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SYMMETRY IN COMPLEX CONTACT GEOMETRY

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ABSTRACT. We first show that a locally symmetric normal complex contact metric manifold is locally isometric to the complex projective space with the standard Fubini-Study metric. We then study reflections in the integral submanifolds of the vertical subbundle of a regular normal complex contact metric manifold. If the reflections are isometries, the manifold fibers over a locally symmetric space. Moreover, if the normal complex contact metric manifold is Kähler, then the manifold fibers over a quaternionic symmetric space. On the other hand, if the complex contact structure is given by a global holomorphic contact form, then the manifold fibers over a locally symmetric complex symplectic manifold.

1. Introduction. In real contact geometry the question of locally symmetric contact metric manifolds has a long history and a short answer. By 1962 Okumura [**11**] had proved that a locally symmetric Sasakian manifold is locally isometric to the sphere $S^{2n+1}(1)$ and in 2006 Boeckx and Cho [**3**] proved that a locally symmetric contact metric manifold is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1}\times S^n(4)$, the tangent sphere bundle of Euclidean space. Various studies and generalizations of this question were made in the intervening years. Perhaps most importantly, since the locally symmetric condition is very restrictive, Takahashi [**13**] introduced the notion of a locally ϕ-symmetric space for Sasakian manifolds by restricting the locally symmetric condition to the contact subbundle and showed that these manifolds locally fiber over Hermitian symmetric spaces. The first author and Vanhecke [**2**] showed that this condition is equivalent to reflections in the integral curves of the characteristic (Reeb) vector field being isometries. Subsequently, to extend the notion to contact metric manifolds, Boeckx and Vanhecke [**4**] took this reflection idea as the definition of a strongly locally ϕ-symmetric space; a contact metric manifold satisfying the condition of restricting local symmetric to the contact subbundle is called a weakly

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locally ϕ-symmetric space. For a general discussion of these ideas in real contact geometry we refer the reader to [**1**].

In this paper we begin the study of these ideas for complex contact manifolds. We first show that a locally symmetric normal complex contact metric manifold is locally isometric to the complex projective space, $\mathbf{C}P^{2n+1}(4)$, with the Fubini-Study metric of constant holomorphic curvature $+4$. We then study reflections in the integral submanifolds of the vertical subbundle of a normal complex contact metric manifold. In complex contact geometry the vertical subbundle is generally assumed to be integrable, and we suppose that the induced foliation is regular. When such reflections are isometries, we show that the manifold fibers locally over a locally symmetric space. Moreover, if the normal complex contact metric manifold is Kähler, then the manifold fibers over a quaternionic symmetric space. Wolf [**14**] established a correspondence between quaternionic symmetric spaces and certain complex contact manifolds. On the other hand, if the complex contact structure is given by a global holomorphic contact form, then the manifold fibers over a locally symmetric complex symplectic manifold. See Foreman [**7**] for examples and further discussion.

2. Complex contact geometry. A *complex contact manifold* is a complex manifold M of odd complex dimension $2n+1$ together with an open covering $\{\mathcal{O}\}$ of coordinate neighborhoods such that:

1) On each $\mathcal{O}$ there is a holomorphic 1-form θ such that $\theta \wedge (d\theta)^n \neq 0$.

2) On $\mathcal{O} \cap \mathcal{O}' \neq \varnothing$ there is a non-vanishing holomorphic function f such that $\theta' = f\theta$.

The complex contact structure determines a non-integrable subbundle $\mathcal{H}$ by the equation $\theta = 0$; $\mathcal{H}$ is called the *complex contact subbundle* or the *horizontal subbundle*.

On the other hand, if M is an Hermitian manifold with almost complex structure J, Hermitian metric g and open covering by coordinate neighborhoods $\{\mathcal{O}\}$, it is called a *complex almost contact metric manifold* if it satisfies the following two conditions:

1) In each $\mathcal{O}$ there exist 1-forms u and $v = u \circ J$ with dual unit vector fields U and $V = -JU$ and $(1,1)$ tensor fields G and $H = GJ$ such

that

$$G^2 = H^2 = -I + u \otimes U + v \otimes V,$$

$$GJ = -JG, \qquad GU = 0, \qquad g(X, GY) = -g(GX, Y).$$

2) On $\mathcal{O} \cap \mathcal{O}' \neq \varnothing$,

$$u' = Au - Bv, \qquad v' = Bu + Av,$$

$$G' = AG - BH, \qquad H' = BG + AH$$

where A and B are functions with $A^2 + B^2 = 1$.

A complex contact manifold admits a complex almost contact metric structure for which the local contact form θ is $u - iv$ to within a nonvanishing complex-valued function multiple [**9**]. The local tensor fields G and H are related to du and dv by

$$du(X, Y) = \widehat{G}(X, Y) + (\sigma \wedge v)(X, Y),$$
$$dv(X, Y) = \widehat{H}(X, Y) - (\sigma \wedge u)(X, Y)$$

for some 1-form σ and where $\widehat{G}(X, Y) = g(X, GY)$ and $\widehat{H}(X, Y) = g(X, HY)$. Moreover, on $\mathcal{O} \cap \mathcal{O}'$ it is easy to check that $U' \wedge V' = U \wedge V$ and hence we have a global vertical bundle $\mathcal{V}$ orthogonal to $\mathcal{H}$ which is generally assumed to be integrable; in this case, σ takes the form $\sigma(X) = g(\nabla_X U, V)$, ∇ being the Levi-Civita connection of g. The subbundle $\mathcal{V}$ can be thought of as the analogue of the characteristic or Reeb vector field of real contact geometry. We refer to a complex contact manifold with a complex almost contact metric structure satisfying these conditions as a *complex contact metric manifold.*

In the case that the complex contact structure is given by a global holomorphic 1-form θ, u and v may be taken globally such that $\theta = u - iv$ and $\sigma = 0$.

Ishihara and Konishi [**8**] introduced a notion of normality for complex contact structures. Their notion is the vanishing of the two tensor fields

S and T given by

$$\begin{aligned}S(X,Y) = {} & [G,G](X,Y) + 2\widehat{G}(X,Y)U - 2\widehat{H}(X,Y)V \\ & + 2(v(Y)HX - v(X)HY) + \sigma(GY)HX - \sigma(GX)HY \\ & + \sigma(X)GHY - \sigma(Y)GHX, \\ T(X,Y) = {} & [H,H](X,Y) - 2\widehat{G}(X,Y)U + 2\widehat{H}(X,Y)V \\ & + 2(u(Y)GX - u(X)GY) + \sigma(HX)GY - \sigma(HY)GX \\ & + \sigma(X)GHY - \sigma(Y)GHX,\end{aligned}$$

where $[G,G]$ and $[H,H]$ denote the Nijenhuis tensors of G and H, respectively. However, this notion seems to be too strong; among its implications is that the underlying Hermitian manifold (M,g) is Kähler. Thus, while indeed one of the canonical examples of a complex contact manifold, the odd-dimensional complex projective space, is normal in this sense, the complex Heisenberg group, is not. Korkmaz [**10**] generalized the notion of normality, and we adopt her definition here. A complex contact metric structure is *normal* if

$$\begin{aligned} S(X,Y) = T(X,Y) = 0, &\quad \text{for every } X, Y \in \mathcal{H}, \\ S(U,X) = T(V,X) = 0, &\quad \text{for every } X. \end{aligned}$$

Even though the definition appears to depend upon the special nature of U and V, it respects the change in overlaps, $\mathcal{O} \cap \mathcal{O}'$, and is a global notion. With this notion of normality both odd-dimensional complex projective space and the complex Heisenberg group with their standard complex contact metric structures are normal.

We now give expressions for the covariant derivatives of the structures tensors on a normal complex contact metric manifold; for proofs, see Korkmaz [**10**].

$$\nabla_X U = -GX + \sigma(X)V, \tag{2.1}$$

$$\nabla_X V = -HX - \sigma(X)U. \tag{2.2}$$

A complex contact metric manifold is normal if and only if the covariant derivatives of G and H have the following forms.

$$\begin{aligned} g((\nabla_X G)Y, Z) & \\ = {} & \sigma(X)g(HY,Z) + v(X)\,d\sigma(GZ,GY) \\ & - 2v(X)g(HGY,Z) - u(Y)g(X,Z) - v(Y)g(JX,Z) \\ & + u(Z)g(X,Y) + v(Z)g(JX,Y), \end{aligned} \tag{2.3}$$

$$\begin{aligned}(2.4)\quad & g((\nabla_X H)Y,Z) \\ &= -\sigma(X)g(GY,Z) - u(X)\,d\sigma(HZ,HY) \\ &\quad - 2u(X)g(GHY,Z) + u(Y)g(JX,Z) - v(Y)g(X,Z) \\ &\quad + u(Z)g(X,JY) + v(Z)g(X,Y).\end{aligned}$$

For the underlying Hermitian structure we have

$$\begin{aligned}g((\nabla_X J)Y,Z) &= u(X)\big(d\sigma(Z,GY) - 2g(HY,Z)\big) \\ &\quad + v(X)\big(d\sigma(Z,HY) + 2g(GY,Z)\big).\end{aligned} \tag{2.5}$$

The differential of σ enjoys the following properties.

$$d\sigma(JX,Y) = -d\sigma(X,JY), \tag{2.6}$$

$$d\sigma(GY,GX) = d\sigma(X,Y) - 2u\wedge v(X,Y)\,d\sigma(U,V), \tag{2.7}$$

$$d\sigma(HY,HX) = d\sigma(X,Y) - 2u\wedge v(X,Y)\,d\sigma(U,V), \tag{2.8}$$

$$d\sigma(U,X) = v(X)d\sigma(U,V), \quad d\sigma(V,X) = -u(X)\,d\sigma(U,V). \tag{2.9}$$

We will also need the basic curvature properties of normal complex contact metric manifolds which we will list here and again refer to [**10**] for their proofs. Our convention for the curvature is

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$
$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

First of all, we have $R(U,V)V = -2d\sigma(U,V)U$ and a similar expression for $R(V,U)U$, either of which gives the sectional curvature $R(U,V,V,U) = -2\,d\sigma(U,V)$. Now for X and Y horizontal we have the following (see [**10**] for details).

$$R(X,U)U = X, \quad R(X,V)V = X, \tag{2.10}$$

$$R(X,Y)U = 2(g(X,JY) + d\sigma(X,Y))V, \tag{2.11}$$

$$R(X,Y)V = -2(g(X,JY) + d\sigma(X,Y))U, \tag{2.12}$$

$$R(X,U)V = \sigma(U)GX + (\nabla_U H)X - JX, \tag{2.13}$$

$$R(X,V)U = -\sigma(V)HX + (\nabla_V G)X + JX, \tag{2.14}$$

$$R(X,U)Z = -g(X,Z)U - g(JX,Z)V + d\sigma(HZ,HX)V. \tag{2.15}$$

For a general discussion of complex contact manifolds we refer to [**1**, Chapter 12].

3. Locally symmetric normal complex contact manifolds. In this section we give a characterization in complex contact geometry of complex projective space of constant holomorphic curvature $+4$, $\mathbf{C}P^{2n+1}(4)$.

Theorem 1. *Let M^{2n+1} be a locally symmetric normal complex contact metric manifold. Then M^{2n+1} is locally isometric to $\mathbf{C}P^{2n+1}(4)$. Thus, in the complete, simply connected case the manifold is globally isometric to $\mathbf{C}P^{2n+1}(4)$.*

Proof. We begin with the observation that, since our manifold is locally symmetric it is semi-symmetric, i.e., $R \cdot R = 0$, so that

$$R(R(X,Y)X_1, X_2, X_3, X_4) + R(X_1, R(X,Y)X_2, X_3, X_4) \\ + R(X_1, X_2, R(X,Y)X_3, X_4) + R(X_1, X_2, X_3, R(X,Y)X_4) = 0.$$

Taking $X_4 = U$, $X_1 = X_3 = Y = V$, X_2 and X horizontal and using (2.10), we have

$$R(X_2, X, U, V) + g(R(V,U)V, R(X,V)X_2) \\ + R(X_2, V, U, X) - R(X, V, U, X_2) = 0.$$

Using $R(U,V)V = -2\,d\sigma(U,V)U$ and (2.14) in the second term and applying the first Bianchi identity to the fourth term we have

$$2R(X_2, X, U, V) - 2\,d\sigma(U,V)g(-\sigma(V)HX + (\nabla_V G)X + JX, X_2) = 0.$$

Then (2.3) and (2.11) give

$$2(g(X_2, JX) + d\sigma(X_2, X)) \\ - d\sigma(U,V)(d\sigma(GX_2, GX) - 2g(HGX, X_2) + g(X_2, JX)) = 0.$$

Finally, using (2.7), we have

$$[2 + d\sigma(U,V)][g(X_2, JX) + d\sigma(X_2, X)] = 0.$$

This gives us two cases to consider:

$$2 + d\sigma(U, V) = 0$$

and

$$g(X_2, JX) + d\sigma(X_2, X) = 0.$$

In the first case first note that, since $R(U,V)V = -2d\sigma(U,V)U$, $d\sigma(U,V) = -2$ implies that

$$R(U, V, V, U) = 4.$$

Also, from $R(U,V)V = -2d\sigma(U,V)U$, we have $R(U,V,V,Y) = 0$ for a horizontal unit vector field Y. From the local symmetry condition, $\nabla R = 0$, differentiation with respect to X gives

$$\begin{aligned} R(-GX, V, V, Y) + R(U, -HX, V, Y) + R(U, V, -HX - \sigma(X)U, Y) \\ + R(U, V, V, \nabla_X Y) = 0. \end{aligned}$$

By using (2.10), (2.13) and that $R(U,V)V$ and $R(V,U)U$ are vertical, we obtain

$$\begin{aligned} - g(GX, Y) + g(\sigma(U)GHX + (\nabla_U H)HX - JHX, Y) \\ + R(Y, HX, U, V) - 2\, d\sigma(U, V)g(U, \nabla_X Y) = 0, \end{aligned}$$

which, by (2.4), (2.9) and (2.11), is equivalent to

$$\begin{aligned} - g(GX, Y) - d\sigma(HY, -X) + 2g(HGHX, Y) - g(JHX, Y) \\ + 2g(Y, JHX) + 2d\sigma(Y, HX) - 2\, d\sigma(U, V)g(GX, Y) = 0. \end{aligned}$$

Then it follows that

$$-2g(GY, X) + d\sigma(HY, X) + 2\, d\sigma(Y, HX) + 2\, d\sigma(U, V)g(GY, X) = 0.$$

Setting $X = GY$, recalling that $H = GJ$ and using (2.7), we obtain

$$-2 + 3d\sigma(Y, JY) + 2d\sigma(U, V) = 0. \tag{3.1}$$

Since in this first case we have $d\sigma(U,V) = -2$, (3.1) yields

$$d\sigma(Y, JY) = 2 = -2g(Y, J^2Y) = -2\Omega(Y, JY),$$

where $\Omega(X,Y) = g(X, JY)$ denotes the fundamental 2-form of the almost Hermitian structure. Similarly, $d\sigma(U,V) = -2$ immediately gives $d\sigma(U, JU) = 2g(U,U)$ and $d\sigma(V, JV) = 2g(V,V)$. Now, for a general vector field, write $X = X' + u(X)U + v(X)V$ and compute $d\sigma(X, JX)$, giving us $d\sigma(X, JX) = 2g(X,X)$ for all X. Linearizing using (2.6) then gives us that

$$d\sigma = -2\Omega. \tag{3.2}$$

Equation (2.5) now implies that M^{2n+1} is Kähler.

On the other hand, from formula (2.11), we have

$$R(HX, GX, U, HX) = 2[-g(HX, HX) + d\sigma(HX, GX)]g(V, HX) = 0,$$

where X is a unit horizontal vector field. We now do a computation differentiating this with respect to X without using (3.2) so that (3.3) below will be available to us in treating the second case as well. Using (2.11) and (2.12) along with (2.2), it follows that

$$\begin{aligned} 0 &= R(u(\nabla_X HX)U + v(\nabla_X HX)V, GX, U, HX) \\ &\quad + R(HX, u(\nabla_X GX)U + v(\nabla_X GX)V, U, HX) \\ &\quad - R(HX, GX, GX, HX) \\ &\quad + 2[g(HX, -HX) + d\sigma(HX, GX)]g(V, \nabla_X HX) \\ &= R(V, GX, U, HX) + R(HX, U, U, HX) \\ &\quad - R(HX, GX, GX, HX) - 2 + 2d\sigma(HX, GX). \end{aligned}$$

Continuing the computation using (2.14) and (2.3), we have

$$0 = d\sigma(JX, X) + 2 - R(HX, GX, GX, HX) + 2\,d\sigma(HX, GX).$$

Moreover, for $X = GY$, this formula gives

$$-d\sigma(HY, GY) + 2 - R(JY, Y, Y, JY) - 2\,d\sigma(JY, Y) = 0.$$

Therefore, by virtue of (3.1), we have

$$\begin{aligned} R(Y,JY,JY,Y) &= 2 + 2\,d\sigma(Y,JY) - d\sigma(HY,GY) \\ &= 4d\sigma(Y,JY) + 2\,d\sigma(U,V). \end{aligned}$$

However, we know that $R(U,V,V,U) = -2d\sigma(U,V)$, and hence

$$R(Y,JY,JY,Y) = 4d\sigma(Y,JY) - R(U,V,V,U). \tag{3.3}$$

Returning to the first case, substituting (3.2) into (3.3), we have

$$R(Y,JY,JY,Y) = 4.$$

Of course, $R(Y,JY,JY,Y) = 4$ for Y horizontal and $R(U,V,V,U) = 4$ do not in general imply that the manifold has constant holomorphic curvature $+4$. Thus, we must compute the holomorphic sectional curvature for a general vector $X = X' + u(X)U + v(X)V$. Suppose both the horizontal and vertical holomorphic sectional curvatures have value μ. Then a long computation using normality gives

$$\begin{aligned} R(X,JX,JX,X) = \mu(&|X'|^4 + (u(X)^2 + v(X)^2)^2) \\ &- 4|X'|^2(u(X)^2 + v(X)^2) \\ &+ 6(u(X)^2 + v(X)^2)\,d\sigma(X',JX'), \end{aligned}$$

but for us $\mu = 4$ and $d\sigma = -2\Omega$ giving $R(X,JX,JX,X) = 4$ for all X. Thus, the complex contact metric manifold M is locally isometric to $\mathbf{C}P^{2n+1}(4)$.

To eliminate the second case, note that $R(Z,U,V,U) = 0$ for horizontal Z. Differentiating with respect to X in the same manner as above we obtain

$$\begin{aligned} 2\,d\sigma(U,V)g(Z,HX) &- 2\,d\sigma(GX,Z) \\ &+ d\sigma(JX,HZ) - 2g(JZ,GX) = 0. \end{aligned} \tag{3.4}$$

However, we now have $g(X_2,JX) + d\sigma(X_2,X) = 0$, giving

$$(2d\sigma(U,V) + 1)g(X,HZ) = 0.$$

Therefore, $d\sigma(U,V) = -1/2$, and hence $R(U,V,V,U) = 1$.

We now return again to $R\cdot R=0$ and set $X_4=Y=U$, $X_1=X$ and $X_2=X_3=JX$. Then computing using (2.15) and our other relations, we have

$$\begin{aligned}-4-d\sigma(GX,HX)+d\sigma(HX,GX)&\\ &-d\sigma(HX,GX)^2-2\,d\sigma(GX,HX)\\ &+2\,d\sigma(X,JX)+2\,d\sigma(X,JX)\,d\sigma(GX,HX)\\ &+R(X,JX,JX,X)=0.\end{aligned}$$

Using (2.7) and the condition for the second case we have

$$R(X,JX,JX,X)=1.$$

As noted above, equation (3.3) is available and now gives

$$1=4d\sigma(Y,JY)-1,$$

but $g(X_2,JX)+d\sigma(X_2,X)=0$ which then would imply

$$1=4(-g(Y,J^2Y))-1\quad\text{or}\quad 2=4,$$

a contradiction.

4. Reflections in the vertical foliation. As we have seen, the condition of local symmetry for a normal complex contact metric manifold is extremely strong. We therefore consider a weaker condition in terms of local reflections in the integral submanifolds of the vertical subbundle of a normal complex contact metric manifold. To do this, we first recall the notion of a local reflection in a submanifold. Given a Riemannian manifold (M,g) and a submanifold N, *local reflection* in N, φ_N, is defined as follows. For $m\in M$, consider the minimal geodesic from m to N meeting N orthogonally at p. Let X be the unit vector at p tangent to the geodesic in the direction toward m. Then φ_N maps $m=\exp_p(tX)\to\exp_p(-tX)$. In [**6**] Chen and Vanhecke gave the following necessary and sufficient conditions for a reflection to be isometric.

Theorem. *Let (M,g) be a Riemannian manifold and N a submanifold. Then the reflection φ_N is a local isometry if and only if*

1. *N is totally geodesic;*
2. a) *$(\nabla^{2k}_{X\cdots X}R)(X,Y)X$ is normal to N,*

b) *$(\nabla^{2k+1}_{X\cdots X}R)(X,Y)X$ is tangent to N and*

c) *$(\nabla^{2k+1}_{X\cdots X}R)(X,V)X$ is normal to N*

for all vectors X, Y normal to N and vectors V tangent to N and all $k \in \mathbf{N}$.

In regard to reflections it is worth noting that, on a normal complex contact metric manifold, a geodesic that is initially orthogonal to $\mathcal{V}$ remains orthogonal to $\mathcal{V}$; without normality this is not true.

Proposition. *Let γ be a geodesic on a normal complex contact metric manifold. If $\gamma'(0)$ is a horizontal vector, then $\gamma'(s)$ is horizontal for all s.*

Proof. We have immediately

$$\gamma' g(\gamma', U) = g(\gamma', -G\gamma' + \sigma(\gamma')V) = \sigma(\gamma')v(\gamma'),$$
$$\gamma' g(\gamma', V) = g(\gamma', -H\gamma' - \sigma(\gamma')U) = -\sigma(\gamma')u(\gamma').$$

Multiplying the first equation by $u(\gamma')$, the second by $v(\gamma')$ and adding, we see that $u(\gamma')^2 + v(\gamma')^2$ is constant along the geodesic; but at $s = 0$ this is zero giving $g(\gamma'(s), U) = g(\gamma'(s), V) = 0$, completing the proof.

Let M^{2n+1} be a normal complex contact metric manifold. Since the vertical subbundle $\mathcal{V}$ is integrable, we will suppose that this is a regular foliation, i.e., each point has a neighborhood such that any integral submanifold of $\mathcal{V}$ passing through the neighborhood passes through only once. Then M^{2n+1} fibers over a manifold M' of real dimension $4n$. An easy computation shows that the horizontal parts of the Lie derivatives $\pounds_U g$ and $\pounds_V g$ vanish. Thus the metric is projectable and we denote by g' the metric on the base, ∇' its Levi-Civita connection and R' its curvature. For vectors fields X, Y, etc., on the base we denote by X^*, Y^*, etc., their horizontal lifts to M.

Theorem 2. *Let M^{2n+1} be a normal complex contact metric manifold, and suppose that the foliation induced by vertical subbundle is* *regular. If reflections in the integral submanifolds of the vertical subbundle are isometries, then the manifold fibers over a locally symmetric space.*

Proof. By a result of Cartan [**5**, pages 257–258] it is sufficient to show that $g'((\nabla'_X R')(X,Y)X,Y) = 0$ for orthonormal pairs $\{X,Y\}$ on the base manifold M'. First note that from the fundamental equations of a Riemannian submersion, see e.g. [**12**],

$$\nabla_{X^*}Y^* = (\nabla'_X Y)^* + u(\nabla_{X^*}Y^*)U + v(\nabla_{X^*}Y^*)V.$$

Consequently, from the equations for the curvature of a Riemannian submersion,

$$\begin{aligned} R(X^*, Y^*, Z^*, W^*) & \\ &= R'(X,Y,Z,W) \\ &\quad + 2(u(\nabla_{X^*}Y^*)u(\nabla_{Z^*}W^*) + v(\nabla_{X^*}Y^*)v(\nabla_{Z^*}W^*)) \\ &\quad - u(\nabla_{Y^*}Z^*)u(\nabla_{X^*}W^*) + v(\nabla_{Y^*}Z^*)v(\nabla_{X^*}W^*) \\ &\quad + u(\nabla_{X^*}Z^*)u(\nabla_{Y^*}W^*) - v(\nabla_{X^*}Z^*)v(\nabla_{Y^*}W^*). \end{aligned}$$

From this, using the normality, we have

$$\begin{aligned} R(X^*,Y^*)X^* = {} & (R'(X,Y)X)^* \\ & + 3(g(GX^*,Y^*)GX^* + g(HX^*,Y^*)HX^*). \end{aligned} \tag{4.1}$$

Now, since the reflections in the integral submanifolds of $\mathcal{V}$ are isometries, we have by the above theorem of Chen and Vanhecke that

$$g((\nabla_{X^*}R)(X^*,Y^*)X^*,Y^*) = 0$$

and our task is to expand this using (4.1).

$$\begin{aligned} (\nabla_{X^*}R)(X^*,Y^*)X^* = {} & \nabla_{X^*}(R'(X,Y)X)^* \\ & - 3\nabla_{X^*}(g(GX^*,Y^*)GX^* + g(HX^*,Y^*)HX^*) \\ & - R((\nabla'_X X)^*, Y^*)X^* - R(X^*, (\nabla'_X Y)^* + g(GX^*,Y^*)U \\ & + g(HX^*,Y^*)V)X^* - R(X^*,Y^*)(\nabla'_X X)^*. \end{aligned}$$

Terms in $R(X^*, U)X^*$ and $R(X^*, V)X^*$ are vertical by (2.11) and (2.12). Expanding the remaining terms, using (2.3) and (2.4) where necessary, we obtain

$$0 = g((\nabla_{X^*}R)(X^*, Y^*)X^*, Y^*) = g'((\nabla'_X R')(X, Y)X, Y)$$

and hence that the base manifold M' is locally symmetric.

Theorem 3. *Let M^{2n+1} be a normal complex contact metric manifold whose vertical foliation is regular and whose underlying Hermitian structure is Kähler. If reflections in the integral submanifolds of the vertical subbundle are isometries, then M^{2n+1} fibers over a quaternionic symmetric space.*

Proof. Since M^{2n+1} is Kähler, taking $X = U$ in equation (2.5) we have that $d\sigma(Z, GY) = 2g(HY, Z)$, and hence replacing Y by $-GY$ with Y horizontal we see that $d\sigma$ is equal to minus twice the fundamental 2-form when restricted to horizontal vectors. Thus, for X and Y horizontal, we have $d\sigma(X, Y) = -2\Omega(X, Y)$, and hence taking X and Y to be basic with respect to the fibration (see, e.g., [**12**]) we readily have the following Lie derivatives.

$$\begin{aligned}
(\pounds_U \widehat{G})(X, Y) &= Ug(X, GY) = \sigma(U)\widehat{H}(X, Y),\\
(\pounds_U \widehat{H})(X, Y) &= Ug(X, HY) = -\sigma(U)\widehat{G}(X, Y) - d\sigma(HX, HY)\\
&= -\sigma(U)\widehat{G}(X, Y) - 2\Omega(X, Y),\\
(\pounds_U \Omega)(X, Y) &= Ug(X, JY) = d\sigma(X, GY) = 2\widehat{H}(X, Y)
\end{aligned}$$

and similar expressions for Lie derivatives with respect to V. Then, it follows that

$$\begin{aligned}
(\pounds_U(\widehat{G} \wedge \widehat{G}))(X, Y, Z, W) &= 2\sigma(U)(\widehat{H} \wedge \widehat{G})(X, Y, Z, W),\\
(\pounds_U(\widehat{H} \wedge \widehat{H}))(X, Y, Z, W) &= -2\sigma(U)(\widehat{G} \wedge \widehat{H})(X, Y, Z, W)\\
&\quad - 4(\Omega \wedge \widehat{H})(X, Y, Z, W),\\
(\pounds_U(\Omega \wedge \Omega))(X, Y, Z, W) &= 4(\widehat{H} \wedge \Omega)(X, Y, Z, W).
\end{aligned}$$

Adding, we see that

$$\pounds_U(\widehat{G} \wedge \widehat{G} + \widehat{H} \wedge \widehat{H} + \Omega \wedge \Omega) = 0$$

and similarly for the Lie derivative with respect to V. Therefore, the 4-form $\Lambda = \widehat{G} \wedge \widehat{G} + \widehat{H} \wedge \widehat{H} + \Omega \wedge \Omega$ is projectable giving an almost quaternionic structure Λ' on the base manifold M'. It remains to show that Λ' is parallel. For this, it is enough to show that $(\nabla_X \Lambda)(Y_1, Y_2, Y_3, Y_4) = 0$ for horizontal vector fields $X, Y_1, \dots, Y_4$. From (2.3), we have for horizontal vector fields

$$(\nabla_X \widehat{G})(Y, Z) = g(Y, (\nabla_X G)Z) = \sigma(X) g(HZ, Y) = \sigma(X) \widehat{H}(Y, Z).$$

Similarly, from (2.4) and (2.5), we have

$$(\nabla_X \widehat{H})(Y, Z) = -\sigma(X) \widehat{G}(Y, Z), \quad (\nabla_X \Omega)(Y, Z) = 0.$$

Then

$$(\nabla_X \Lambda)(Y_1, Y_2, Y_3, Y_4) = 2\sigma(X)(\widehat{H} \wedge \widehat{G} - \widehat{G} \wedge \widehat{H})(Y_1, Y_2, Y_3, Y_4) = 0.$$

That the base manifold is symmetric follows from Theorem 2.

Theorem 4. *Let M^{2n+1} be a normal complex contact metric manifold whose vertical foliation is regular and whose complex contact structure is given by a global holomorphic contact form. If reflections in the integral submanifolds of the vertical subbundle are isometries, then M^{2n+1} fibers over a locally symmetric complex symplectic manifold.*

Proof. We noted in Section 2 that, when the complex contact structure is given by a global, holomorphic 1-form, u and v may be taken globally such that $\theta = u - iv$ and $\sigma = 0$. Thus, $\widehat{G}$ and $\widehat{H}$ are closed 2-forms, and the Lie derivatives of $\widehat{G}$ and $\widehat{H}$, with respect to U and V, vanish. Also, computing as in the preceding proof, we have $\pounds_U \Omega = \pounds_V \Omega = 0$. Therefore, each of the 2-forms $\widehat{G}$, $\widehat{H}$ and Ω, projects to a closed 2-form on M', say $\widehat{G}'$, $\widehat{H}'$ and Ω', respectively. The projectability of Ω and g from the complex manifold M^{2n+1} gives M' a complex structure for which g' is an Hermitian metric. Let $\Psi = \widehat{G}' - i\widehat{H}'$. Now, since Ψ is the projection of $d\theta = du - idv$, it is a closed holomorphic 2-form and, from the ranks of $\widehat{G}'$ and $\widehat{H}'$, $\Psi^n \neq 0$, giving us a complex symplectic structure on M'. Again, the rest of the result follows from Theorem 2.

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