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NONDECREASING SOLUTIONS OF A QUADRATIC INTEGRAL EQUATION OF URYSOHN-STIELTJES TYPE

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ABSTRACT. We prove an existence theorem for a quadratic integral equation of Urysohn-Stieltjes type in the space of continuous functions. The quadratic integral equation studied contains as a special case numerous integral equations encountered in the theory of radioactive transfer, neutron transport and the kinetic theory of gases. The concept of measure of noncompactness and a fixed point theorem due to Darbo are the main tools in carrying out our proof.

1. Introduction. Quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport, and in traffic theory. Especially, the so-called quadratic integral equation of Chandrasekher type can very often be encountered in many applications (cf. [**19, 20**, **28–30, 42, 45, 47**]). Moreover, a type of quadratic integral equation arises in the design of bandlimited signals for binary communication using simple memoryless correlation detection, when the signals are disturbed by additive white Gaussian noise. It is shown that a bandlimited signal can be designed which eliminates intersymbol interfererence for signaling at Nyquist rate; this signal is a solution to a quadratic integral equation see [**1, 3, 21, 29, 45, 51**].

In the last 35 years or so, many authors have studied the existence of solutions for several classes of nonlinear quadratic integral equations. For example, Anichini and Conti [**1**], Argyros [**4**], Banaś et al. [**8, 10, 15**], Banaś and Martinon [**11**], Banaś and O'Regan [**13**], Banaś and Rzepka [**16, 17**], Benchohra and Darwish [**18**], Caballero et al. [**22, 23, 26**], Darwish [**31–37**], Darwish and Henderson [**38**], Darwish and

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Ntouyas [**39, 40**], Darwish and Sadarangani [**41**], Hu and Yan [**46**], Leggett [**48**], Liu and Kang [**49**], Stuart [**53**] and Spiga et al. [**52**].

The classical theory of integral operators and equations can be generalized with the help of Stieltjes integrals having kernels dependent upon one or two variables. Such an approach was presented and developed in many research papers and books (cf. [**5–7, 14, 24, 43, 50**]) and the references therein.

The aim of this paper is to investigate the existence of monotonic solutions of a so-called quadratic integral equation of Urysohn-Stieltjes type, namely,

$$x(t) = h(t) + k\,x^2(t) + f(t, x(t)) \int_0^1 u(t, s, x(s))\, d_s g(t, s), \quad t \in [0, 1], \tag{1}$$

where $k \geq 0$. Let us recall that the function $f = f(t, x)$ involved in equation (1) generates the superposition operator F defined by $(Fx)(t) = f(t, x(t))$, where $x = x(t)$ is an arbitrary function defined on $[0, 1]$, see [**2**].

We remark that:

- If $k = 0$, $f(t, x) = 1$ and $u(t, s, x(s)) = (t/t + s)|x(s)|\,\phi(s)$ in equation (1), then we have an equation studied by Caballero et al. in [**27**].

- If $f(t, x) = 1$ in equation (1), then we have an equation studied by Caballero in [**43**].

Using the concept of measure of noncompactness related to monotonicity, introduced by Banaś and Olszowy [**12**], and a fixed point theorem due to Darbo [**44**], we show that equation (1) has solutions belonging to $C(I)$ and are nondecreasing on the interval I.

2. Auxiliary facts and results. This section collects some definitions and results which will be needed further on. Assume that $(E, \|.\|)$ is a real Banach space with zero element θ. Let $B(x, r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If X is a nonempty subset of E we denote by $\overline{X}$ and $\operatorname{Conv} X$ the closure and the convex closed closure of X, respectively. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E, and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the concept of a measure of noncompactness [**9**]:

Definition 2.1. A mapping $\mu : \mathcal{M}_E \to [0, +\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1) The family $\ker \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_E$.

2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

3) $\mu(\overline{X}) = \mu(\mathrm{Conv}\, X) = \mu(X)$.

4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\, \mu(X) + (1-\lambda)\, \mu(Y)$ for $0 \leq \lambda \leq 1$.

5) If $X_n \in \mathcal{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $\cap_{n=1}^{\infty} X_n \neq \phi$.

We recall the fixed point theorem due to Darbo [**44**]. Before quoting this theorem, we need the following definition:

Definition 2.2. Let M be a nonempty subset of a Banach space E and $T : M \to E$ a continuous operator that transforms bounded sets onto bounded ones. We say that T satisfies the Darbo condition (with constant $K \geq 0$) with respect to a measure of noncompactness μ if, for any bounded subset X of M, we have

$$\mu(TX) \leq K\, \mu(X).$$

If T satisfies the Darbo condition with $K < 1$, then it is called a contraction operator with respect to μ.

Theorem 2.3. *Let Q be a nonempty, bounded, closed and convex subset of space E, and let*

$$H : Q \longrightarrow Q$$

be a contraction with respect to the measure of noncompactness μ. Then H has a fixed point in the set Q.

Remark 2.4 [**9**]. Under the assumptions of Theorem 2.3, the set $\mathrm{Fix}\,H$ of fixed points of H belonging to Q is a member of $\ker\mu$. In fact, as $\mu(H(\mathrm{Fix}\,H)) = \mu(\mathrm{Fix}\,H) \leq K\,\mu(\mathrm{Fix}\,H)$ and $0 \leq K < 1$, we deduce that $\mu(\mathrm{Fix}\,H) = 0$.

In what follows we will work in the Banach space $C[0,1]$ consisting of all real functions defined and continuous on $[0,1]$. The space $C[0,1]$ is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0,1]\}.$$

For convenience, we write $I = [0,1]$ and $C(I) = C[0,1]$. Now, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in the next section (see [**11, 12**]).

Let us fix a nonempty and bounded subset X of $C(I)$. For $x \in X$ and $\varepsilon \geq 0$ denoted by $\omega(x,\varepsilon)$, the modulus of continuity of the function x, i.e.,

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,\ s \in I,\ |t-s| \leq \varepsilon\}.$$

Further, let us put

$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X,\varepsilon).$$

Define

$$d(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t,\ s \in I,\ t \leq s\}$$

and

$$d(X) = \sup\{d(x) : x \in X\}.$$

Observe that $d(X) = 0$ if and only if all the functions belonging to X are nondecreasing on I.

Now, let us define function μ on the family $\mathcal{M}_{C(I)}$ by the formula

$$\mu(X) = \omega_0(X) + d(X).$$

The function μ is a measure of noncompactness in the space $C(I)$ [**12**]. Moreover, the kernel $\ker\mu$ consists of all sets X belonging to $\mathcal{M}_{C(I)}$ such that all functions from X are equicontinuous and nondecreasing on the interval I.

Now, we recall some auxiliary facts related to functions of bounded variation and the Stieltjes integral, [**25**]. Let x be a given real function defined on the interval I. The symbol $\vee_0^1 x$ will denote the variation of x on the interval I, defined by

$$\bigvee_0^1 x = \sup_P \Big\{ \sum_{i=1}^n |x(t_i) - x(t_{i-1})| : \\ P = \{0 = t_0 < t_1 < \ldots < t_n = 1\} \text{ is a partition of } I \Big\}.$$

If $\vee_0^1 x$ is finite, then we say that x is of bounded variation on I. We have

(i) $\vee_0^1 x = \vee_0^1(-x)$

(ii) $\vee_0^1(x+y) \leq \vee_0^1 x + \vee_0^1 y$

(iii) $\vee_0^1(x-y) \leq \vee_0^1 x + \vee_0^1 y$

(iv) $\left|\vee_0^1 x - \vee_0^1 y\right| \leq \vee_0^1(x-y)$.

For other properties of functions of bounded variation see [**43, 50**]. Let $g : I \times I \to \mathbf{R}$ be a function; then the symbol $\vee_{t=a}^b g(t,s)$ indicates the variation of the function $t \to g(t,s)$ on the interval $[a,b] \subset I$. Now, let us assume that x, $\varphi : I \to \mathbf{R}$ are bounded functions. Then under some extra conditions, ([**43, 50**]), we can define the Stieltjes integral $\int_0^1 x(t)\, d\varphi(t)$ of function x with respect to the function φ. In this case, we say that x is Stieltjes integrable on interval I with respect to the function φ. If x is continuous and φ is of bounded variation on the interval I, then x is Stieltjes integrable with respect to φ on I. Moreover, under the assumption that x and φ are of bounded variation on the interval I, the Stieltjes integral $\int_0^1 x(t)\, d\varphi(t)$ exists if and only if the functions x and φ have no common points of discontinuity. Finally, we recall a few properties of the Stieltjes integral which will be used later. These properties are contained in the following lammas (cf. [**43, 50**]).

Lemma 2.5. *If x is Stieltjes integrable on I with respect to a function φ of bounded variation, then*

$$\left| \int_0^1 x(t)\, d\varphi(t) \right| \le \Big(\sup_{0\le t\le 1} |x(t)| \Big) \bigvee_0^1 \varphi.$$

Moreover, the following inequality holds:

$$\left| \int_0^1 x(t)\, d\varphi(t) \right| \le \int_0^1 |x(t)|\, d\Big(\bigvee_0^t \varphi \Big).$$

Corollary 2.6. *If x is a Stieltjes integrable function with respect to a nondecreasing function φ, then*

$$\left| \int_0^1 x(t)\, d\varphi(t) \right| \le \Big(\sup_{0\le t\le 1} |x(t)| \Big)\ (\varphi(1) - \varphi(0)).$$

Lemma 2.7. *Let x_1 and x_2 be the Stieltjes integrable functions on I with respect to a nondecreasing function φ and such that $x_1(t) \le x_2(t)$ for $t \in I$. Then*

$$\int_0^1 x_1(t)\, d\varphi(t) \le \int_0^1 x_2(t)\, d\varphi(t).$$

Corollary 2.8. *Let x be the Stieltjes integrable function on I with respect to a nondecreasing function φ and such that $x(t) \ge 0$ for all $t \in I$. Then*

$$\int_0^1 x(t)\, d\varphi(t) \ge 0.$$

Lemma 2.9. *Let φ_1 and φ_2 be nondecreasing functions on I with $\varphi_2 - \varphi_1$ a nondecreasing function. If x is a Stieltjes integrable on I and $x(t) \ge 0$ for $t \in I$, then*

$$\int_0^1 x(t)\, d\varphi_1(t) \le \int_0^1 x(t)\, d\varphi_2(t).$$

We will need later the Stieltjes integral of the form $\int_0^1 x(s)\,d_s g(t,s)$ where g is a function of two variables, $g: I \times I \to \mathbf{R}$, and the symbol d_s indicates that the integration is taken with respect to s.

3. Main theorem. In this section, we will study equation (1) assuming that the following assumptions are satisfied:

a_1) $h : I \to \mathbf{R}$ is a continuous, nondecreasing and nonnegative function on I.

a_2) f : I $\times$ $\mathbf{R} \to \mathbf{R}$ is continuous and there exists a nonnegative constant c such that

$$|f(t,x) - f(t,y)| \leq c\,|x-y|$$

for all $t \in I$ and x, $y \in \mathbf{R}$. Moreover, $f : I \times \mathbf{R}_+ \to \mathbf{R}_+$.

a_3) The superposition operator F satisfies for any nonnegative function x the condition

$$d(Fx) \leq c\,d(x),$$

where c is the same constant as in a_2).

a_4) $g : I \times I \to \mathbf{R}$ satisfies the following conditions:

(i) The function $s \to g(t,s)$ is a nondecreasing function on I for each $t \in I$.

(ii) For all t_1, $t_2 \in I$ such that $f_1 < t_2$, the function $s \to g(t_2,s) - g(t_1,s)$ is a nondecreasing function on I.

(iii) The functions $t \to g(t,0)$ and $t \to g(t,1)$ are continuous on I.

a_5) $u : I \times I \times \mathbf{R} \to \mathbf{R}$ is a continuous function such that $u : I \times I \times \mathbf{R}_+ \to \mathbf{R}_+$, and, for an arbitrary fixed $s \in I$ and $x \in \mathbf{R}_+$, the function $t \to u(t,s,x)$ is nondecreasing on I.

a_6) The function u satisfies the following conditions:

(i) There exists a continuous nondecreasing function $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$|u(t,s,x)| \leq \phi(|x|)$$

for each t, $s \in I$ and $x \in \mathbf{R}$.

(ii) For any $\mu > 0$, there exists a continuous nondecreasing function $\psi_\mu : \mathbf{R}_+ \to \mathbf{R}_+$ with $\psi_\mu(0) = 0$, such that, for each $s \in I$, $x \in \mathbf{R}$ with

$|x| \le \mu$ and, for all t_1, $t_2 \in I$, $t_1 < t_2$, we have

$$|u(t_2, s, x) - u(t_1, s, x)| \le \psi_\mu(t_2 - t_1).$$

a_7) The inequality

$$\|h\| + k\, r^2 + (c\, r + m)\, T\, \phi(r) \le r$$

has a positive solution r_0 such that $2\, r_0\, k + c\, \phi(r_0)\, T < 1$, where $m = \max_{t \in I} f(t, 0)$ and $T = \sup\{\vee_{s=0}^{1} g(t, s) : t \in I\}$, see Remark 3.3 below.

Proposition 3.1 [**25**]. *Assume that the function $g : I \times I \to \mathbf{R}$ satisfies a_4)* (ii) *and* (iii)*. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for t_1, $t_2 \in I$, $t_1 < t_2$ with $t_2 - t_1 \le \delta$, we have*

$$\bigvee_{s=0}^{1} (g(t_2, s) - g(t_1, s)) \le \varepsilon.$$

Proposition 3.2 [**25**]. *Assume that the function $g : I \times I \to \mathbf{R}$ satisfies a_4)* (ii) *and* (iii) *and the function $s \to g(t, s)$ is of bounded variation on I for each $t \in I$. Then the function $t \to \vee_{s=0}^{1} g(t, s)$ is continuous on I.*

Remark 3.3. As every nondecreasing function is of bounded variation, in view of Proposition 3.2 and the compactness of the interval I, there exists a constant $T > 0$ such that $\vee_{s=0}^{1} g(t, s) \le T$ for every $t \in I$, if g satisfies assumption a_4).

Now, we are in a position to state and prove our main result in the paper.

Theorem 3.4. *Let assumptions a_1)–a_7) be satisfied. Then equation* (1) *has at least one solution $x \in C(I)$ which is nondecreasing on the interval I.*

Proof. Let $M : \mathbf{R}_+ \to \mathbf{R}_+$ be

$$M(\varepsilon) = \sup\left\{ \bigvee_{s=0}^{1} (g(t_2, s) - g(t_1, s)) : t_1,\, t_2 \in I,\, t_1 < t_2,\, t_2 - t_1 \le \varepsilon \right\}.$$

Now Proposition 3.1 implies $M(\varepsilon) \to 0$ as $\varepsilon \to 0$. Denote by $\mathcal{F}$ the operator associated with the right-hand side of equation (1), i.e., equation (1) takes the form

$$x = \mathcal{F}\, x, \tag{1}$$

where

$$(\mathcal{F}\, x)(t) = h(t) + k\, x^2(t) + f(t, x(t)) \int_0^1 u(t, s, x(s))\, d_s g(t, s), \quad t \in I. \tag{2}$$

Solving equation (1) is equivalent to finding a fixed point of the operator $\mathcal{F}$ defined on the space $C(I)$.

First we claim that if $x \in C(I)$ then $\mathcal{F}x \in C(I)$. To establish this claim, it suffices to show that if $x \in C(I)$ then $\mathcal{U}x \in C(I)$, where

$$(\mathcal{U}\, x)(t) = \int_0^1 u(t, s, x(s))\, d_s g(t, s).$$

Let us fix $\varepsilon > 0$ and take $t_1,\, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \le \varepsilon$. Let $x \in C(I)$ so there exists a $\mu > 0$ with $\|x\| \le \mu$. Then we have

$$\begin{aligned}
&|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)| \\
&\quad = \left| \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \right| \\
&\quad \le \left| \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) - \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) \right| \\
&\qquad + \left| \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \right| \\
&\quad \le \int_0^1 |u(t_2, s, x(s)) - u(t_1, s, x(s))|\, d_s \left(\bigvee_{p=0}^{s} g(t_2, p) \right)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 |u(t_1,s,x(s))|\, d_s\Big(\bigvee_{p=0}^{s}(g(t_2,p)-g(t_1,p))\Big)\\
&\le \psi_\mu(t_2-t_1)\bigvee_{p=0}^{1} g(t_2,p)+\phi(\|x\|)\bigvee_{p=0}^{1}(g(t_2,p)-g(t_1,p))\\
&\le \psi_\mu(\varepsilon)\cdot T+\phi(\|x\|)\, M(\varepsilon).
\end{aligned}
$$

Thus, we obtain the following estimate

$$\omega(\mathcal{U}x,\varepsilon)\le \psi_\mu(\varepsilon)\cdot T+\phi(\|x\|)\, M(\varepsilon).$$

Now we have $\omega(\mathcal{U}x,\varepsilon)\to 0$ as $\varepsilon\to 0$. So our claim is established.

Second, we show that $\mathcal{F}$ is a continuous operator on the space $C(I)$. In order to prove this result it is sufficient to prove the continuity of operator $\mathcal{U}$ on $C(I)$. To do this, fix $\varepsilon>0$, and take an arbitrary $x\in C(I)$ with $\|x-y\|\le\varepsilon$. Then, for a fixed $t\in I$, we have

$$
\begin{aligned}
&|(\mathcal{U}x)(t)-(\mathcal{U}y)(t)|\\
&\qquad=\Big|\int_0^1 u(t,s,x(s))\, d_sg(t,s)-\int_0^1 u(t,s,y(s))\, d_sg(t,s)\Big|\\
&\qquad\le \int_0^1 |u(t,s,x(s))-u(t,s,y(s))|\, d_s\Big(\bigvee_{p=0}^{s} g(t,p)\Big)\\
&\qquad\le \beta(\varepsilon)\bigvee_{p=0}^{1} g(t,p)\\
&\qquad\le \beta(\varepsilon)\cdot T,
\end{aligned}
$$

where $\beta(\varepsilon)$ is given by

$$
\begin{aligned}
\beta(\varepsilon)=\sup\{&|u(t,s,x_1)-u(t,s,x_2)| : t,\, s\in I,\\
&x_1,\, x_2\in[-\|x\|-\varepsilon,\|x\|+\varepsilon], |x_1-x_2|\le\varepsilon\}.
\end{aligned}
$$

By virtue of the uniform continuity of the function $u(t,s,x)$ on the set $I\times I\times[-\|x\|-\varepsilon,\|x\|+\varepsilon]$, we have that $\beta(\varepsilon)\to 0$ as $\varepsilon\to 0$. This fact and the last inequality prove that the operator $\mathcal{U}$ is continuous, and consequently the operator $\mathcal{F}$ is continuous. Now, let us fix an arbitrary

$x \in C(I)$. Then, in view of our assumptions, we get

$$\begin{aligned}
|(\mathcal{F}x)(t)| &= \left| h(t) + k\, x^2(t) + f(t, x(t)) \int_0^1 u(t, s, x(s))\, d_s g(t, s) \right| \\
&\le \|h\| + k\, \|x\|^2 + |f(t, x(t))| \\
&\quad \times \int_0^1 |u(t, s, x(s))|\, d_s \left(\bigvee_{p=0}^{s} g(t, p) \right) \\
&\le \|h\| + k\, \|x\|^2 + |f(t, x(t)) - f(t, 0) + f(t, 0)| \\
&\quad \times \int_0^1 |u(t, s, x(s))|\, d_s \left(\bigvee_{p=0}^{s} g(t, p) \right) \\
&\le \|h\| + k\, \|x\|^2 + [c\, \|x\| + m] \\
&\quad \times \int_0^1 \phi(\|x\|)\, d_s \left(\bigvee_{p=0}^{s} g(t, p) \right) \\
&\le \|h\| + k\, \|x\|^2 + [c\, \|x\| + m]\, \phi(\|x\|) \left(\bigvee_{p=0}^{1} g(t, p) \right) \\
&\le \|h\| + k\, \|x\|^2 + [c\, \|x\| + m]\, \phi(\|x\|) \cdot T.
\end{aligned}$$

Thus, if $\|x\| \le r_0$, we obtain from a_7) that

$$\|h\| + k\, r_0^2 + [c\, r_0 + m]\, T\, \phi(r_0) \le r_0.$$

Consequently, the operator $\mathcal{F}$ transforms the ball B_{r_0} into itself.

Further, let us consider operator $\mathcal{F}$ on the subset $B_{r_0}^+$ of B_{r_0} defined in the following way:

$$B_{r_0}^+ = \{x \in B_{r_0} : x(t) \ge 0, \text{ for } t \in I\}.$$

Then $B_{r_0}^+$ is a nonempty, bounded, closed and convex subset in $C(I)$. In view of these facts and assumptions a_1), a_2) and a_5), we conclude that $\mathcal{F}$ transforms the set $B_{r_0}^+$ into itself.

We *claim* that operator $\mathcal{F}$ is continuous on $B_{r_0}^+ \subset C(I)$. To establish this claim, let us fix $\varepsilon > 0$ and take arbitrary $x, y \in B_{r_0}^+$ such that

$\|x-y\| \le \varepsilon$. Then, for $t \in I$, we have the following estimates

$$
\begin{aligned}
&|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \\
&\le k\,|x(t)-y(t)|\,|x(t)+y(t)| + \left| f(t,x(t)) \int_0^1 u(t,s,x(s))\,d_s g(t,s) \right. \\
&\qquad\qquad\qquad\qquad \left. - f(t,y(t)) \int_0^1 u(t,s,y(s))\,d_s g(t,s) \right| \\
&\le 2\,k\,r_0\,|x(t)-y(t)| + \left| f(t,x(t)) \int_0^1 u(t,s,x(s))\,d_s g(t,s) \right. \\
&\qquad\qquad\qquad\qquad \left. - f(t,y(t)) \int_0^1 u(t,s,x(s))\,d_s g(t,s) \right| \\
&\quad + \left| f(t,y(t)) \int_0^1 u(t,s,x(s))\,d_s g(t,s) \right. \\
&\qquad\quad \left. - f(t,y(t)) \int_0^1 u(t,s,y(s))\,d_s g(t,s) \right| \\
&\le 2\,k\,r_0\,\|x-y\| + |f(t,x(t)) - f(t,y(t))| \\
&\quad \times \int_0^1 |u(t,s,x(s))|\,d_s g(t,s) \\
&\quad + |f(t,y(t))| \int_0^1 |u(t,s,x(s)) - u(t,s,y(s))|\,d_s g(t,s) \\
&\le 2\,k\,r_0\,\|x-y\| + c\|x-y\|\phi(\|x\|) \\
&\quad \times \bigvee_{p=0}^{1} g(t,p) + [c\,\|x\| + m]\,\beta(\varepsilon) \bigvee_{p=0}^{1} g(t,p) \\
&\le 2\,k\,r_0\,\varepsilon + c\,\varepsilon\,\phi(r_0)\,T + [c\,r_0 + m]\,\beta(\varepsilon)\,T.
\end{aligned}
$$

By virtue of the uniform continuity of the function $u(t,s,x)$ on the set $I \times I \times [-r_0-\varepsilon, r_0+\varepsilon]$, it is easy to see that $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$. From the above estimate, we have

$$\|\mathcal{F}x - \mathcal{F}y\| \le 2\,k\,r_0\,\varepsilon + c\,\varepsilon\,\phi(r_0)\,T + [c\,r_0 + m]\,\beta(\varepsilon)\,T,$$

which implies the continuity of operator $\mathcal{F}$ on the set $B_{r_0}^{+}$.

Now, let us take a nonempty set $X \subset B_{r_0}^{+}$. Fix arbitrarily the number $\varepsilon > 0$, and choose $x \in X$ and $t_1, t_2 \in I$ such that $|t_2 - t_1| \le \varepsilon$.

Assume, without loss of generality, that $t_2 \geq t_1$. Then, in view of our assumptions, we obtain

$$
\begin{aligned}
&|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| \\
&\qquad \leq |h(t_2) - h(t_1)| + k\,|x(t_2) - x(t_1)|\,|x(t_2) + x(t_1)| \\
&\qquad\quad + \bigg| f(t_2, x(t_2)) \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad\qquad - f(t_1, x(t_1)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \bigg| \\
&\qquad \leq \omega(h, \varepsilon) + 2\,k\,r_0\,\omega(x, \varepsilon) \\
&\qquad\quad + \bigg| f(t_2, x(t_2)) \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad\qquad - f(t_2, x(t_2)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) \bigg| \\
&\qquad\quad + \bigg| f(t_2, x(t_2)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad\qquad - f(t_1, x(t_1)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) \bigg| \\
&\qquad\quad + \bigg| f(t_1, x(t_1)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad\qquad - f(t_1, x(t_1)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \bigg| \\
&\qquad \leq \omega(h, \varepsilon) + 2\,k\,r_0\,\omega(x, \varepsilon) \\
&\qquad\quad + |f(t_2, x(t_2))| \int_0^1 |u(t_2, s, x(s)) - u(t_1, s, x(s))|\, d_s g(t_2, s) \\
&\qquad\quad + [|f(t_2, x(t_2)) - f(t_1, x(t_2))| + |f(t_1, x(t_2)) - f(t_1, x(t_1))|] \\
&\qquad\quad \times \int_0^1 |u(t_1, s, x(s))| + |f(t_1, x(t_1))| \\
&\qquad\quad \times \int_0^1 |u(t_1, s, x(s))|\, d_s (g(t_2, s) - g(t_1, s)) \\
&\qquad \leq \omega(h, \varepsilon) + 2\,k\,r_0\,\omega(x, \varepsilon) \\
&\qquad\quad + [c\,\|x\| + m]\,\psi_\mu(t_2 - t_1) \bigvee_{p=0}^{1} g(t_2, p)
\end{aligned}
$$

$$
\begin{aligned}
&+ [\gamma_{r_0}(f,\varepsilon) + c\,\omega(x,\varepsilon)]\,\phi(\|x\|)\bigvee_{p=0}^{1} g(t_2,p) \\
&+ [c\,\|x\| + m]\,\phi(\|x\|)\Big(\bigvee_{p=0}^{1}(g(t_2,p) - g(t_1,p))\Big) \\
&\le \omega(h,\varepsilon) + 2\,k\,r_0\,\omega(x,\varepsilon) + [\gamma_{r_0}(f,\varepsilon) + c\,\omega(x,\varepsilon)]\phi(r_0)\,T \\
&\quad + [c\,r_0 + m]\,[\psi_\mu(\varepsilon)\,T + \phi(r_0)\,M(\varepsilon)],
\end{aligned}
$$

where

$$\gamma_{r_0}(f,\varepsilon) = \sup\left\{|f(s,x) - f(t,x)| : s,\ t \in I,\ x \in [0,r_0],\ |s-t| \le \varepsilon\right\}.$$

Thus from the last inequality, we get

$$
\begin{aligned}
|(\mathcal{F}x)(s) - (\mathcal{F}x)(t)| \le\ & \omega(h,\varepsilon) + 2\,k\,r_0\,\omega(x,\varepsilon) \\
& + [\gamma_{r_0}(f,\varepsilon) + c\,\omega(x,\varepsilon)]\,\phi(r_0)\,T \\
& + [c\,r_0 + m]\,[\psi_\mu(\varepsilon)\,T + \phi(r_0)\,M(\varepsilon)].
\end{aligned}
$$

In view of the uniform continuity of function f on the set $I \times [0,r_0]$ and from the last inequality, we have

$$\omega_0(\mathcal{F}X) \le (2\,k\,r_0 + c\,\phi(r_0)\,T)\,\omega_0(X). \tag{3}$$

In what follows, fix arbitrary $x \in X$ and t_1, $t_2 \in I$ such that $t_1 \le t_2$. Then we have

(4)

$$
\begin{aligned}
&|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| - [(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)] \\
&\quad = \Bigg| h(t_2) + k\,x^2(t_2) + f(t_2,x(t_2)) \\
&\qquad \times \int_0^1 u(t_2,s,x(s))\,d_s g(t_2,s) - h(t_1) - k\,x^2(t_1) - f(t_1,x(t_1)) \\
&\qquad\qquad \times \int_0^1 u(t_1,s,x(s))\,d_s g(t_1,s)\Bigg| \\
&\qquad - \Bigg[h(t_2) + k\,x^2(t_2) + f(t_2,x(t_2))\int_0^1 u(t_2,s,x(s))\,d_s g(t_2,s)
\end{aligned}
$$

$$
\begin{aligned}
&-h(t_1) - k\,x^2(t_1) - f(t_1, x(t_1)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s)\Big] \\
&\le |h(t_2) - h(t_1)| - [h(t_2) - h(t_1)] \\
&\quad + k\,(|x(t_2) - x(t_1)| - [x(t_2) - x(t_1)])\ [x(t_2) + x(t_1)] \\
&\quad + \Big| f(t_2, x(t_2)) \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad - f(t_1, x(t_1)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big| \\
&\quad + \Big[f(t_2, x(t_2)) \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad - f(t_1, x(t_1)) \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big] \\
&\le 2\,k\,r_0\,d(x) + |f(t_2, x(t_2))| \\
&\qquad \times \Big| \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big| \\
&\quad + |f(t_2, x(t_2)) - f(t_1, x(t_1))| \\
&\qquad \times \Big| \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big| \\
&\qquad\qquad - f(t_2, x(t_2)) \Big[\int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad\qquad\qquad - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big] \\
&\qquad\qquad - [f(t_2, x(t_2)) - f(t_1, x(t_1))] \\
&\quad \times \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \le 2\,k\,r_0\,d(x) + f(t_2, x(t_2)) \\
&\quad \times \Big| \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big| \\
&\qquad + \{ |f(t_2, x(t_2)) - f(t_1, x(t_1))| - [f(t_2, x(t_2)) - f(t_1, x(t_1))] \}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \\
&\quad - f(t_2, x(t_2)) \Big[\int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad\qquad - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big] \\
&\le 2\, k\, r_0\, d(x) + d(Fx)\, \phi(\|x\|) \bigvee_{p=0}^{1} g(t_1, p) \\
&\quad + f(t_2, x(t_2)) \Big\{ \Big| \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big| \\
&\qquad\qquad - \Big[\int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) \\
&\qquad\qquad\qquad - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \Big] \Big\}.
\end{aligned}
$$

Next we will prove that

$$
\int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \ge 0.
$$

In fact, we have

$$
\begin{aligned}
&\int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s) \\
&\quad = \int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) - \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) \\
&\qquad + \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) - \int_0^1 u(t_1, s, x(s))\, d_s g(t_1, s)
\end{aligned} \tag{5}
$$

Moreover,

$$
\begin{aligned}
&\int_0^1 u(t_2, s, x(s))\, d_s g(t_2, s) - \int_0^1 u(t_1, s, x(s))\, d_s g(t_2, s) \\
&\qquad = \int_0^1 [u(t_2, s, x(s)) - u(t_1, s, x(s))]\, d_s g(t_2, s).
\end{aligned}
$$

So assumption a_5) and Corollary 2.8 yield

$$(6)\quad \int_0^1 u(t_2,s,x(s))\,d_s g(t_2,s) - \int_0^1 u(t_1,s,x(s))\,d_s g(t_2,s) \geq 0.$$

On the other hand,

$$\begin{aligned}\int_0^1 u(t_1,s,x(s))\,d_s g(t_2,s) &- \int_0^1 u(t_1,s,x(s))\,d_s g(t_1,s)\\ &= \int_0^1 u(t_1,s,x(s))\,d_s(g(t_2,s)-g(t_1,s)).\end{aligned}$$

But we have that $g(t_2,s)-g(t_1,s)$ is a nondecreasing function (assumption a_4) (ii)), $u(t_1,s,x)\geq 0$ (assumption a_5)) and $g(t_2,s)$, $g(t_1,s)$ are nondecreasing functions (assumption a_4) (i)). From these facts and Lemma 2.9, we deduce that

$$(7)\quad \int_0^1 u(t_1,s,x(s))\,d_s g(t_2,s) - \int_0^1 u(t_1,s,x(s))\,d_s g(t_1,s) \geq 0.$$

Now (5), (6) and (7) imply

$$\int_0^1 u(t_2,s,x(s))\,d_s g(t_2,s) - \int_0^1 u(t_1,s,x(s))\,d_s g(t_1,s) \geq 0.$$

This together with (4) yields

$$d(\mathcal{F}x) \leq 2\,k\,r_0\,d(x) + \phi(r_0)\,T\,d(Fx).$$

Therefore,

$$d(\mathcal{F}x) \leq (2\,k\,r_0 + c\,\phi(r_0)\,T)\,d(x)$$

and consequently,

$$(8)\quad d(\mathcal{F}X) \leq (2\,k\,r_0 + c\,\phi(r_0)\,T)\,d(X).$$

Finally, from (3) and (8) and the definition of the measure of noncompactness μ, we obtain

$$\mu(\mathcal{F}X) \leq (2\,k\,r_0 + c\,\phi(r_0)\,T)\,\mu(X).$$

Now, the above obtained inequality, together with the fact that $(2\,k\,r_0 + c\,\phi(r_0)\,T) < 1$ enables us to apply Theorem 2.3; then equation (1) has at least one solution $x \in C(I)$. Also, such a solution is nondecreasing in view of Remark 2.4 and the definition of the measure of noncompactness μ given in Section 2.

Remark 3.5. The result of our main theorem holds for the quadratic integral equation

$$x(t) = h(t) + k\,x^n(t) + f(t, x(t)) \int_0^1 u(t, s, x(s))\, d_s g(t, s) \tag{9}$$

with $n \in \mathbf{N}$ provided that assumption $a_6)$ is changed to: the inequality

$$\|h\| + k\,r^n + (cr + m)T\,\phi(r) \le r$$

has a positive solution r_0 such that

$$n\,r_0^{n-1}k + c\,\phi(r_0)T < 1.$$

The case when $n = 1$ is more easier since we can rewrite the equation in the form

$$x(t) = \frac{h(t)}{1-k} + \frac{f(t, x(t))}{1-k} \int_0^1 u(t, s, x(s))\, d_s g(t, s).$$

4. Example. Consider the function $g : I \times I \to \mathbf{R}$ defined by

$$g(t, s) = \begin{cases} t \cdot \ln(t + s/t) & \text{for } t \in (0, 1],\ s \in I, \\ 0 & \text{for } t = 0,\ s \in I. \end{cases}$$

The function $s \to g(t, s)$ is nondecreasing for each $t \in I$. In fact,

$$\frac{d}{ds}\left[t \ln\left(\frac{t+s}{t}\right)\right] = \frac{t}{t+s} \ge 0, \quad t,\ s \in I.$$

By this fact we have that

$$\bigvee_{s=0}^{1} g(t, s) \le \ln 2.$$

In order to prove that $g(t,s)$ satisfies assumptions $a_4)$ (ii) and $a_4)$ (iii), we fix t_1, $t_2 \in I$, $t_1 \leq t_2$ and we get

$$g(t_2,s) - g(t_1,s) = \begin{cases} t_2 \cdot \ln(t_2 + s/t_2) & \text{for } t_1 = 0, \\ t_2 \cdot \ln(t_2 + s/t_2) - t_1 \cdot \ln(t_1 + s/t_1) & \text{for } t_1 > 0. \end{cases}$$

It is clear that the function $t \to g(t_2,s) - g(t_1,s)$ is nondecreasing on I. Moreover, functions $g(t,0)$ and $g(t,1)$ are continuous on I.

As $d_s g(t,s) = t/(t+s)$ our integral equation (1) takes the form

$$(10)\quad x(t) = h(t) + k\, x^2(t) + f(t,x(t)) \int_0^1 \frac{t}{t+s} u(t,s,x(s))\, ds, \quad t \in I.$$

Equation (10) is a generalization of a famous equation in transport theory, the so-called Chandrasekhar H-equation. In fact, taking $h(t) = 1$, $k = 0$, $f(t,x) = x$ and $u(t,s,x) = \phi(s)\, x(s)$ in equation (10), we obtain Chandrasekhar's H-equation [**10, 21, 28**–**30, 47, 48, 52**].

Note that, to apply our analysis, we have to impose an additional condition that the characteristic function ϕ is continuous nondecreasing and satisfies $\phi(0) = 0$.

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