ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 2, 2012

NOTES ON NEW (ANTISYMMETRIZED) ALGEBRAS

SEUL HEE CHOI AND KI-BONG NAM

ABSTRACT. We define the simple non-associative algebra $N(e^{A_S}, q, n, t)_k$ and its simple subalgebras in this work. We also prove that the anti-symmetrized algebra $N(e^{A_S}, q, n, t)^-_{[k]}$ is simple. There are various papers on finding all the derivations of an associative algebra, a Lie algebra and a non-associative algebra (see [**3, 5–7, 9, 12, 14–16**]). We also find all the derivations $\mathrm{Der}_{\mathrm{anti}}(N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]})$ of the anti-symmetrized algebra $N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]}$, and every derivation of the algebra is outer in this paper.

1. Preliminaries. Let $\mathbf{N}$ be the set of all non-negative integers and $\mathbf{Z}$ the set of all integers. Let $\mathbf{N}^+$ be the set of all positive integers. Let $\mathbf{F}$ be a field of characteristic zero and $\mathbf{F}^\bullet$ the set of all non-zero elements in $\mathbf{F}$. For fixed integers $i_1, \dots, i_m$, we define S_m as the set $\{x_1^{i_1} \cdots x_m^{i_m}, x_1^{i_1} \cdots x_{m-1}^{i_{m-1}}, \dots, x_2^{i_2} \cdots x_m^{i_m}, \dots, x_1^{i_1}, \dots, x_m^{i_m}\}$. Throughout the paper, n and t are given non-negative integers, and m denotes a non-negative integer such that $m \le n+t$. For any subset S of S_m and $q \le n$, we can define the $\mathbf{F}$-algebra $\mathbf{F}[e^{\pm[S]}, q, n, t] := \mathbf{F}[e^{\pm[S]}, \ln(x_1)^{\pm 1}, \dots, \ln(x_q)^{\pm 1}, x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_{n+1}, \dots, x_{n+t}]$ spanned by

$$\mathbf{B} = \{e^{a_1 s_1} \cdots e^{a_r s_r} \ln(x_1)^{d_1} \cdots \ln(x_q)^{d_q} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} | s_1, \dots, s_r \in S,$$
$$a_1, \dots, a_r, d_1, \dots, d_q \in \mathbf{Z},\ j_1, \dots, j_n \in \mathbf{Z}, j_{n+1}, \dots, j_{n+t} \in \mathbf{N}\}$$

where, throughout the paper, we put $\ln(x_u)^{d_u} := (\ln(x_u))^{d_u}$, $1 \le u \le q$. Note that, if $t \ge 1$, then $\mathbf{F}[e^{\pm[S]}, q, n, t]$ is a semi-group ring not a group ring (see [**17**]). We then denote $\partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r}$ as the composition of the partial derivatives $\partial_{h_1}, \dots, \partial_{h_r}$ on $\mathbf{F}[e^{\pm[S]}, q, n, t]$ and ∂_h^0, $1 \le h \le n+t$, denotes the identity map on $\mathbf{F}[e^{\pm[S]}, q, n, t]$ where $0 \le h_1, \dots, h_r \le$

2010 AMS *Mathematics subject classification.* Primary 17B40, 17B56.
Keywords and phrases. Simple, non-associative algebra, anti-symmetrized algebra, m-abelian, derivation.
Received by the editors on December 30, 2008, and in revised form on September 7, 2009.
DOI:10.1216/RMJ-2012-42-2-529

$n+t$. For any $\alpha_u \in S \subset S_m$, let A_{α_u} be an additive subgroup of $\mathbf{F}$ such that A_{α_u} contains $\mathbf{Z}$. For $q \le n$, we define the (free) $\mathbf{F}$-vector space $N(e^{A_S}, q, n, t)_k$ (respectively $N(e^{A_S}, q, n, t)_{k^+}$) whose basis is the set

$$(1)\quad \mathbf{B}_1 = \{e^{a_1 s_1} \cdots e^{a_r s_r} \ln(x_1)^{d_1} \cdots \ln(x_q)^{d_q} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r} \mid a_1 \in A_{\alpha_1}, \ldots, a_r, d_1, \ldots, d_q \in A_{\alpha_r}, s_1, \ldots, s_r \in S,\ h_1, \ldots, h_r \le n+t, p_1 + \cdots + p_r \le k \in \mathbf{N} \quad (\text{resp. } \mathbf{N}^+)\}.$$

If we define the multiplication $*$ on $N(e^{A_S}, q, n, t)_k$ as follows:

$$(2)\qquad f\partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r} * g\partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q} = f(\partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r}(g))\partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q}$$

for any $f\partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r}, g\partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q} \in N(e^{A_S}, q, n, t)_k$ (respectively $N(e^{A_S}, q, n, t)_{k^+}$), then we define the combinatorial algebra $N(e^{A_S}, q, n, t)_k$ (respectively $N(e^{A_S}, q, n, t)_{k^+}$) whose product is $*$ in (2) (see [**5, 6, 14, 16**]). The non-associative subalgebra $N(e^{A_S}, q, n, t)_{\langle k\rangle}$ of the algebra $N(e^{A^i_S}, q, n, t)_k$ is spanned by

$$(3)\quad \{f\partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r} \mid f \in \mathbf{B}, 1 \le h_1, \ldots, h_r \le n+t, p_1 + \cdots + p_r = k \le \in \mathbf{N}^+\}.$$

We define the non-associative subalgebra $N(e^{A_S}, q, n, t)_{[k^+]}$ (respectively $N(e^{A_S}, q, n, t)_{[k]}$) of the algebra $N(e^{A^i_S}, q, n, t)_k$ is spanned by

$$(4)\quad \{f\partial_h^k \mid f \in \mathbf{B},\ 1 \le h \le n,\ k \in \mathbf{N}^+ (\text{resp. for a fixed } k \in \mathbf{N}^+)\}.$$

For an algebra A and $l \in A$, an element $l_1 \in A$ is a right (respectively left) identity of l, if $l * l_1 = l$ (respectively $l_1 * l = l$) holds. The set of all right identities of $N(e^{A_S}, q, n, t)_{[1]}$ is $\{\sum_{1\le u\le n+t} x_u\partial_u + \sum_{1\le u\le n+t} c_u\partial_u \mid c_u \in \mathbf{F}\}$. There is no left identity of $N(e^{A_S}, q, n, t)_{k^+}$ (see [**10, 13, 17**]). The algebra $N(e^{A_S}, n, t)_k$ has the left identity 1. If A is an associative $\mathbf{F}$-algebra, then the anti-symmetrized algebra A is a Lie algebra relative to the commutator $[x, y] := xy - yx$ (see [**1, 18**]). For a general nonassociative $\mathbf{F}$-algebra N we define in the same way its anti-symmetrized algebra N^-. In case N^- is a Lie algebra we shall say that N is Lie admissible. For $S \subset N^-$, an element l is ad-diagonal with respect to S, if for any $l_1 \in S$, $[l, l_1] = cl_1$ holds where $c \in \mathbf{F}$. For a given basis B of an anti-symmetrized algebra N^-, the toral $\mathrm{tor}_{N^-}(B) = \mathrm{tor}\,(B)$ of B is n if there is a linearly independent maximal set $\{l_1, \dots l_n\}$ of ad-diagonal elements relative to B. For an anti-symmetrized algebra N^-, we define $\mathrm{Tor}\,(N^-)$ as follows:

$$\mathrm{Tor}\,(N^-) = \max\{\mathrm{tor}\,(B) \mid B \text{ is a basis of } N^-\}.$$

An anti-symmetrized algebra N^- is n-toral, if $Tor(N^-) = n$. For an algebra A, the abelian hull AH of its anti-symmetrized algebra A^- is the maximal abelian subalgebra of A^-. An anti-symmetrized algebra A^- is h-abelian, if the dimension of the abelian hull of A^- is h (see [**11**]). The algebra $N(e^{A_S}, q, n, t)_{[1]}$ is Lie admissible (see [**1, 16, 19**]). For all $\alpha \in S_m$, if A_α is $\mathbf{Z}$, then the algebra $N(e^{A_{S_m}}, q, n, t)_k$ is $\mathbf{Z}^{2^m}$-graded as follows:

$$N(e^{A_{S_m}}, q, n, t)_k = \bigoplus_{(a_1, \dots, a_{m^2})} N_{(a_1, \dots, a_{m^2})} \tag{5}$$

where $N_{(a_1, \dots, a_{2^m})}$ is the vector subspace of $N(e^{A_{S_m}}, q, n, t)_k$ spanned by

$$\{e^{a_1 s_1} \cdots e^{a_r s_r} \ln(x_1)^{d_1} \cdots \ln(x_q)^{d_q} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_1^{u_1} \cdots \partial_{n+t}^{u_{n+t}} \mid d_1, \dots, d_q \in \mathbf{Z}, j_1, \dots, j_n \in \mathbf{Z}, j_{n+1}, \dots, j_{n+t}, u_1, \dots, u_{n+t} \in \mathbf{N}\}.$$

This implies that $N(e^{A_S}, q, n, t)_k$ and $N(e^{A_S}, q, n, t)_{k^+}$ are appropriate graded subalgebras of the algebra $N(e^{A_{S_m}}, q, n, t)_k$. The algebra $N(0, q, n, t)_{[k]}$ (respectively its anti-symmetrized algebra) is $\mathbf{Z}^n \times (\mathbf{N} \cup \{-1, \dots, -k\})^t$-graded as follows:

$$N(0, q, n, t)_{[k]} = \bigoplus_{(j_1, \dots, j_{n+t})} N'_{(j_1, \dots, j_{n+t})} \tag{6}$$

where $N'_{(j_1, \dots, j_{n+t})}$ is the vector subspace of $N(0, q, n, t)_{[k]}$ (respectively its anti-symmetrized algebra) spanned by

$$\{\ln(x_1)^{d_1} \cdots \ln(x_q)^{d_q} x_1^{j_1} \cdots x_u^{j_u + k} x_{u+1}^{j_{u+1}} \cdots x_{n+t}^{j_{n+t}} \partial_u^k \mid d_1, \dots, d_q \in \mathbf{Z}, 1 \le u \le n+t\}.$$

Thus, throughout the paper, N_0 and N_0' denote the $(0,\dots,0)$-homogeneous components of $N(e^{As},q,n,t)_k$ (respectively its anti-symmetrized algebra) and $N(0,q,n,t)_{[k]}$ (respectively its anti-symmetrized algebra) respectively. For basis elements $e^{a_1s_1}\cdots e^{a_rs_r}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}}\partial_{t_1}^{p_1}\cdots\partial_{t_r}^{p_r}$, $e^{a_1s_1'}\cdots e^{a_rs_r'}\, x_1^{j_1'}\cdots x_{n+t}^{j_{n+t}'}\partial_{t_1'}^{p_1}\cdots\partial_{t_r'}^{p_r}$ of $N(e^{As_m},q,n,t)_k$, we define the lexicographic order $>_o$ as follows:

(7)

$$
\begin{aligned}
&e^{a_1s_1}\cdots e^{a_rs_r}\ln(x_1)^{d_1}\cdots\ln(x_q)^{d_q}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}}\partial_{t_1}^{p_1}\cdots\partial_{t_r}^{p_r} >_o\\
&e^{a_1s_1'}\cdots e^{a_rs_r'}\ln(x_1)^{d_1'}\cdots\ln(x_q)^{d_q'}x_1^{j_1'}\cdots x_{n+t}^{j_{n+t}'}\partial_{t_1'}^{p_1}\cdots\partial_{t_r'}^{p_r} \text{ if and only if}\\
&a_1 > a_1',\ \text{or, } a_1 = a_1' \text{ and } a_2 > a_2',\ \text{or, } \cdots,\\
&\text{or } a_1 = a_1',\cdots,p_{r-1} = p_{r-1}',\ \text{and } p_r > p_r'.
\end{aligned}
$$

Thus we can define the order $>_o$ on the algebra $N(e^{As_m},q,n,t)_k$. By (5) and (6), we can define the order $>_c$ on each homogeneous component of $N(e^{As_m},q,n,t)_k$ and $N(0,q,n,t)_{[k]}$ using the order $>_o$. Throughout the paper, for any basis element $e^{a_1s_1}\cdots e^{a_rs_r}\ln(x_1)^{d_1}\cdots\ln(x_q)^{d_q}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}}\partial_{t_1}^{p_1}\cdots\partial_{t_r}^{p_r}$ of $N(e^{As_m},q,n,t)_k$, d_v, $1\le v\le q$, is called the power of the natural logarithmic function $\ln(x_v)$. For any element l of $N(e^{As},q,n,t)_k$ (respectively its subalgebra or subalgebra its anti-symmetrized algebra), $H(l)$ denotes the number of different homogeneous components of $N(e^{As},q,n,t)_{[k]}$ (respectively its subalgebra or subalgebra its anti-symmetrized algebra) such that the homogeneous components contain a non-zero term of l. Note that the set of all right annihilators of $N(e^{As},q,n,t)_k$ (respectively its appropriate subalgebras) is the subalgebra T_{n+t} of $N(e^{As},q,n,t)_k$ that is spanned by $\{\partial_{t_1}^{p_1}\cdots\partial_{t_r}^{p_r}\mid 1\le t_1,\dots,t_r\le n+t, p_1,\dots,p_r\in k\}$.

3. Simplicities.

Theorem 1. *The algebra $N(0,q,n,t)_{[k+]}$ and the subalgebra $N(0,q,n,t)_{[k]}^-$ of the anti-symmetrized algebra $N(0,q,n,t)_{[k+]}^-$ are simple. The matrix ring $M_{n+t}(\mathbf{F})$ is a subalgebra of $N(0,q,n,t)_{[k+]}$. The matrix ring $M_{n+t}(\mathbf{F})$ is a subalgebra of $N(0,q,n,t)_{[k+]}$, and the algebra $sl_{n+t}(\mathbf{F})$ is a Lie subalgebra of $N(0,q,n,t)_{[k+]}^-$.*

Proof. The proof of simplicities of the non-associative algebra $N(0,q,n,t)_{[k+]}$ is easy, so it is enough to prove that its anti-symmetrized algebra $N(0,q,n,t)_{[k]}{}^{-}$ is simple. It is enough to show that an ideal generated by a non-zero element of T_{n+t} is the algebra $N(0,q,n,t)_{[k]}{}^{-}$, and an ideal generated by a non-zero element of $N(0,q,n,t)_{[k]}{}^{-}$ contains an element of T_{n+t}. Let I be a non-zero ideal of $N(0,q,n,t)^{-}_{[k]}$. First, let us show that the ideal I of $N(0,q,n,t)^{-}_{[k+]}$ generated by a non-zero element of T_{n+t} is the algebra $N(0,q,n,t)^{-}_{[k]}$. Since the algebra $N(0,0,n,t)^{-}_{[k]}$ is simple, the ideal I contains the algebra $N(0,0,n,t)^{-}_{[k]}$. For any $f = ln(x_1)^{d_1}\cdots\widehat{\ln(x_u)^{d_u}}\ln(x_{u+1})^{d_{u+1}}\cdots\ln(x_q)^{d_q}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}} \in \mathbf{F}[0,q,n,t]$, by $[\partial_u^k, x_u^k f\partial_u^k] = k!f\partial_u^k + x_u^k\partial_u^k(f)\partial_u^k$ and $[x^k\partial_u^k, f\partial_u^k] = -k!f\partial_u^k + x_u^k\partial_u^k(f)\partial_u^k$, we have that $f\partial_u^k \in N(0,q,n,t)^{-}_{[k]}$ where $\widehat{\ln(x_u)^{d_u}}$ means that the term $\ln(x_u)^{d_u}$ is omitted (see [**13, 14**]). This implies that $\ln(x_1)^{d_1}\cdots\widehat{\ln(x_u)^{d_u}}\ln(x_{u+1})^{d_{u+1}}\cdots\ln(x_q)^{d_q}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}}\partial_u^k$ is an element of the ideal I. Let us prove that the element $\ln(x_1)^{d_1}\cdots\ln(x_q)^{d_q}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}}\partial_u^k \in I$ by induction on d_u. If d_u is zero, then we have already proved this case. By $[x_u^{u+j_u}\partial_u, \ln(x_1)^{d_1}\cdots\ln(x_q)^{d_q}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}}\partial_u^k] \in I$ and induction, we have that $\ln(x_1)^{d_1}\cdots\ln(x_q)^{d_q}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}}\partial_u^k$ is an element of the ideal I. This implies that the ideal I is the algebra $N(0,q,n,t)^{-}_{[k]}$. Now we can assume that the ideal I contains a non-zero element l which is not an element of $N(0,0,n,t)^{-}_{[k]}$. By the gradation (6), we know that every term of l is in a homogeneous component of $N(0,q,n,t)^{-}_{[k]}$. Note that the element

$$(8)\qquad [l, ln(x_1)^{d'_{11}}\cdots\ln(x_q)^{d'_{1q}}x_u^k\partial_u^k] \neq 0$$

is an element of I such that the powers of natural logarithmic functions of $[l,\ln(x_1)^{d'_{11}}\cdots\ln(x_q)^{d'_{1q}}x_u^k\partial_u^k]$ are positive where $d'_{11},\ldots,d'_{1q}$ are sufficiently large positive integers (see [**13**]). Thus, without loss of generality, we can assume that all the powers of the natural logarithmic functions of l are positive. Let us prove that the algebra is simple by induction on $H(l)$ of l. Let us assume that $H(l) = 1$ and l is in the $(0,\ldots,0)$-homogeneous component N_0' of $N(0,q,n,t)^{-}_{[k]}$. l can be written as follows:

$$(9)\quad l = c_{d_{11},\ldots,d_{1q},0,\ldots,u,0,\ldots,0,u}\ln(x_1)^{d_{11}}\cdots\ln(x_q)^{d_{1q}}x_u^k\partial_u^k + \cdots \\ + c_{d_{11},\ldots,d_{1q},0,\ldots,v,0,\ldots,0,v}\ln(x_1)^{d_{h1}}\cdots\ln(x_q)^{d_{hq}}x_v^k\partial_v^k$$

where $c_{d_{11},\dots,d_{1q},0,\dots,u,0,\dots,0,u} \ln(x_1)^{d_{11}} \dots \ln(x_q)^{d_{1q}} x_u^k \partial_u^k$ is the maximal element of l using the orders $>_o$ and $>_c$ with appropriate coefficients (see [**5, 13**]). Now let us prove that the theorem on the number of non-zero terms of l. If l has one term, then we have that

$$l = c_{d_{11},\dots,d_{1q},0,\dots,u,0,\dots,0,u} \ln(x_1)^{d_{11}} \cdots \ln(x_q)^{d_{1q}} x_u^k \partial_u^k. \tag{10}$$

If d_{1u} is zero, then by

$$[\ln(x_1)^{-d_{11}} \cdots \ln(x_q)^{-d_{1q}} \partial_u^k, c_{d_{11},\dots,d_{1q},0,\dots,u,0,\dots,0,u} \ln(x_1)^{d_{11}} \cdots \ln(x_q)^{d_{1q}} x_u^k \partial_u^k] = k! \partial_u^k, \tag{11}$$

we have that $\partial_u^k \in I$. Thus, $I = N(0,q,n,t)_{[k]}{}^-$. This implies that the algebra is simple. If d_{1u} is non-zero, then by

$$\begin{aligned} l_1 &= [x_u^k \partial_u^k, l] - \alpha_1 \ln(x_1)^{d_{11}} \cdots \ln(x_q)^{d_{1q}} x_u^k \partial_u^k] \\ &= [x_u^k \partial_u^k, c_{d_{11},\dots,d_{1q},0,\dots,u,0,\dots,0,u} \ln(x_1)^{d_{11}} \cdots \ln(x_q)^{d_{1q}} x_u^k \partial_u^k] \\ &\quad - \alpha_1 \ln(x_1)^{d_{11}} \cdots \ln(x_q)^{d_{1q}} x_u^k \partial_u^k, \end{aligned} \tag{12}$$

we have a non-zero element l_1 such that the order of l_1 is strictly less than the order of l where we take an appropriate scalar α_1. Repeating similar procedures of (12), we have an element l_1' of I such that every term of l_1' has no natural logarithmic function $\ln(x_u)$, $1 \le u \le q$, and the order of l_1' is less than the order of l_1. By similar procedures of (11), we have a non-zero element of I such that the element is in $N(0,0,n,t)_{[k]}^-$. This implies that the algebra is simple. By induction, let us assume that if $H(l) = p$, then I is the algebra $N(0,q,n,t)_{[k]}^-$. Let us assume that $H(l)$ is $p+1$. By using the orders $<_o$ and $<_c$, l can be written as follows:

(13)

$$l = c_{d_{\lambda,1},\dots,d_{\lambda,q},j_1,\dots,j_{n+t},u} \ln(x_1)^{\lambda_{11}} \cdots \ln(x_q)^{\lambda_{1q}} x_1^{j_1} \cdots x_u^{j_u+k} x_{u+1}^{j_{u+1}} \cdots x_{n+t}^{j_{n+t}} \partial_u^k + \cdots + c_{d_{\lambda,1},\dots,d_{\lambda,q},j_1,\dots,j_{n+t},u} \ln(x_1)^{\lambda_{r1}} \cdots \ln(x_q)^{\lambda_{rq}} x_1^{j_1} \cdots x_v^{j_v+k} x_{v+1}^{j_{v+1}} \cdots x_{n+t}^{j_{n+t}} \partial_v^k + \#$$

where the term $\ln(x_1)^{\lambda_{11}} \cdots \ln(x_q)^{\lambda_{1q}} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_u^k$ is the maximal term of l with respect to the orders $>_o$ and $>_c$, and # is the sum of k different homogeneous components which are not equal to the $(j_1, \dots, j_{n+t})$-homogeneous component with appropriate coefficients. By (8), we can assume that all the powers of natural logarithmic functions of l are non-negative integers. Without loss of generality, we can assume that λ_{11} is a non-zero integer, otherwise either we can take the first non-zero integer $\lambda_{1,\mu}$, $1 \leq \mu \leq q$, or l is such an element of $N(0,0,n,t)^-_{[k]}$ (in this case there is nothing to prove). As a similar calculation of (12), we have an element l_1'' of I such that $H(l_1'') \leq p+1$ and $l_1'' <_0 l$. Since we have an element $l_1'' \in I$, by repeating this calculation, we have an element $l_r'' \in I$ such that every term of the $(j_1, \dots, j_{n+t})$-homogeneous component does not have a natural logarithmic function. If $H(l_r'') = 1$, then l_r'' is an element of $N(0,0,n,t)^-_{[k]}$, i.e., the ideal I is the algebra $N(0,q,n,t)^-_{[k]}$. Thus, there is nothing to prove. If $H(l_r'') \leq r+1$, then we can find an appropriate element ∂_v^k such that

$$l''_{r+1} = [\partial_v^k, [\partial_v^k, \dots, [\partial_v^k, l_r''] \dots] \tag{14}$$

of I where we applied the Lie bracket appropriate number of times so that l''_{r+1} is a non-zero element of I with no term of the $(j_1, \dots, j_{n+t})$-homogeneous component N_0'. This implies that $H(l_q) \leq r$. This implies that the ideal I is $N(0,q,n,t)^-_{[k]}$, i.e., the algebra $N(0,q,n,t)^-_{[k]}$ is simple. The remaining proofs of the theorem are obvious. Therefore we have proven the theorem. □

Theorem 2. *The algebra $N(e^{A_S}, q, n, t)_{[k+]}$ and the anti-symmetrized subalgebra $N(e^{A_S}, q, n, t)^-_{[k]}$ of the anti-symmetrized algebra $N(e^{A_S}, q, n, t)^-_{[k+]}$ are simple. The matrix ring $M_{n+t}(\mathbf{F})$ is a subalgebra of $N(e^{A_S}, q, n, t)_{[k+]}$, and the algebra $sl_{n+t}(\mathbf{F})$ is a Lie subalgebra of $N(e^{A_S}, q, n, t)^-_{[k+]}$.*

Proof. It is easy to prove that the algebra $N(e^{A_S}, q, n, t)_{[k+]}$ is simple (see [**12**]). Let us prove that its anti-symmetrized algebra $N(e^{A_S}, q, n, t)^-_{[k]}$ is simple. By (5), the anti-symmetrized algebra $N(e^{A_S}, q, n, t)^-_{[k]}$ is a graded algebra depending on the cardinality $|A_S|$ of A_S. This implies that, without loss of generality, we can put that the algebra is $\mathbf{Z}^p$-graded, i.e., $N(e^{A_S}, q, n, t)^-_{[k]} = \oplus_{(a_1, \dots, a_p)} N_{(a_1, \dots, a_p)}$

as (5). Let I be a non-zero ideal of the algebra $N(e^{As}, q, n, t)^{-}_{[k]}$ and l a non-zero element of I. Let us prove the theorem by induction on $H(l)$ of l. Let us assume that $H(l)$ is one. If l is in the $(0, \dots, 0)$-homogeneous component N_0, then the ideal generated by l is the algebra by Theorem 1, i.e., we have proved the theorem. If l is not an element of N_0, then we can assume that l is an element of $N_{(a_1, \dots, a_n)}$ such that $a_1 \neq 0$. We have that $[e^{-a_1 x_1} \cdots e^{-a_k x_k} \partial_1^k, l]$ is a non-zero element of N_0, and the element has a term in the homogeneous component N_0. For this case, by Theorem 1, we have proven that the algebra is simple. By induction, let us assume that the algebra $N(e^{As}, q, n, t)^{-}_{[k]}$ is simple when $H(l)$ is p. Let us assume that $H(l)$ is $p + 1$. First let us assume that l has a term in N_0. Note that N_0 is the subalgebra $N(0, q, n, t)^{-}_{[k]}$ of the algebra $N(e^{As}, q, n, t)^{-}_{[k]}$, and it is simple. Using the gradation of $N(0, q, n, t)^{-}_{[k]}$, l can be written as follows:

(15)
$$l = \#_1 + c_{d_{\lambda,1}, \dots, d_{\lambda,q}, j_1, \dots, j_{n+t}, u} \ln(x_1)^{\lambda_{11}} \cdots \ln(x_q)^{\lambda_{1q}} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_u^k + \cdots + c_{d_{\lambda,1}, \dots, d_{\lambda,q}, j_1, \dots, j_{n+t}, u} \ln(x_1)^{\lambda_{r1}} \cdots \ln(x_q)^{\lambda_{rq}} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_v^k + \#_2$$

where $\ln(x_1)^{\lambda_{11}} \cdots \ln(x_q)^{\lambda_{1q}} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_u^k$ is the maximal term of l in the $(j_1, \dots, j_u - k, j_{u+1}, \dots, j_{n+t})$-homogeneous component with respect to the orders $>_o$ and $>_c$ in (6), $\#_1$ is the sum of terms of l which are not in N_0, $\#_1$ is the sum of k different homogeneous components of l with appropriate coefficients, and $\#_2$ is the sum of remaining terms of $(j_1, \dots, j_u - k, j_{u+1}, \dots, j_{n+t})$-homogeneous component which are in N_0'. Since $c_{d_{\lambda,1}, \dots, d_{\lambda,q}, j_1, \dots, j_{n+t}, u} \ln(x_1)^{\lambda_{11}} \cdots \ln(x_q)^{\lambda_{1q}} x_1^{j_1} x_{n+t}^{j_{n+t}} \partial_u^k + \cdots + c_{d_{\lambda,1}, \dots, d_{\lambda,q}, j_1, \dots, j_{n+t}, u} \ln(x_1)^{\lambda_{r1}} \cdots \ln(x_q)^{\lambda_{rq}} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_v^k + \#_2$ is in N_0' and the algebra N_0' is simple, by Theorem 1, we have a non-zero element $l_1 \in I$ such that $l_1 = \#_3 + c\partial_u^k$ where $\#_3$ is the sum of the remaining terms of l_1 which does not contain a term of N_0, $H(l_1) \leq p + 1$, and $c \in \mathbf{F}^{\bullet}$. Furthermore, without loss of generality, we can assume that $\#_3$ contains a term in the $(a_1, \dots, a_p)$-homogeneous component such that $a_1 \neq 0$. This implies that $[\partial_1^k, l_1] = l_2$ with $H(l_2) = k$. Thus, by Theorem 1 and by induction, we have proven that the algebra is simple. Let us assume that l does not have a term of N_0, it has a non-zero term in the $(a_1, \dots, a_p)$-homogeneous component, and

$e^{a_1x_1}\cdots e^{a_px_p}ln(x_1)^{d_1}\cdots\ln(x_q)^{d_q}x_1^{j_1}\cdots x_{n+t}^{j_{n+t}}\partial_1^k$ is the maximal term of l with respect to $>_o$. Then $[e^{-a_1x_1}\cdots e^{-a_px_p}\partial_1^k, l] = l_3$ is a non-zero element of I. Since $H(l_3)$ is less than or equal to $p+1$ and it has a term of N_0, we have already proven that the ideal I is $N(e^{A_S}, q, n, t)_{[k]}^-$, i.e., the algebra is simple. The remaining proofs of the theorem are obvious. Therefore we have proven the theorem. □

Corollary 1. *The Lie algebra* $N(e^{A_S}, q, n, t)_{[1]}^-$ *is simple.*

Proof. The proof of the corollary is straightforward by Theorem 2. □

Theorem 3. *The non-associative algebras* $N(e^{A_S},q,n,t)_k$, $N(e^{A_S},q, n,t)_{k^+}$, $N(e^{A_S},q,n,t)_{\langle k\rangle}$, $N(e^{A_S},q,n,t)_{[k^+]}$, *and* $N(e^{A_S},q,n,t)_{[k]}$ *are simple. The matrix ring* $M_{n+t}(\mathbf{F})$ *is a subalgebra of* $N(e^{A_S},q,n,t)_k$, $N(e^{A_S},q,n,t)_{k^+}$, $N(e^{A_S},q,n,t)_{\langle k\rangle}$, $N(e^{A_S},q,n,t)_{[k^+]}$ *and* $N(e^{A_S},q,n,t)_{[k]}$.

Proof. Since the algebra $N(e^{A_S}, q, n, t)_k$ is simple, the remaining proofs of the theorem are easy (see [**12**]). So they are omitted. □

Proposition 1. *The dimension of the abelian hull AH of the algebra* $N(e^{\pm x^r}, 0, 0, 1)_{[2+]}{}^-$ *is 2, i.e., it is 2-abelian.*

Proof. It is easy to prove that the finite dimensional maximal abelian subalgebra $\langle\partial^2, x\partial^2\rangle$ of $N(e^{\pm x^r}, 0, 0, 1)_{[2+]}{}^-$ is spanned by ∂^2 and $x\partial^2$. This completes the proof of the proposition. □

Corollary 2. *There is no non zero anti-symmetrized algebra homomorphism from the anti-symmetrized algebra* $N(e^{\pm x^r}, 0, 0, 1)_{[2+]}^-$ *to the anti-symmetrized algebra* $N(0, 0, 0, 1)_{[2+]}^-$, *i.e., the algebras* $N(e^{\pm x^r}, 0, 0, 1)_{[2+]}^-$ *and* $N(0, 0, 0, 1)_{[2+]}^-$ *are not isomorphic.*

Proof. Let us assume that there is a non-zero homomorphism (called retraction) θ from $N(e^{\pm x^r}, 0, 0, 1)_{[2+]}^-$ to $N(0, 0, 0, 1)_{[2+]}^-$. By Proposition 1, we have that $\theta(\partial) = c_1\partial + c_2\partial^2$ where $c_1, c_2 \in \mathbf{F}^\bullet$. If one of the c_1 and c_2 is non-zero, then the element $\theta(e^{\pm x^r}\partial^2)$ cannot be an element of $N(0,0,0,1)^-_{[2+]}$. This contradiction shows that θ is the zero map between them. So we have proved the corollary. □

Theorem 4. *If k is not equal to m, and if L_1 is a k-abelian anti-symmetrized algebra L_1 and L_2 is an m-abelian anti-symmetrized algebra L_2, then they are not isomorphic. The abelian hull is iso-invariant (see* [**2, 8**]*).*

Proof. Without loss of generality, we can assume that $n > m$. If there is an isomorphism θ from L_1 to L_2, then L_2 has a k-dimensional abelian subalgebra. This contradiction shows that there is no isomorphism between them. The remaining proof of the theorem is straightforward by definitions of an abelian hull and an isomorphism. □

Corollary 3. *If k_1 is not equal to k_2, then the algebras $N(e^{\pm x^r},0,0,1)^-_{[k_1^+]}$ and $N(e^{\pm x^r},0,0,1)^-_{[k_2^+]}$ are not isomorphic.*

Proof. Since $N(e^{\pm x^r},0,0,1)^-_{[k_1^+]}$ is k_1-abelian and $N(e^{\pm x^r},0,0,1)^-_{[k_2^+]}$ is k_2-abelian, by Theorem 4 they are not isomorphic. Thus we have proved the corollary. □

Corollary 4. *If one of n and t is not zero, then there is no non-zero homomorphism from one of the algebras $N(e^{As},q,n,t)^-_k$, $N(e^{As},q,n,t)^-_{k+}$, $N(e^{As},q,n,t)^-_{\langle k\rangle}$, $N(e^{As},q,n,t)^-_{[k+]}$ and $N(e^{As},q,n,t)^-_{[k]}$ to the algebra $N(e^{\pm x^r},0,0,1)^-_{[k+]}$.*

Proof. Since the dimensions of the abelian hulls of $N(e^{As},q,n,t)^-_k$, $N(e^{As},q,n,t)^-_{k+}$, $N(e^{As},q,n,t)^-_{\langle k\rangle}$, $N(e^{As},q,n,t)^-_{[k+]}$ and $N(e^{As},q,n,t)^-_{[k]}$ are infinite and the dimension of the abelian hull of $N(e^{\pm x^r},0,0,1)^-_{[k+]}$ is finite, there is no non-zero homomorphism from one of the algebras $N(e^{As},q,n,t)^-_k$, $N(e^{As},q,n,t)^-_{k+}$, $N(e^{As},q,n,t)^-_{\langle k\rangle}$, $N(e^{As},q,n,t)^-_{[k+]}$, and $N(e^{As},q,n,t)^-_{[k]}$ to the algebra $N(e^{\pm x^r},0,0,1)^-_{[k+]}$. This completes the proof of the corollary. □

4. Derivations of the anti-symmetrized algebra $N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]}$.

Note 1. For any basis elements $e^{px^r} x^i \partial$ and $e^{px^r} x^i \partial^2$ of $N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]}$, and given $c \in \mathbf{F}$, if we define an $\mathbf{F}$-linear map D_c from the algebra $N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]}$ to itself as follows:

(16)

$$D_c(\partial) = 0, \qquad D_c(\partial^2) = 0, \qquad D_c(x^i \partial) = 0, \qquad D_c(x^i \partial^2) = 0,$$
$$D_c(e^{kx^r} x^i \partial^j) = \delta_{1,j} kce^{kx^r} x^i \partial + \delta_{2,j} kce^{kx^r} x^i \partial^2,$$

then the map D_c can be linearly extended to an anti-symmetrized algebra derivation of $N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]}$ where $\delta_{1,j}$ and $\delta_{2,j}$ are Kronecker delta and $1 \leq j \leq 2$ (see [**4, 7, 9**]). □

Lemma 1. *For any $D \in \mathrm{Der}_{\mathrm{anti}}(N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]})$, $D = D_c$ holds where D_c is the derivation as shown in Note* 1 *where $c \in \mathbf{F}$.*

Proof. Let D be the derivation in the lemma. Since the algebra $N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]}$ is $\mathbf{Z}$-graded, $D(\partial)$ and $D(x\partial)$ is the sum of terms in different homogeneous components of (5). Thus $D(\partial)$ and $D(\partial^2)$ can be written as follows:

$$D(\partial) = \sum_{i \geq 0} a_{i,1} e^{px^r} x^i \partial + \sum_{i \geq 0} a_{i,2} e^{px^r} x^i \partial^2$$

and

$$D(\partial^2) = \sum_{i \geq 0} b_{i,1} e^{px^r} x^i \partial + \sum_{i \geq 0} b_{i,2} e^{px^r} x^i \partial^2$$

with appropriate coefficients. Since ∂ centralizes ∂^2, we have that

(17)
$$
\begin{aligned}
&\Big[\sum_{i\geq 0} a_{i,1}e^{px^r}x^i\partial + \sum_{i\geq 0} a_{i,2}e^{px^r}x^i\partial^2, \partial^2\Big] \\
&\qquad + \Big[\partial, \sum_{i\geq 0} b_{i,1}e^{px^r}x^i\partial + \sum_{i\geq 0} b_{i,2}e^{px^r}x^i\partial^2\Big] \\
&\quad = -\sum_{i\geq 0} p^2r^2a_{i,1}e^{px^r}x^{i+2r-2}\partial \\
&\qquad - \sum_{i\geq 0} pr(i+r-1)a_{i,1}e^{px^r}x^{i+r-2}\partial \\
&\qquad - \sum_{i\geq 1} \mathrm{pria}_{i,1}e^{px^r}x^{i+r-2}\partial - \sum_{i\geq 2} i(i-1)a_{i,1}e^{px^r}x^{i-2}\partial \\
&\qquad - \sum_{i\geq 0} p^2r^2a_{i,2}e^{px^r}x^{i+2r-2}\partial^2 \\
&\qquad - \sum_{i\geq 0} pr(i+r-1)a_{i,2}e^{px^r}x^{i+r-2}\partial^2 \\
&\qquad - \sum_{i\geq 1} \mathrm{pria}_{i,2}e^{px^r}x^{i+r-2}\partial^2 - \sum_{i\geq 2} i(i-1)a_{i,2}e^{px^r}x^{i-2}\partial^2 \\
&\qquad + \sum_{i\geq 0} prb_{i,1}e^{px^r}x^{i+r-1}\partial + \sum_{i\geq 1} ib_{i,1}e^{px^r}x^{i-1}\partial \\
&\qquad + \sum_{i\geq 0} prb_{i,2}e^{px^r}x^{i+r-1}\partial^2 + \sum_{i\geq 1} b_{i,2}e^{px^r}x^{i-1}\partial^2 = 0
\end{aligned}
$$

with appropriate coefficients. Note that since the algebra is $\mathbf{Z}$-graded, it is enough to assume that non-zero terms of $D(\partial)$ and $D(\partial^2)$ are in the homogeneous components N_0 or N_p where $p \neq 0$. By (17), we have that $a_{i,1}$, $a_{i,2}$, $b_{i,1}$ and $b_{i,2}$ are zeroes, $i \geq 0$, and $D(\partial)$ and $D(\partial^2)$ are also zeroes. Since $D([\partial, x\partial])$ is zero, we are able to prove that $D(x\partial) = c_{0,1}\partial + c_{0,2}\partial^2$. Since $D([\partial, x^2\partial]) = 2D(x\partial) = c_{0,1}\partial + c_{0,2}\partial^2$, we are also able to prove that $D(x^2\partial) = 2c_{0,1}x\partial + 2c_{0,2}x\partial^2 + t_1\partial + t_2\partial^2$. Since $x\partial$ is an ad-diagonal element with respect to the element $x^2\partial$, we have that $c_{0,2}$, t_1 and t_2 are zeroes. This implies that $D(x\partial) = c_{0,1}x\partial$ and $D(x^2\partial) = 2c_{0,1}x\partial$ hold with appropriate coefficients. By induction on i of $x^i\partial^2$ and $D([x^i\partial, x^{i+1}\partial]) = D(x^{2i}\partial)$, we have that

$D(x^{2i}\partial) = 2ic_{0,1}x^{2i-1}\partial$. Similarly, we can prove that $D(x^{2i-1}\partial) = (2i-1)c_{0,1}x^{2i-2}\partial$. This implies that $D(x^i\partial) = ic_{0,1}x^{i-1}\partial$ for all i. By $D([\partial, x\partial^2]) = 0$, we have that $D(x\partial^2) = g_{0,1}\partial_1 + g_{0,2}\partial^2$ with appropriate coefficients. By $D([x\partial, x\partial^2]) = D(x\partial^2)$, we can prove that $c_{0,1} = g_{0,1} = g_{0,2} = 0$. Thus, $D(x^i\partial) = 0$ for all i and $D(x\partial^2) = 0$. By induction on i of $x^i\partial^2$ and $D([x^2\partial, x^{i-1}\partial^2]) = (i-1)D(x^i\partial^2) - 2D(x^{i-1}\partial)$, we can also prove that $D(x^i\partial^2) = 0$ for all i. Assume that

$$D(e^{x^r}x^i\partial) = \sum_{k\geq 0} u_{k,1}e^{px^r}x^k\partial + \sum_{k\geq 0} u_{k,2}e^{px^r}x^k\partial^2,$$
$$D(e^{x^r}x^{i+1}\partial) = \sum_{k\geq 0} w_{k,1}e^{px^r}x^k\partial + \sum_{k\geq 0} w_{k,2}e^{px^r}x^k\partial^2$$

with appropriate coefficients. By $D([x\partial, e^{x^r}x^i\partial]) = rD(e^{x^r}x^{r+i}\partial) + (i-1)D(e^{x^r}x^i\partial)$ and $D([\partial, e^{x^r}x^{i+1}\partial]) = rD(e^{x^r}x^{r+i}\partial) + (i+1)D(e^{x^r}x^i\partial)$, we have that $D([x\partial, e^{x^r}x^i\partial]) - (i-1)D(e^{x^r}x^i\partial) = D([\partial, e^{x^r}x^{i+1}\partial]) - (i+1)D(e^{x^r}x^i\partial)$. This implies that $w_{k+1,1} = u_{k,1}$, $k \geq 0$, $w_{0,1} = 0$, and $u_{k,2} = w_{k,2} = 0$, $k \geq 0$. These imply that

$$\begin{aligned} D(e^{x^r}x^{i+1}\partial) &= \sum_{k\geq 1} w_{k,1}e^{px^r}x^k\partial \\ &= x\left(\sum_{k\geq 0} u_{k,1}e^{px^r}x^k\partial\right) = xD(e^{x^r}x^i\partial). \end{aligned}$$

Let us put that $D(e^{x^r}x^i\partial) = xD(e^{x^r}x^{i-1}\partial) = x^2D(e^{x^r}x^{i-1}\partial) = \cdots = x^iD(e^{x^r}\partial)$, $u_{0,1} = \cdots = u_{i-1,1} = 0$, and $D(e^{x^r}\partial) = \sum_{k\geq i} u_{k,1}e^{px^r}\partial_1$. By $D([\partial, e^{x^r}\partial]) = rD(e^{x^r}x^{r-1}\partial) = rx^{r-1}(i-1)D(e^{x^r}\partial)$, we can prove that $p = 1$. We can also prove that $u_{k,1} = 0$, $k \geq i+1$. So we have that $D(e^{x^r}\partial) = u_{i,1}e^{x^r}\partial_1 = ce^{x^r}\partial_1$ with $c = u_{i,1}$. By $D([x\partial^2, e^{x^r}\partial]) = r^2D(e^{x^r}x^{2r-1}\partial) + r(r-1)D(e^{x^r}x^{2r-1}\partial) - D(e^{x^r}\partial^2)$, we have that $D(e^{x^r}\partial^2) = ce^{x^r}\partial^2$. By $D([e^{x^r}\partial, e^{x^r}x\partial]) = D(e^{2x^r}\partial)$, we have that $D(e^{2x^r}\partial) = 2ce^{2x^r}\partial$. By $D([e^{x^r}\partial, e^{x^r}x^{i+1}\partial]) = (i+1)D(e^{2x^r}x^i\partial)$, we also have that

$$D(e^{2x^r}x^i\partial) = 2ce^{2x^r}x^i\partial.$$

Thus, we need to consider the following two cases:

Case I. Put $p = 2m$. We have that $D([e^{mx^r}\partial, e^{mx^r}x^{i+1}\partial]) = (i+1)D(e^{2mx^r}x^i\partial)$. By induction on p of $e^{px^r}x^i\partial$, we are also able to prove that

$$D(e^{px^r}x^i\partial) = pce^{px^r}x^i\partial.$$

Case II. Put $p = 2m+1$. Then we have that

$$\begin{aligned}D([e^{mx^r}\partial, e^{(m+1)x^r}x^{i+1}\partial]) &= rD(e^{(2m+1)x^r}x^{r+i}\partial)\\ &\quad + (i+1)D(e^{(2m+1)x^r}x^i\partial)\end{aligned} \tag{18}$$

and

$$\begin{aligned}D([e^{mx^r}x^{i+1}\partial, e^{(m+1)x^r}\partial]) &= rD(e^{(2m+1)x^r}x^{r+i}\partial)\\ &\quad - (i+1)D(e^{(2m+1)x^r}x^i\partial).\end{aligned} \tag{19}$$

By induction on p of $e^{px^r}x^i\partial$ and by (18)–(19), we are also able to prove that

$$D(e^{px^r}x^i\partial) = pce^{px^r}x^i\partial.$$

By $D([e^{px^r}\partial, x\partial^2]) = D(e^{px^r}\partial^2) - p^2r^2D(e^{px^r}x^{2r-2}\partial) - pr(r-1)\times D(e^{px^r}x^{r-2}\partial)$, we have that $D(e^{px^r}\partial^2) = pce^{px^r}\partial^2$. By

$$\begin{aligned}D([x^{i+1}\partial^2, e^{px^r}\partial]) &= p^2r^2D(e^{px^r}x^{i+2r-1}\partial)\\ &\quad + pr(r-1)D(e^{px^r}x^{i+r-1}\partial) - (i+1)D(e^{px^r}x^i\partial^2),\end{aligned}$$

we also have that $D(e^{px^r}x^i\partial^2) = pce^{px^r}x^i\partial^2$. Thus D can be linearly extended to the derivation D_c as shown in Note 1. Therefore we have proved the lemma. □

Theorem 5. *For any $D \in \mathrm{Der}_{\mathrm{anti}}(N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]})$, D is the linear sum of the derivations D_c as shown in Note 1 where $c \in \mathbf{F}$. Every derivation of the algebra $N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]}$ is outer.*

Proof. The proofs of the theorem are straightforward by Lemma 1 and Note 1, and the fact that D_c is not inner. □

Corollary 5. *The dimension of* $\mathrm{Der}_{\mathrm{anti}}(N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]})$ *of the algebra $N(e^{\pm x^r}, 0, 0, 1)^-_{[2+]}$ is two. For any derivation D of* $\mathrm{Der}_{\mathrm{anti}}(N(e^{\pm x^r},$

$0,0,1)^{-}_{[2+]})$, $D(N'_0) = 0$ *holds where* N'_0 *is the zero-homogeneous component of* $N(e^{\pm x^r}, 0, 0, 1)^{-}_{[2+]}$.

Proof. The proofs of the corollary are straightforward by Theorem 5 and Note 1. □

REFERENCES

1. Mohammad H. Ahmadi, Ki-Bong Nam and Jonathan Pakianathan, *Lie admissible non-associative algebras*, Algebra Colloq. **12**, World Scientific, 2005.

2. G. Brown, *Properties of a 29-dimensional simple Lie algebra of characteristic three*, Math. Ann. **261** (1982), 487–492.

3. Seul Hee Choi, Jeong-Sig Lee and Ki-Bong Nam, *W-type and H-type non-associative algebras using additive maps* I, Int. J. Algebra **1** (2007), 335–346.

4. Seul Hee Choi, Jongwoo Lee and Ki-Bong Nam, *Derivations of a restricted Weyl type algebra containing the polynomial ring*, Commun. Algebra **36** (2008), 3435–3446.

5. Seul Hee Choi and Ki-Bong Nam, *The derivation of a restricted Weyl type non-associative algebra*, Hadronic Journal **28** (2005), 287–295.

6. ———, *Weyl type non-associative algebra using additive groups* I, Algebra Colloq. **14** (2007), 479–488.

7. ———, *Derivations of a restricted Weyl type algebra* I, Rocky Mountain Math. J. **37** (2007), 1813–1830.

8. I.N. Herstein, *Noncommutative rings*, Carus Math. Mono. **15**, Mathematical Association of America, Washington, DC, 1994.

9. T. Ikeda, N. Kawamoto and Ki-Bong Nam, *A class of simple subalgebras of generalized W algebras*, Proc. Inter. Conf. at Pusan, 1998, A.C. Kim, ed., Walter de Gruyter Co. KG, 2000.

10. V.G. Kac, *Description of filtered Lie algebra with which graded Lie algebras of Cartan type are associated*, Izv. Akad. Nauk SSSR **38** (1974), 832–834.

11. Naoki Kawamoto, Atsushi Mitsukawa, Ki-Bong Nam and Moon-Ok Wang, *The automorphisms of generalized Witt type Lie algebras*, J. Lie Theory **13** (2003), 571–576.

12. Jongwoo Lee and Ki-bong Nam, *Non-Associative algebras containing the matrix ring*, Linear Algebra Appl. **429** (2008), 72–78.

13. Ki-Bong Nam, *Generalized W and H type Lie algebras*, Algebra Colloquium **6**, (1999), 329–340.

14. ———, *On some non-associative algebras using additive groups*, Southeast Asian Bull. Math. **27** (2003), 493–500.

15. Ki-Bong Nam and Seul Hee Choi, *On the derivations of non-associative Weyl-type algebras*, Southeast Asian Bull. Math. **31** (2007), 341–348.

16. Ki-Bong Nam and Moon-Ok Wang, *Notes on some non-associative algebras*, J. Appl. Algebra Discrete Structured **1** (2003), 159–164.

17. D.S. Passman, *Simple Lie algebras of Witt type*, J. Algebra **206** (1998), 682–692.

18. A.N. Rudakov, *Groups of automorphisms of infinite-dimensional simple Lie algebras*, Math. USSR-Izvestija **3** (1969), 707–722.

19. R.D. Schafer, *Introduction to nonassociative algebras*, Dover, New York, 1995.

Dept. of Mathematics, Jeonju University, Chon-ju 560-759, Korea
Email address: chois@jj.ac.kr

Department of Mathematics and Computer science, University of Wisconsin–Whitewater, Whitewater, WI 53190
Email address: namk@uww.edu