delta, t)]\, dS_p\, dt \right| \\
&\qquad\quad + C\eta \left| \int_0^T \int_\Gamma \delta u(q^\delta, t)\, dS_p dt \right| \\
&\qquad \le C\eta \Big(\int_0^T \int_\Gamma |u(p,t) - u(q^\delta,t)|^2 dS_p\, dt \Big)^{1/2} \\
&\qquad\quad + C\eta\delta \Big(\int_0^T \int_{\partial\Omega} |u|^2 dS_{q^\delta}\, dt \Big)^{1/2} \\
&\qquad \le C\eta \Big(\delta \int_0^T \int_{\Omega_2} |\nabla u|^2 dx\, dt \Big)^{1/2} + o(1) \\
&\qquad \le C\eta \cdot \frac{\delta^{1/2}}{\sigma^{1/2}} + o(1) \\
&\qquad = o(1).
\end{aligned}$$

Here we have used (2.18) in the first inequality, (2.21) in the third inequality and Lemma 2.1 in the fourth inequality. Now, substituting (2.27) into (2.26) and passing to the limit in (2.26), we arrive at

$$\int_0^T \left[-\int_{\Omega_1} w\zeta_t \, dx + k\int_{\Omega_1} \nabla w \cdot \nabla \zeta \, dx + \int_\Gamma \beta w \zeta \, dS_p \right] dt \\ - \int_{\Omega_1} \varphi \zeta(x,0) \, dx = \int_0^T \int_{\Omega_1} f\zeta \, dx \, dt,$$

which means w is a weak solution of (2.9) with the boundary condition understood as $k(\partial w/\partial \nu) + \beta w = 0$. □

Proof of (iii). By (2.5) and (2.7), as $\delta \to 0$, u is bounded in $W_2^{1,0}(\Omega_1 \times (0,T))$, and thus u converges to some w weakly in this space and also in $L^2(\Omega_1 \times (0,T))$ after passing to a subsequence.

We claim that u converges to w strongly in $L^2(\Omega_1 \times (0,T))$. Actually, by (2.5), for any $\varepsilon > 0$, there exists a small $\tau > 0$ such that

$$\int_0^\tau \int_\Omega |u|^2 dx \, dt \le \varepsilon. \tag{2.28}$$

On the other hand, in view of (2.7), for this fixed $\tau > 0$

$$\int_0^\tau \left[k\int_{\Omega_1} |\nabla u|^2 dx + \sigma \int_{\Omega_2} |\nabla u|^2 dx + \eta \int_{\partial\Omega} u^2 dS_p \right] dt \le C.$$

Note that u is in $C([0,T]; H^1(\Omega))$; thus, there exists a $\tau' \in (0,\tau)$ such that

$$k\int_{\Omega_1} |\nabla u(\tau')|^2 dx + \sigma \int_{\Omega_2} |\nabla u(\tau')|^2 dx + \eta \int_{\partial\Omega} u^2(\tau') \, dS_p \le \frac{C}{\tau}$$

for some constant C independent of δ. As for (2.6), we also have

$$\begin{aligned}\int_{\tau'}^{T}\int_{\Omega}|u_t|^2dx\,dt&\\ &+\max_{\tau'\le t\le T}\left[k\int_{\Omega_1}|\nabla u|^2dx+\sigma\int_{\Omega_2}|\nabla u|^2dx+\eta\int_{\partial\Omega}u^2dS_p\right]\\ &\le C\bigg(k\int_{\Omega_1}|\nabla u(\tau')|^2dx\\ &\quad+\sigma\int_{\Omega_2}|\nabla u(\tau')|^2dx+\eta\int_{\partial\Omega}u^2(\tau')\,dS_p\bigg)\\ &\quad+C\int_{\tau'}^{T}\int_{\Omega}f^2dx\,dt\\ &\le\frac{C}{\tau}.\end{aligned}$$

This in combination with (2.5) leads to

$$\|u\|_{W_2^{1,1}(\Omega_1\times(\tau,T))}\le\frac{C}{\tau}.$$

Thus, $u\to w$ strongly in $L^2(\Omega_1\times(\tau,T))$ as $\delta\to 0$. Due to (2.28), $u\to w$ strongly in $L^2(\Omega_1\times(0,T))$, which is the claim.

If α is finite, then we still have (2.23). By (2.7) and Cauchy's inequality

$$\left|\frac{\sigma}{\delta}\int_0^T\int_{\partial\Omega}\nu\cdot\nu_{\overline{A}}u(q^\delta,t)\zeta(q^\delta,t)\,dS_{q^\delta}\,dt\right|\le\frac{C}{\sqrt{\eta}}=o(1).$$

The first two integrals in (2.23) are bounded functionals on $u\in W_2^{1,0}(\Omega_1\times(0,T))$; the same can be said about the boundary integral there because of the existence of a bounded trace operator from $W_2^{1,0}(\Omega_1\times(0,T))$ to $L^2(\partial\Omega_1\times(0,T))$. Now, by the convergence of u to w, we can pass the limit in (2.23) to get

$$\begin{aligned}\int_0^T\left[-\int_{\Omega}w\zeta_t\,dx+k\int_{\Omega_1}\nabla w\cdot\nabla\zeta\,dx+\int_{\Gamma}\alpha\nu\cdot\nu_{\overline{A}}w\zeta\,dS_p\right]dt&\\ -\int_{\Omega_1}\varphi(x)\zeta(x,0)\,dx&\\ =\int_0^T\int_{\Omega_1}f\zeta\,dx\,dt.&\end{aligned}$$

This shows that w is a "generalized solution" of (2.9) with the boundary condition understood as $k(\partial w/\partial \nu) + \alpha\nu \cdot \nu_{\overline{A}} w = 0$, as defined in [**6**, Section 5, Chapter III]. (Note that, at this point, we only know that $w \in W_2^{1,0}(\Omega_1 \times (0,T))$.) On the other hand, by the eigenexpansion method, (2.9) with the boundary condition just mentioned has a weak solution (in $W_2^{1,1}(\Omega_1 \times (0,T))$, as defined in the same spirit of (2.2)); this weak solution is a generalized one. Since we have the uniqueness of generalized solutions [**6**], w must be the weak solution. We have proved (iii) in the case of finite α.

Now, if $\alpha = \infty$, we take the test function ζ in (2.2) such that it is equal to zero on $\partial\Omega_1 \times (0,T)$ and when $t = T$. Then we have

$$\int_0^T \left[-\int_{\Omega_1} u\zeta_t \, dx + k\int_{\Omega_1} \nabla u \nabla \zeta \, dx \right] dt - \int_{\Omega_1} \varphi\zeta(x,0)\, dx \\ = \int_0^T \int_{\Omega_1} f\zeta \, dx \, dt.$$

Passing to the limit $\delta \to 0$, we are led to

$$\int_0^T \left[-\int_{\Omega_1} w\zeta_t \, dx + k\int_{\Omega_1} \nabla w \cdot \nabla \zeta \, dx \right] dt - \int_{\Omega_1} \varphi\zeta(x,0)\, dx \\ = \int_0^T \int_{\Omega_1} f\zeta \, dx \, dt.$$

We argue that the trace of w on $\partial\Omega_1 \times (0,T)$ is equal to zero (and hence w is the weak solution of (2.9) with Dirichlet condition) as follows. From (2.7) and (2.21), it follows that

$$\int_0^T \int_\Gamma u^2 dS_p \, dt \le C\int_0^T \int_{\partial\Omega} u^2 dS_p \, dt + C\delta \int_0^T \int_{\Omega_2} |\nabla u|^2 \, dx \, dt \\ \le \frac{C}{\eta} + \frac{C\delta}{\sigma} \to 0.$$

This and the convergence of u to w imply

$$\int_0^T \int_\Gamma w^2 dS_p \, dt = \lim_{\delta \to 0} \int_0^T \int_\Gamma wu \, dS_p \, dt = 0.$$

The proof of (iii) is complete. □

Recall that we say the coating Ω_2 is optimally aligned if, at every $x \in \Omega_2$, the vector $\overline{px}$ is an eigenvector of the thermal tensor $(a_{ij}(x))$ corresponding to the smallest eigenvalue $\sigma_{\min}(x)$, where p is the projection of x onto $\partial\Omega_1$, i.e., $x = p + \tau\nu(p)$. $\tau > 0$.

Theorem 2.3. *Assume that* $\Gamma \in C^2$, $a_{ij} \in C^1(\overline{\Omega}_2)$, $f \in L^2(Q_T)$ *and* $\varphi \in H^1(\Omega)$ *with* f *and* φ *being fixed on* Ω *as* $\delta \to 0$. *Suppose that the coating* Ω_2 *is optimally aligned, and that*

$$\sigma_{\min}(x) = \sigma\gamma(x), \tag{2.29}$$

where σ *is a bounded parameter and* $\gamma(x) > 0$ *is a fixed* C^1 *continuous function defined on* $\overline{\Omega}_2$. *Let the largest eigenvalue of* (a_{ij}) *be bounded on* Ω_2 *as* Ω_2 *shrinks.*

(i) *If*

$$\lim_{\delta \to 0^+} \eta = 0,$$

then the solution $u(x,t) \to w(x,t)$ *strongly in* $L^2(\Omega_1 \times (0,T))$ *as* $\delta \to 0^+$, *where* $w(x,t)$ *is the solution of the Neumann problem* (2.8).

(ii) *If*

$$\lim_{\delta \to 0^+} \eta = \beta \in (0, +\infty), \quad \textit{and} \quad \lim_{\delta \to 0^+} \frac{\sigma}{\delta} = \alpha \in [0, \infty],$$

then the solution $u(x,t) \to w(x,t)$ *strongly in* $L^2(\Omega_1 \times (0,T))$ *as* $\delta \to 0^+$, *where* $w(x,t)$ *is the solution of*

$$\begin{cases} w_t - k\Delta w = f & x \in \Omega_1,\ t > 0, \\ w = \varphi(x) & x \in \Omega_1,\ t = 0, \\ k(\partial w/\partial \nu) + [(\beta\alpha\gamma)/(\beta + \alpha\gamma)]w = 0 & x \in \Gamma, \quad t > 0, \end{cases}$$

where the boundary condition is interpreted as $k(\partial w/\partial \nu) + \beta w = 0$ *if* $\alpha = \infty$.

(iii) *If* $\beta = \infty$ *and* $\alpha \in [0, \infty]$, *then in* (ii), *the convergence of* u *to* w *holds strongly in* $L^2(\Omega_1 \times (0,T)$; *the boundary condition is understood as* $k(\partial w/\partial \nu) + \alpha\gamma w = 0$ *if* α *is finite, as* $w = 0$ *if otherwise.*

Proof. The proof is very similar to that of Theorem 2.2; here, we only present some modifications to the proof of (ii) of Theorem 2.2, to prove (ii) of the current theorem in the case of finite α.

We first note that Lemma 2.1 still holds. Based on (2.4), it is clear that the volume element on Ω_2 at $F(p,\tau)$ is

$$dx_1 \cdots dx_n = (1 + O(1)\tau)\ dS_p\, d\tau,$$

and the surface element on $\partial\Omega$ at $F(p,\delta)$ is

$$dS_{p^\delta} = (1 + O(1)\delta)\ dS_p.$$

For any $\zeta \in C^1(\overline{\Omega}_1 \times [0,T])$ that is equal to zero when $t = T$, extend ζ along $\nu(p)$ by $\zeta(p,\tau,t) = \zeta(p,t)$. By using the test function $\phi\zeta$ and observing that $\nabla\phi(x) = [(\partial\phi)/(\partial\tau)]\nu(p) = -[\nu(p)]/\delta$ is an eigenvector of $(a_{ij}(x))$ corresponding to $\sigma_{\min}(x)$, we have

$$\begin{aligned}\int_0^T \int_{\Omega_2} a_{ij} u_{x_i} \phi_{x_j} \zeta \, dx\, dt &= -\frac{\sigma}{\delta} \int_0^T \int_{\Omega_2} \gamma(x)\zeta \frac{\partial u}{\partial \nu}\, dx\, dt \\ &= -\frac{\sigma}{\delta} \int_0^T \int_\Gamma \int_0^\delta \gamma(p,\tau)\zeta(p,\tau) \\ &\qquad \times \frac{\partial u}{\partial \tau}(1 + O(\tau))\, d\tau\, dS_p\, dt.\end{aligned}$$

By Taylor's expansion and Lemma 2.1, we see that estimate (2.22) is replaced by

$$\begin{aligned}\int_0^T \int_{\Omega_2} a_{ij} u_{x_i} \phi_{x_j} \zeta \, dx\, dt &= \frac{\sigma}{\delta} \int_0^T \int_\Gamma \zeta(p,t)\gamma(p)u(p,t)\, dS_p\, dt \\ &\quad - \frac{\sigma}{\delta} \int_0^T \int_{\partial\Omega} \gamma(p) u(p^\delta,t)\zeta(p,t) dS_{p^\delta} + o(1).\end{aligned}$$

We thus get the analog of (2.23)

$$\begin{aligned}\int_0^T \Big[-\int_{\Omega_1} u\zeta_t\, dx + k\int_{\Omega_1} \nabla u \cdot \nabla \zeta\, dx + \frac{\sigma}{\delta}\int_\Gamma \gamma(p)u(p,t)\zeta(p,t)\, dS_p \\ - \frac{\sigma}{\delta}\int_{\partial\Omega} \gamma(p)u(p^\delta,t)\zeta(p^\delta,t)\, dS_{p^\delta}\Big]\, dt \\ - \int_{\Omega_1} \varphi\zeta(x,0)\, dx = \int_0^T \int_{\Omega_1} f\zeta\, dx\, dt + o(1).\end{aligned}$$

We next extend $\gamma(x)$ to Ω_1 such that $\gamma(x) > 0$ in Ω_1. Then (2.24) is replaced by

$$\int_0^T \bigg[-\int_{\Omega_1} u\zeta_t\gamma\,dx + k\int_{\Omega_1} \nabla u\cdot\nabla(\gamma\zeta)\,dx + \eta\int_{\partial\Omega} \gamma(p)u(p^\delta,t)\zeta(p^\delta,t)\,dS_{p^\delta}\bigg]\,dt - \int_{\Omega_1}\varphi\zeta(x,0)\gamma\,dx = \int_0^T\int_{\Omega_1} f\zeta\gamma\,dx\,dt + o(1).$$

The remainder of the proof may now be safely left to the reader. □

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