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REFLECTIONS AND A GENERALIZATION OF THE MAZUR-ULAM THEOREM

OSAMU HATORI, KIYOTAKA KOBAYASHI,
TAKESHI MIURA AND SIN-EI TAKAHASI

ABSTRACT. In this paper, we will generalize the Mazur-Ulam theorem which states that every bijective isometry between two normed spaces is affine. To do this, we introduce a notion of metricoid spaces, which is a generalization of metric space. Finally, we give a representation of surjections from $C^+(X)$ onto $C^+(Y)$ which preserve certain subdistances.

1. Introduction and preliminaries. Let N_1 and N_2 be two normed linear spaces. It was proved by Mazur and Ulam [**2**] that every bijective isometry T from N_1 onto N_2 is affine, that is, $T((f+g)/2) = (T(f)+T(g))/2$ holds for every f, $g \in N_1$. Using the idea of Vogt [**6**], a simple proof of this result was given by Väisälä [**5**]. In the proof, reflection played an essential role. In this paper, we will generalize the Mazur-Ulam theorem. To do this, we will introduce a notion of subdistances and metricoid spaces. We will give some examples of (super reflective) metricoid groups. In the final section, we will give representations of surjections from $C^+(X)$ onto $C^+(Y)$ that preserve certain subdistances. Here, $C^+(K)$ denotes the set of all *real-valued* continuous functions f on a compact Hausdorff space K such that $f(x) > 0$ for every $x \in K$.

Denote the set of all real numbers by $\mathbf{R}$. We denote by $\mathbf{R}^+$ the set of all non-negative real numbers.

Definition 1.1. Let G be a set and $\delta: G \times G \to \mathbf{R}^+$ a map which satisfies that

(1) $\delta(f,g) = 0$ if and only if $f = g$.

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A self-map T on G is said to be δ-*isometric* if $\delta(T(f),T(g)) = \delta(f,g)$ holds for every pair f and $g \in G$. We call δ a *subdistance* if

(2) for every pair f and $g \in G$, there exists a $K(f,g) \in \mathbf{R}^+$ such that the inequality $\delta(T(f),f) \leq K(f,g)$ holds for every bijective δ-isometry T on G with $T(g) = g$.

We call the pair (G,δ) a *metricoid space.*

Every metric space is a metricoid space. In fact, if (G,δ) is a metric space, then by a simple calculation we see that $\delta(T(f),f) \leq 2\delta(f,g)$ holds for every $f,g \in G$ and every δ-isometry T on G with $T(g) = g$. So, every metric space (G,δ) is a metricoid space.

Example 1.1. Let A be a unital semi-simple commutative Banach algebra with the maximal ideal space M_A, and let A^{-1} be the set of all invertible elements of A. Let us consider a function $\delta\colon A^{-1}\times A^{-1} \to \mathbf{R}^+$ defined by

$$\delta(f,g) = r\left(\frac{f}{g} - 1\right), \quad (f,g \in A^{-1}),$$

where $r(a)$ denotes the spectral radius of $a \in A$. In this case, it is trivial that δ satisfies (1). To see that δ satisfies (2), let $f,g \in A^{-1}$ and T be a δ-isometry on A^{-1} with $T(g) = g$. Take a point $\phi_0 \in M_A$ such that

$$r\left(\frac{T(f)}{f} - 1\right) = \left|\frac{\widehat{T(f)}(\phi_0)}{\widehat{f}(\phi_0)} - 1\right|,$$

where $\widehat{a}$ denotes the Gelfand transform of $a \in A$. Then we have

$$\begin{aligned}
\delta(T(f),f) &= \left|\frac{\widehat{T(f)}(\phi_0)}{\widehat{f}(\phi_0)} - 1\right| = \left|\frac{\widehat{T(f)}(\phi_0)}{\widehat{f}(\phi_0)} - \frac{\widehat{g}(\phi_0)}{\widehat{f}(\phi_0)} + \frac{\widehat{g}(\phi_0)}{\widehat{f}(\phi_0)} - 1\right| \\
&\leq \frac{|\widehat{g}(\phi_0)|}{|\widehat{f}(\phi_0)|}\left|\frac{\widehat{T(f)}(\phi_0)}{\widehat{g}(\phi_0)} - 1\right| + \left|\frac{\widehat{g}(\phi_0)}{\widehat{f}(\phi_0)} - 1\right| \\
&\leq r(g)r(f^{-1})\, r\left(\frac{T(f)}{g} - 1\right) + r\left(\frac{g}{f} - 1\right) \\
&= r(g)r(f^{-1})\delta(T(f),T(g)) + \delta(g,f) \\
&= r(g)r(f^{-1})\delta(f,g) + \delta(g,f)
\end{aligned}$$

since T is δ-isometric and $T(g) = g$. By taking

$$K(f, g) = r(g)r(f^{-1})\delta(f, g) + \delta(g, f),$$

we see that δ satisfies (2). On the other hand, δ is not a metric on A^{-1} since δ is not symmetric. Here, we note that surjective maps $S: A^{-1} \to B^{-1}$ that satisfy $\delta(S(f), S(g)) = \delta(f, g)$ for all $f, g \in A^{-1}$ are characterized in [**1**, Theorem 4.1] in terms of homeomorphisms between maximal ideal spaces, where B is another unital semi-simple commutative Banach algebra (cf. [**3**, Theorem 3.2]).

Definition 1.2. Let (G, δ) be a metricoid space and $h \in G$. A self-map ρ on G is called a *reflection* of G at h if the following conditions (3), (4), (5) and (6) hold.

(3) $\rho(h) = h$,

(4) $\rho^2 = \mathrm{Id}$, the identity map,

(5) ρ is δ-isometric, and

(6) there is a constant $L(h) > 1$ such that $\delta(\rho(f), f) \geq L(h)\delta(f, h)$ for every $f \in G$.

Example 1.2. Let N be a normed space. Let $f, g \in N$ be arbitrary, and put $h = (f + g)/2$. Then the map $\rho: N \to N$ defined by $\rho(u) = 2h - u$ ($u \in N$) is a reflection on N at h that satisfies $\rho(f) = g$ and $\rho(g) = f$.

Remark 1.1. We see that a reflection ρ of G at h is a bijective δ-isometry since $\rho^2 = \mathrm{Id}$. Moreover, we see that h is the only fixed point of ρ. In fact, if $a \in G$ such that $\rho(a) = a$, then putting $f = a$ in (6), $0 \geq L(h)\delta(a, h)$ holds, and so we get $a = h$.

Definition 1.3. Let (G, δ) be a metricoid space. We denote by $R(G; h)$ the set of all reflections of G at $h \in G$. A metricoid space (G, δ) is said to be *reflective* if $R(G; h) \neq \varnothing$ for every $h \in G$.

Definition 1.4. Let H be a subgroup of the group of all δ-isometries from G onto itself. We call H a *bijective δ-isometry group* on G. For

$f \in G$, put

$$\lambda(f;H) = \sup\{\delta(S(f),f) : S \in H\}.$$

Lemma 1.1. *Let (G,δ) be a metricoid space, H a bijective δ-isometry group on G and $h \in G$ with $\lambda(h;H) < \infty$. If there is a $\rho \in R(G;h)$ such that $\rho H \rho \subset H$, then h is a common fixed point of all δ-isometries in H.*

Proof. Suppose that there is a $\rho \in R(G;h)$ such that $\rho H \rho \subset H$. Pick $S \in H$ arbitrarily, and set $U = \rho S^{-1} \rho S$. Then $U \in H$ by hypothesis. Note that S is a δ-isometry. Note also that ρ is a δ-isometry with $\rho(h) = h$ since ρ is a reflection on G at h. We now obtain the following inequality.

$$\begin{aligned}\delta(U(h),h) &= \delta(\rho S^{-1}\rho S(h),h) = \delta(\rho S^{-1}\rho S(h),\rho(h))\\ &= \delta(S^{-1}\rho S(h),h) = \delta(\rho S(h),S(h))\\ &\geq L(h)\delta(S(h),h).\end{aligned}$$

Since $U \in H$, and since $S \in H$ was arbitrary, it follows that $\lambda(h;H) \geq L(h)\lambda(h;H)$. Then $\lambda(h;H) = 0$ since $L(h) > 1$, which implies that h is a common fixed point of all δ-isometries in H. $\square$

Definition 1.5. Let (G,δ) be a metricoid space. We define

$$\frac{f \circ g}{2} = \{h \in G : \text{there exists a } \rho \in R(G;h) \text{ such that } \rho(f) = g\}$$

for each f, $g \in G$.

Lemma 1.2. *Let (G_1,δ_1) and (G_2,δ_2) be two metricoid spaces and T a bijective (δ_1,δ_2)-isometry of G_1 onto G_2, that is,*

$$\delta_2(T(f),T(g)) = \delta_1(f,g), \quad (f,g \in G_1).$$

Let f, $g \in G_1$ with

$$\text{(1.1.1)} \qquad \frac{f \circ g}{2} \neq \varnothing \quad \text{and} \quad \frac{T(f) \circ T(g)}{2} \neq \varnothing.$$

Then both $(f \circ g)/2$ and $(T(f) \circ T(g))/2$ consist of single elements and

$$T\left(\frac{f \circ g}{2}\right) = \frac{T(f) \circ T(g)}{2}. \tag{1.1.2}$$

Proof. Let $f, g \in G_1$ with (1.1.1). Let H denote the set of all bijective δ_1-isometries S from G_1 onto itself such that $S(f) = f$ and $S(g) = g$. Then H is a bijective δ_1-isometry group on G_1 such that both f and g are common fixed points of all δ_1-isometries in H. Pick $a \in (f \circ g)/2$ arbitrarily. There exists a $\rho \in R(G_1; a)$ such that $\rho(f) = g$. For each $S \in H$, we have

$$\rho S \rho(f) = \rho S g = \rho(g) = f.$$

As in the same way, we see that $\rho S \rho(g) = g$, and so $\rho S \rho \in H$ for every $S \in H$. Note also that $\lambda(a; H) < \infty$. In fact, if $S \in H$, then

$$\delta_1(S(a), a) \leq K(a, f)$$

holds since δ_1 is subdistance, and hence $\lambda(a, H) \leq K(a, f) < \infty$. By Lemma 1.1, we have that a is a common fixed point of all δ_1-isometries in H.

Finally, pick $b \in (T(f) \circ T(g))/2$ arbitrarily. There exists a $\theta \in R(G_2; b)$ such that $\theta T(f) = T(g)$. Put $U = \rho T^{-1} \theta T$. Then we see that U is a bijective δ_1-isometry on G_1. Since $\theta T(f) = T(g)$ and $\rho(g) = f$, we have

$$U(f) = \rho T^{-1} \theta T(f) = \rho T^{-1} T(g) = \rho(g) = f.$$

As in the same way, we also have $U(g) = g$, and so $U \in H$. Since a is a common fixed point of all δ_1-isometries in H, it follows that $U(a) = a$. Since $\rho^2 = \mathrm{Id}$, we have

$$\theta T(a) = T \rho \rho T^{-1} \theta T(a) = T \rho U(a) = T \rho(a) = T(a).$$

Since b is the only fixed point of θ (see Remark 1.1), we obtain $T(a) = b$. Since $a \in (f \circ g)/2$ and $b \in (T(f) \circ T(g))/2$ were arbitrary, both $(f \circ g)/2$ and $(T(f) \circ T(g))/2$ consist of single points, and (1.1.2) holds. □

Corollary 1.3. *Let (G, δ) be a metricoid space and $f, g \in G$. Then either $(f \circ g)/2$ is the empty set or a singleton.*

Proof. Consider the case where $G_1 = G_2$, $\delta_1 = \delta_2$ and $T = \mathrm{Id}$ in Lemma 1.2. Then we see that $(f \circ g)/2$ is the empty set or a singleton. □

Remark 1.2. Let N be a normed space. Set $h = (f+g)/2$ for each $f, g \in N$. By Example 1.2, $h \in (f \circ g)/2$. According to Corollary 1.3, we have that $(f \circ g)/2$ is a singleton, that is,

$$\frac{f \circ g}{2} = \{h\} = \{(f+g)/2\} \quad \text{for every } f, g \in N.$$

Thus, we may regard $(f \circ g)/2$ as $(f+g)/2$.

Let N_1 and N_2 be normed spaces. Recall that a map $T \colon N_1 \to N_2$ is affine if $T((1-t)f + tg) = (1-t)T(f) + tT(g)$ for all $f, g \in N_1$ and for all $t \in \mathbf{R}$.

Lemma 1.4. *Let N_1 and N_2 be normed spaces. A continuous map $T \colon N_1 \to N_2$ satisfies*

$$T\left(\frac{f \circ g}{2}\right) = \frac{T(f) \circ T(g)}{2} \tag{1.1.3}$$

for every $f, g \in N_1$ if and only if T is affine.

Proof. If T is affine, then $T((f+g)/2) = (T(f) + T(g))/2$ for all $f, g \in N_1$, and therefore, T satisfies (1.1.3) by Remark 1.2. Conversely, suppose that T satisfies (1.1.3). Set $S(f) = T(f) - T(0)$ for each $f \in N_1$. Identifying $(f \circ g)/2$ with $(f+g)/2$ (see Remark 1.2), we see that $S((f+g)/2) = (S(f) + S(g))/2$ holds for all $f, g \in N_1$. We will prove that S is real-linear. To do this, pick $f, g \in N_1$ arbitrarily. Since $S(0) = 0$, it follows that

$$S(f) = S\left(\frac{2f}{2}\right) = \frac{S(2f) + S(0)}{2} = \frac{S(2f)}{2},$$

which shows that $S(2f) = 2S(f)$. Therefore,

$$S(f+g) = S\left(\frac{2(f+g)}{2}\right) = 2S\left(\frac{f+g}{2}\right) = S(f) + S(g),$$

which proves that S is additive. We next show that $S(mf/2^n) = mS(f)/2^n$ holds for every integer m and natural number n. Since $2S(f) = S(2f)$, we also have $S(f/2) = S(f)/2$. Inductively, we can prove that $S(f/2^n) = S(f)/2^n$ for every natural number n. Suppose that $S(mf) = mS(f)$. It follows from the additivity of S that

$$S((m+1)f) = S(mf) + S(f) = mS(f) + S(f) = (m+1)S(f).$$

By induction, we see that $S(mf) = mS(f)$ for every natural number m. Since S is additive, we see that $S(-f) = -S(f)$. It follows that

$$S(-mf) = -S(mf) = (-m)S(f)$$

for every natural number m. From the above, we have that

$$S\left(\frac{mf}{2^n}\right) = \frac{1}{2^n}S(mf) = \frac{m}{2^n}S(f)$$

for every integer m and natural number n. Since S is continuous, we see that $S(cf) = cS(f)$ for every $c \in \mathbf{R}$. Thus, S is real-linear, as claimed. In particular, $S((1-t)f + tg) = (1-t)S(f) + tS(g)$. Since $T(f) = S(f) + T(0)$, we conclude that T is affine. $\square$

Definition 1.6. For a map T from a metricoid space (G_1, δ_1) into another one (G_2, δ_2), T is said to be *affine* if

$$T\left(\frac{f \circ g}{2}\right) = \frac{T(f) \circ T(g)}{2}$$

holds for every pair $f,\, g \in G_1$.

Definition 1.7. A metricoid space (G, δ) is said to be *strongly reflective* if $(f \circ g)/2 \neq \varnothing$ for every $f,\, g \in G$.

Remark 1.3. If G is strongly reflective, then G is reflective. In fact, suppose that G is strongly reflective and take an element $f \in G$ arbitrarily. Then $(f \circ f)/2 \neq \varnothing$ by hypothesis, and hence there is a $\rho \in R(G; h)$ with $\rho(f) = f$ for some $h \in G$. Since h is the only fixed point of ρ by Remark 1.1, it follows that $f = h$. This implies that $R(G; f) \neq \varnothing$, and hence G is reflective.

Definition 1.8. Let (G, δ) be a metricoid space. (G, δ) is said to be *internally reflective metricoid group*, or *metricoid group* in short, if G has a group structure such that

(7) $\delta(hf^{-1}h, hg^{-1}h) = \delta(f, g)$ for every f, g, $h \in G$, and

(8) for each $h \in G$ there exists a constant $L(h) > 1$ such that $\delta(hf^{-1}h, f) \geq L(h)\delta(f, h)$ for every $f \in G$.

Remark 1.4. A metricoid group (G, δ) is reflective. In fact, pick $h \in G$ arbitrarily and put $\rho_h(f) = hf^{-1}h$ for every $f \in G$. In this case, we can easily see that ρ_h is a reflection on G at h, and so (G, δ) is reflective.

Definition 1.9. Let (G, δ) be a metricoid group. For each f, $g \in G$, we put

$$M(f, g) = \{h \in G : \rho_h(f) = g\},$$

where ρ_h is the reflection on G at $h \in G$ defined by $\rho_h(f) = hf^{-1}h$ for $f \in G$. A metricoid group (G, δ) is said to be *super reflective* if

(9) $M(f, g) \neq \varnothing$ for every f, $g \in G$.

Remark 1.5. By definition, we see that $M(f, g) \subseteq (f \circ g)/2$ holds for every f, $g \in G$. Therefore, if a metricoid group (G, δ) is super reflective, then (G, δ) is strongly reflective.

It is possible that the case where $(f \circ g)/2 \neq \varnothing$ and $M(f, g) = \varnothing$ occurs. However, if $M(f, g) \neq \varnothing$, then $(f \circ g)/2 = M(f, g)$ by Corollary 1.3.

2. A generalization of the Mazur-Ulam theorem. Under the definition of the strong reflectivity, Lemma 1.2 immediately implies the following result.

Theorem 2.1. *Every bijective (δ_1, δ_2)-isometry between strongly reflective metricoid spaces (G_1, δ_1) and (G_2, δ_2) is affine.*

Remark 2.1. If (G, δ) is a strongly reflective metricoid space, then δ must be symmetric, that is to say, $\delta(f, g) = \delta(g, f)$ holds for every $f, g \in G$. To see this, let $f, g \in G$ be arbitrary. By the strong reflectivity of (G, δ), we can choose a $\rho \in R(G; h)$ with $\rho(f) = g$ for some $h \in G$. Therefore, we have

$$\delta(f, g) = \delta(\rho(f), \rho(g)) = \delta(g, f),$$

and hence δ is symmetric.

As a direct corollary to Theorem 2.1, we obtain the following theorem of Mazur and Ulam.

Corollary 2.2 (the Mazur-Ulam theorem [**2**]). *Every bijective isometry between normed spaces is affine.*

Proof. By Remark 1.2, each normed space is strongly reflective. According to Theorem 2.1 and Lemma 1.4, we have the Mazur-Ulam theorem. □

It follows from Remark 1.2 that every normed space is a strongly reflective metricoid space. Moreover, we see that every normed space, as an additive group, is a metricoid group. So, the following result, a special case of Theorem 2.1, is a generalization of the Mazur-Ulam theorem [**2**].

Theorem 2.3. *Every bijective (δ_1, δ_2)-isometry between super reflective metricoid groups (G_1, δ_1) and (G_2, δ_2) is affine.*

We give two examples of super reflective metricoid groups.

Notation. In the remainder of this paper, $C(K)$ denotes the set of all *real-valued* continuous functions f on a compact Hausdorff space K and $C^+(K)$ the *subset* of all $f \in C(K)$ such that $f(x) > 0$ for every $x \in K$. We will regard $C^+(K)$ as a multiplicative group. For each $f \in C(K)$, we put $\|f\| = \sup\{|f(x)| : x \in K\}$. Since $C^+(K)$ is not a linear space, $\|\cdot\|$ is not a norm on $C^+(K)$. However, $\|\cdot\|$ induces a topology on $C^+(K)$. For each $f,\ g \in C^+(K)$, we put

$$\delta_+(f,g) = \Delta(f,g) + \Delta(g,f) \quad \text{and} \quad \delta_\times(f,g) = \Delta(f,g)\Delta(g,f),$$

where

$$\Delta(f,g) = \left\| \frac{f}{g} - 1 \right\|,$$

that is,

$$\delta_+(f,g) = \left\| \frac{f}{g} - 1 \right\| + \left\| \frac{g}{f} - 1 \right\|$$

and

$$\delta_\times(f,g) = \left\| \frac{f}{g} - 1 \right\| \left\| \frac{g}{f} - 1 \right\|$$

for $f, g \in C^+(K)$.

Under the above notation, we have the following result.

Theorem 2.4. *Let $\delta \in \{\delta_+, \delta_\times\}$. Then $(C^+(X), \delta)$ is a super reflective metricoid group with $(f \circ g)/2 = \sqrt{fg}$ for every $f,\ g \in C^+(X)$.*

Proof. Equations (1) and (7) are obviously true for δ. It is enough to prove that (2), (8) and (9) hold.

First, we prove (2). Let $f, g \in C^+(X)$ and T be a bijective δ-isometry on $C^+(X)$ with $T(g) = g$. Then we have by a simple calculation that

$$\begin{aligned} \Delta(T(f), f) &= \left\| \frac{T(f)}{f} - 1 \right\| \\ &\le \left(\left\| \frac{T(f)}{T(g)} - 1 \right\| + 1 \right) \left\| \frac{g}{f} \right\| + 1 \\ &\le \left(\left\| \frac{T(f)}{T(g)} - 1 \right\| + \left\| \frac{T(g)}{T(f)} - 1 \right\| + 1 \right) \left\| \frac{g}{f} \right\| + 1 \\ &= \{\delta_+(T(f), T(g)) + 1\} \left\| \frac{g}{f} \right\| + 1. \end{aligned} \tag{2.2.1}$$

In a similar way to the above, we have that

$$\Delta(f, T(f)) \le \{\delta_+(T(f), T(g)) + 1\} \left\| \frac{f}{g} \right\| + 1. \tag{2.2.2}$$

First, we consider the case where $\delta = \delta_+$. By (2.2.1), we see that

$$\begin{aligned} \Delta(T(f), f) &\le \{\delta_+(T(f), T(g)) + 1\}\left\|\frac{g}{f}\right\| + 1 \\ &= (\delta_+(f,g) + 1)\left\|\frac{g}{f}\right\| + 1 \end{aligned} \tag{2.2.3}$$

since T is assumed to be δ_+-isometry. Similarly to the above, it follows from (2.2.2) that

$$\Delta(f, T(f)) \le (\delta_+(f,g) + 1)\left\|\frac{f}{g}\right\| + 1. \tag{2.2.4}$$

By (2.2.3) and (2.2.4), we conclude that

$$\delta_+(T(f), f) \le (\delta_+(f,g) + 1)\left(\left\|\frac{f}{g}\right\| + \left\|\frac{g}{f}\right\|\right) + 2$$

holds, and so we have proved that (2) holds for $\delta = \delta_+$.

We next consider the case where $\delta = \delta_\times$. If $\|T(f)T(g)^{-1} - 1\| < 1/2$, then by a simple calculation we see that

$$-\frac{1}{3} < \frac{T(g)(x)}{T(f)(x)} - 1 < 1$$

holds for every $x \in X$, and so we have that

$$\delta_+(T(f), T(g)) = \left\|\frac{T(f)}{T(g)} - 1\right\| + \left\|\frac{T(g)}{T(f)} - 1\right\| < \frac{1}{2} + 1 < 2.$$

If $\Delta(T(f), T(g)) = \|T(f)T(g)^{-1} - 1\| \ge 1/2$, then let $x_0 \in X$ be such that

$$\left|\frac{T(f)(x_0)}{T(g)(x_0)} - 1\right| = \left\|\frac{T(f)}{T(g)} - 1\right\|.$$

It follows from an easy calculation that

$$\frac{T(g)(x_0)}{T(f)(x_0)} - 1 \le -\frac{1}{3} \quad \text{or} \quad 1 \le \frac{T(g)(x_0)}{T(f)(x_0)} - 1,$$

and so we get $\Delta(T(g), T(f)) = \|T(g)T(f)^{-1} - 1\| \geq 1/3$. Note that if $s \geq 1/2$ and $t \geq 1/3$, then the inequality $s + t \leq 5st$ holds. Consequently, we see that

$$\Delta(T(f), T(g)) + \Delta(T(g), T(f)) \leq 5\Delta(T(f), T(g))\Delta(T(g), T(f)),$$

which implies that

$$\delta_+(T(f), T(g)) \leq 5\delta_\times(T(f), T(g)) = 5\delta_\times(f, g)$$

since T is assumed to be $\delta_\times$-isometry. In any case, if we put $\alpha(f, g) = \max\{2,\ 5\delta_\times(f, g)\}$, then we have that

$$\delta_+(T(f), T(g)) \leq \alpha(f, g). \tag{2.2.5}$$

It follows from (2.2.1), (2.2.2) and (2.2.5) that

$$\begin{aligned}\delta_\times(T(f), f) = \Delta(T(f), f)\Delta(f, T(f)) \leq &\left\{(\alpha(f, g) + 1)\left\|\frac{g}{f}\right\| + 1\right\} \\ &\times \left\{(\alpha(f, g) + 1)\left\|\frac{f}{g}\right\| + 1\right\}\end{aligned}$$

holds, and hence we have proved that (2) holds for $\delta = \delta_\times$.

Secondly, we prove that (8) holds. Let f, $h \in C^+(X)$, and pick x, $y \in X$ so that

$$\left\|\frac{f}{h} - 1\right\| = \left|\frac{f(x)}{h(x)} - 1\right| \quad \text{and} \quad \left\|\frac{h}{f} - 1\right\| = \left|\frac{h(y)}{f(y)} - 1\right|.$$

To prove (8), it is enough to consider the case when $f \neq h$. Thus, we may and do assume that $f(y) \neq h(y)$. Then we have

$$\left|\frac{f(y)}{h(y)} - 1\right| \leq \left|\frac{f(x)}{h(x)} - 1\right| \quad \text{and} \quad \left|\frac{h(x)}{f(x)} - 1\right| \leq \left|\frac{h(y)}{f(y)} - 1\right|.$$

Since f, $h \in C^+(X)$, it follows from the above inequalities that

$$h(x)|f(y) - h(y)| \leq h(y)|f(x) - h(x)|$$

and

$$f(y)|h(x) - f(x)| \leq f(x)|h(y) - f(y)|$$

hold. We thus obtain

$$\begin{aligned} f(y)h(x)|f(y)-h(y)| &\le f(y)h(y)|f(x)-h(x)| \\ &\le h(y)f(x)|h(y)-f(y)|. \end{aligned}$$

Since $f(y) \neq h(y)$, we see that

$$f(y)h(x) \le f(x)h(y). \tag{2.2.6}$$

We first consider the case where $\delta = \delta_+$. Note that

$$|a^2-1| + \left|\frac{1}{b^2}-1\right| \ge \frac{11}{10}\left(|a-1| + \left|\frac{1}{b}-1\right|\right) \tag{2.2.7}$$

holds for all positive real numbers a, b with $a \ge b$. In fact, we see by a simple calculation that the inequality is equivalent to

$$|a-1|(10a-1) + \left|\frac{1}{b}-1\right|\left(\frac{10}{b}-1\right) \ge 0. \tag{2.2.8}$$

This is obviously true if $b \le 1 \le a$. If $1 \le b \le a$, then we have that

$$\begin{aligned} &|a-1|(10a-1) + \left|\frac{1}{b}-1\right|\left(\frac{10}{b}-1\right) \\ &\qquad = \left(a-\frac{1}{b}\right)\left\{10\left(a+\frac{1}{b}\right)-11\right\} \ge 0. \end{aligned}$$

If $b \le a \le 1$, then a similar argument shows that (2.2.8) holds. From the above, we have proved that the inequality (2.2.7) holds for $0 < b \le a$. So, it follows from (2.2.6) and (2.2.7) that

$$\begin{aligned} \frac{11}{10}\delta_+(f,h) &= \frac{11}{10}\left(\left|\frac{f(x)}{h(x)}-1\right| + \left|\frac{h(y)}{f(y)}-1\right|\right) \\ &\le \left|\frac{f(x)^2}{h(x)^2}-1\right| + \left|\frac{h(y)^2}{f(y)^2}-1\right| \\ &\le \delta_+(hf^{-1}h, f), \end{aligned}$$

which shows that $\delta_+(hf^{-1}h, f) \ge 11\delta_+(f,h)/10$. Hence, δ_+ satisfies (8) by taking $L(h) = 11/10$.

We next consider the case where $\delta = \delta_\times$. It follows from (2.2.6) that

$$2f(y)h(x) \le (f(x) + h(x))(f(y) + h(y)) \tag{2.2.9}$$

holds. Therefore, by (2.2.9), we have that

$$\begin{aligned}
2\delta_\times(f,h) &= 2\left\|\frac{f}{h} - 1\right\| \left\|\frac{h}{f} - 1\right\| \\
&= 2\left|\frac{f(x)}{h(x)} - 1\right|\left|\frac{h(y)}{f(y)} - 1\right| \\
&= 2\left|\frac{(f(x) - h(x))(h(y) - f(y))}{f(y)h(x)}\right| \\
&= 2\left|\frac{(f(x)^2 - h(x)^2)(h(y)^2 - f(y)^2)}{f(y)h(x)(f(x) + h(x))(f(y) + h(y))}\right| \\
&\le \left|\frac{(f(x)^2 - h(x)^2)(h(y)^2 - f(y)^2)}{f(y)^2h(x)^2}\right| \\
&= \left|\frac{f(x)^2}{h(x)^2} - 1\right|\left|\frac{h(y)^2}{f(y)^2} - 1\right| \\
&\le \left\|\frac{f^2}{h^2} - 1\right\| \left\|\frac{h^2}{f^2} - 1\right\|.
\end{aligned}$$

It follows that $2\delta_\times(f,h) \le \delta_\times(hf^{-1}h, f)$ holds. We thus conclude that $\delta_\times$ satisfies (8) by taking $L(h) = 2$.

Finally, we prove (9). Pick $f,\ g \in C^+(X)$ arbitrarily, and put $h = \sqrt{fg}$. Then $h \in M(f,g)$, and so (9) is proved. Therefore, we conclude that $(C^+(X), \delta)$ is a super reflective metricoid group. Recall that $(f \circ g)/2$ is an empty set, or a singleton, by Corollary 1.3. Since $\sqrt{fg} \in M(f,g) \subset (f \circ g)/2$, we have that $(f \circ g)/2 = \sqrt{fg}$, and the proof is complete. $\square$

3. Surjections on $C^+(X)$ which preserve subdistance. In this section we give representations for certain isometries from $C^+(X)$ onto $C^+(Y)$, which preserve subdistance. Put

$$\delta_{\max}(f,g) = \max\{\mathit{\Delta}(f,g), \mathit{\Delta}(g,f)\}, \quad (f, g \in C^+(K)),$$

that is,

$$\delta_{\max}(f,g) = \max\left\{\left\|\frac{f}{g} - 1\right\|, \left\|\frac{g}{f} - 1\right\|\right\}$$

for $f,\ g \in C^+(K)$. Here, K denotes a compact Hausdorff space. We first give a representation for $\delta_{\max}$-isometry.

Theorem 3.1. *Let T be a surjection from $C^+(X)$ onto $C^+(Y)$ such that the equality*

$$\delta_{\max}(T(f), T(g)) = \delta_{\max}(f, g)$$

holds for every $f, g \in C^+(X)$. Then there exist a $w \in C^+(Y)$, an $h \in C^+(Y)$ with $h(Y) \subset \{-1, 1\}$ and a homeomorphism $\Phi: Y \to X$ such that $T(f)(y) = \{w(y)f(\Phi(y))\}^{h(y)}$ holds for every $f \in C^+(X)$ and $y \in Y$.

Proof. Put $\widetilde{T} = T/T(1)$. Then we can easily see that $\widetilde{T}: C^+(X) \to C^+(Y)$ is a surjection such that

$$\delta_{\max}(\widetilde{T}(f), \widetilde{T}(g)) = \delta_{\max}(f, g), \quad (f, g \in C^+(X)). \tag{3.3.1}$$

Note that $\widetilde{T}$ is injective by (3.3.1). It follows that $\widetilde{T}$ is a bijection. Let $S: C(X) \to C(Y)$ be a map defined by

$$S(u) = \log(\widetilde{T}(\exp u)), \quad (u \in C(X)). \tag{3.3.2}$$

We show that $\|S(u) - S(v)\| = \|u - v\|$ holds for every $u,\ v \in C(X)$. To this end, pick $u,\ v \in C(X)$ arbitrarily. Since

$$\exp \|u\| - 1 = \max \left\{ \|\exp u - 1\|,\ \left\| \frac{1}{\exp u} - 1 \right\| \right\},$$

we see that

$$\delta_{\max}(\exp u, \exp v) = \exp \|u - v\| - 1. \tag{3.3.3}$$

A quite similar argument to the above shows that

$$\delta_{\max}(\exp S(u), \exp S(v)) = \exp \|S(u) - S(v)\| - 1. \tag{3.3.4}$$

Since $\exp S(u) = \exp(\log(\widetilde{T}(\exp u))) = \widetilde{T}(\exp u)$, it follows from (3.3.1), (3.3.3) and (3.3.4) that

$$\begin{aligned} \exp \|S(u) - S(v)\| - 1 &= \delta_{\max}(\widetilde{T}(\exp u), \widetilde{T}(\exp v)) \\ &= \delta_{\max}(\exp u, \exp v) = \exp \|u - v\| - 1, \end{aligned}$$

which proves that $\|S(u)-S(v)\| = \|u-v\|$. Thus S is an isometry and it is bijective since $\widetilde{T}$ is. By Corollary 2.2, S is affine. Since $S(0)=0$, we have that S is real-linear. Consequently, S is a bijective real-linear isometry. By the Banach-Stone theorem there exist an $h \in C(Y)$ with $h(Y) \subset \{-1,1\}$ and a homeomorphism $\Phi\colon Y \to X$ such that $S(u)(y) = h(y)u(\Phi(y))$ holds for every $u \in C(X)$ and $y \in Y$. Therefore, for each $f \in C^+(X)$ and $y \in Y$, we have by (3.3.2) that

$$\begin{aligned}\widetilde{T}(f)(y) &= \exp(S(\log f)(y)) = \exp(h(y)\log f(\Phi(y)) \\ &= \{f(\Phi(y))\}^{h(y)},\end{aligned}$$

and hence

$$T(f)(y) = T(1)(y)\{f(\Phi(y))\}^{h(y)}$$

holds for every $f \in C^+(X)$ and $y \in Y$. Put $w(y) = \{T(1)(y)\}^{h(y)}$. Then we see that

$$(w(y))^{h(y)} = \{T(1)(y)^{h(y)}\}^{h(y)} = \{T(1)(y)\}^{h(y)^2} = T(1)(y).$$

We thus conclude that

$$T(f)(y) = (w(y))^{h(y)}\{f(\Phi(y))\}^{h(y)} = \{w(y)f(\Phi(y)\}^{h(y)} \tag{3.3.5}$$

holds for every $f \in C^+(X)$ and $y \in Y$. This completes the proof. □

Remark 3.1. Let $B(H)$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space H. We denote by $B(H)^+_{-1}$ the set of all invertible positive operators of $B(H)$. The Thompson metric δ_T on $B(H)^+_{-1}$ is given by

$$\delta_T(A,B) = \|\log A^{-1/2}BA^{-1/2}\|, \quad (A, B \in B(H)^+_{-1}).$$

In [**4**] Molnár characterized bijective isometries from $B(H)^+_{-1}$ onto itself with respect to the Thompson metric. It seems natural to consider surjections T from $C^+(X)$ onto $C^+(Y)$ that satisfy

$$\|\log T(f) - \log T(g)\| = \|\log f - \log g\|, \quad (f, g \in C^+(X)).$$

If we define $S\colon C(X) \to C(Y)$ by (3.3.2), then we see that S is a bijective isometry. In the same way as in the proof of Theorem 3.1, we can prove that T is of the form (3.3.5).

Corollary 3.2. *Let T be a surjection from $C^+(X)$ onto $C^+(Y)$ such that*

$$\Delta(T(f), T(g)) = \Delta(f, g)$$

holds for every $f, g \in C^+(X)$. Then there exist a $w \in C^+(Y)$ and a homeomorphism $\Phi : Y \to X$ such that $T(f) = w(f \circ \Phi)$ holds for every $f \in C^+(X)$.

Proof. Since $\Delta(T(f), T(g)) = \Delta(f, g)$ for f, $g \in C^+(X)$, we see that

$$\delta_{\max}(T(f), T(g)) = \delta_{\max}(f, g), \quad (f, g \in C^+(X))$$

by the definition of $\delta_{\max}$. It follows from Theorem 3.1 that there exist a $w \in C^+(Y)$, an $h \in C^+(Y)$ with $h(Y) \subset \{-1, 1\}$ and a homeomorphism $\Phi: Y \to X$ such that $T(f)(y) = \{w(y)f(\Phi(y))\}^{h(y)}$ for every $f \in C^+(X)$ and $y \in Y$. So it is enough to show that $h = 1$ on Y. Since $T(1)(y) = w(y)^{h(y)}$ and $T(2)(y) = 2^{h(y)}w(y)^{h(y)}$, we have that $T(1)(y)/T(2)(y) = 1/2^{h(y)}$ for all $y \in Y$, and hence

$$\left\| \frac{1}{2^h} - 1 \right\| = \left\| \frac{T(1)}{T(2)} - 1 \right\| = \left\| \frac{1}{2} - 1 \right\| = \frac{1}{2},$$

which implies that $h = 1$ on Y. □

Finally, we give representations for isometries which preserve subdistance δ_+ or $\delta_\times$.

Theorem 3.3. *Let $\delta \in \{\delta_+, \delta_\times\}$ and T be a surjection from $C^+(X)$ onto $C^+(Y)$ such that the equality*

$$\delta(T(f), T(g)) = \delta(f, g) \tag{3.3.6}$$

holds for every f, $g \in C^+(X)$. Then there exist a $w \in C^+(Y)$ and a homeomorphism $\Phi: Y \to X$ such that $T(f)$ is either of the form

$$T(f) = w(f \circ \Phi), \quad (f \in C^+(X))$$

or of the form

$$T(f) = \frac{1}{w(f \circ \Phi)}, \quad (f \in C^+(X)).$$

To prove this, we fix some notation.

Notation. For a, $b \in \mathbf{R}$, put

$$\gamma_+(a,b) = a+b, \quad \gamma_\times(a,b) = ab.$$

Recall that

$$\delta_+(f,g) = \left\|\frac{f}{g}-1\right\| + \left\|\frac{g}{f}-1\right\|, \quad \delta_\times(f,g) = \left\|\frac{f}{g}-1\right\|\left\|\frac{g}{f}-1\right\|.$$

So, we can rewrite δ_+ and $\delta_\times$ as

$$\delta_+(f,g) = \gamma_+(\mathit{\Delta}(f,g), \mathit{\Delta}(g,f)),$$
$$\delta_\times(f,g) = \gamma_\times(\mathit{\Delta}(f,g), \mathit{\Delta}(g,f))$$

for f, $g \in C^+(K)$, where K denotes a compact Hausdorff space. Let $(\delta,\gamma) \in \{(\delta_+,\gamma_+), (\delta_\times,\gamma_\times)\}$. From Lemma 3.5 to Lemma 3.11, T denotes a surjection from $C^+(X)$ onto $C^+(Y)$ such that (3.3.6) holds for every f, $g \in C^+(X)$.

Lemma 3.4. *No combination of real numbers m, M and n_0 exists such that*

$$(M^n-1)\left(\frac{1}{m^n}-1\right) = \frac{1}{3^n}(3^n-1)^2 \tag{3.3.7}$$

holds for every natural number $n \geq n_0$.

Proof. Suppose, on the contrary, that there exist real numbers m, M and n_0 satisfying (3.3.7) for all $n \geq n_0$. Pick $n \geq n_0$ arbitrarily. By (3.3.7), we see that

$$(M^{2n}-1)\left(\frac{1}{m^{2n}}-1\right) = \frac{1}{3^{2n}}(3^{2n}-1)^2$$

holds. So we have that

$$\begin{aligned}(M^n-1)(M^n+1)\left(\frac{1}{m^n}-1\right)\left(\frac{1}{m^n}+1\right)\\ = \frac{1}{3^n}(3^n-1)^2\frac{1}{3^n}(3^n+1)^2.\end{aligned} \tag{3.3.8}$$

It follows from (3.3.7) and (3.3.8) that

$$(M^n+1)\left(\frac{1}{m^n}+1\right)=\frac{1}{3^n}(3^n+1)^2. \tag{3.3.9}$$

Subtraction of (3.3.7) from (3.3.9) gives

$$\begin{aligned}(M^n+1)&\left(\frac{1}{m^n}+1\right)-(M^n-1)\left(\frac{1}{m^n}-1\right)\\&=\frac{1}{3^n}\left\{(3^n+1)^2-(3^n-1)^2\right\}=4.\end{aligned}$$

On the other hand, since

$$(M^n+1)\left(\frac{1}{m^n}+1\right)-(M^n-1)\left(\frac{1}{m^n}-1\right)=2\left(M^n+\frac{1}{m^n}\right),$$

we now get

$$2\left(M^n+\frac{1}{m^n}\right)=4,\quad\text{and hence}\quad\frac{1}{m^n}=2-M^n.$$

Since $n\ge n_0$ is arbitrary, we have

$$4-4M^n+M^{2n}=(2-M^n)^2=\frac{1}{m^{2n}}=2-M^{2n}.$$

It follows that $M^{2n}-2M^n+1=0$. Consequently, $M^n=1$, which contradicts (3.3.7). This completes the proof. □

Lemma 3.5. *T is a continuous map with respect to the topology induced by $\|\cdot\|$ such that*

$$T((fg)^{1/2})=T(f)^{1/2}T(g)^{1/2} \tag{3.3.10}$$

holds for every f, $g\in C^+(X)$.

Proof. Pick $\{f_n\}\subset C^+(X)$ so that $\|f_n-f\|\to 0$ as $n\to\infty$ for some $f\in C^+(X)$. In this case, we have that

$$\Delta(f,f_n)=\left\|\frac{f}{f_n}-1\right\|\longrightarrow 0$$

and

$$\varDelta(f_n, f) = \left\| \frac{f_n}{f} - 1 \right\| \to 0$$

as $n \to \infty$. Since

$$\delta(T(f_n), T(f)) = \delta(f_n, f) = \gamma(\varDelta(f_n, f), \varDelta(f, f_n)) \longrightarrow 0$$

as $n \to \infty$, we see that $\varDelta(T(f_n), T(f))$ or $\varDelta(T(f), T(f_n))$ converges to 0 as $n \to \infty$. In any case, we have that

$$\varDelta(T(f_n), T(f)) = \left\| \frac{T(f_n)}{T(f)} - 1 \right\| \longrightarrow 0 \quad (\text{as } n \to \infty).$$

This implies that

$$\|T(f_n) - T(f)\| \le \|T(f)\| \left\| \frac{T(f_n)}{T(f)} - 1 \right\| \longrightarrow 0$$

as $n \to \infty$, and so T is continuous.

We will prove that (3.3.10) holds for every $f,\, g \in C^+(X)$. Since $\delta(T(f), T(g)) = \delta(f, g)$ for $f,\, g \in C^+(X)$, we have that T is injective. Recall that for a compact Hausdorff space K, $(C^+(K), \delta)$ is a super reflective metricoid group by Theorem 2.4. It follows from Theorem 2.1 that T is affine, that is,

$$T\left(\frac{f \circ g}{2}\right) = \frac{T(f) \circ T(g)}{2}$$

holds for every $f,\, g \in C^+(X)$ (cf. Definition 1.6). Note that $(f \circ g)/2 = \sqrt{fg}$ by Theorem 2.4, and so we get

$$T((fg)^{1/2}) = T(f)^{1/2} T(g)^{1/2}$$

for every $f,\, g \in C^+(X)$. This completes the proof. □

By Lemma 3.5, T is a surjection from $C^+(X)$ onto $C^+(Y)$ such that

$$T((fg)^{1/2}) = T(f)^{1/2} T(g)^{1/2}$$

holds for every $f,\, g \in C^+(X)$. From Lemma 3.6 to Lemma 3.8, we will use the above property.

Lemma 3.6. *The following equations hold.*
(a) $T(fg) = T(f)T(g)T(1)^{-1}$ *for every* $f,\, g \in C^+(X)$.
(b) $T(f^{-1}) = T(f)^{-1}T(1)^2$ *for every* $f \in C^+(X)$.
(c) $T(f^m) = T(f)^m T(1)^{1-m}$ *for every* $f \in C^+(X)$ *and integer* m.

Proof. (a) By (3.3.10), we have that

$$T(f) = T((f^2 \cdot 1)^{1/2}) = T(f^2)^{1/2}T(1)^{1/2}.$$

This implies that $T(f)^2 = T(f^2)T(1)$, and hence $T(f^2) = T(f)^2T(1)^{-1}$ holds for $f \in C^+(X)$. Then we have that

$$T(fg) = T\Big(\left(f^2g^2\right)^{1/2} \Big) = T(f^2)^{1/2}T(g^2)^{1/2} = T(f)T(g)T(1)^{-1}$$

holds for every $f,\, g \in C^+(X)$.

(b) It follows from (a) that

$$T(1) = T(ff^{-1}) = T(f)T(f^{-1})T(1)^{-1}$$

for $f \in C^+(X)$, which shows that $T(f^{-1}) = T(1)^2T(f)^{-1}$ holds for every $f \in C^+(X)$.

(c) By induction, with (a), we see that $T(f^m) = T(f)^mT(1)^{1-m}$ holds for every $f \in C^+(X)$ and nonnegative integer m. Pick an integer $m \le -1$ arbitrarily. Since $T(f^m) = T((f^{-1})^{-m})$, it follows from the equation above and (b) that

$$\begin{aligned} T(f^m) &= T(f^{-1})^{-m}T(1)^{1+m} \\ &= \left(T(f)^{-1}T(1)^2\right)^{-m} T(1)^{1+m} \\ &= T(f)^mT(1)^{1-m} \end{aligned}$$

for every $f \in C^+(X)$. Since $m \le -1$ is arbitrary, this proves (c). □

Lemma 3.7. *Let* n *be a positive integer. Then the equation*

$$T(f^rg^{1-r}) = T(f)^rT(g)^{1-r} \tag{3.3.11}$$

holds for every f, $g \in C^+(X)$ and every r of the form $r = m/2^n$, where m is a non-negative integer with $0 \leq m \leq 2^n$.

Proof. We prove equation (3.3.11) by induction with respect to n. Equation (3.3.11) holds for $n = 1$ by hypothesis. Suppose that (3.3.11) holds for n. Pick an integer m with $0 \leq m \leq 2^{n+1}$ arbitrarily, and put $r = m/2^{n+1}$. We first consider the case where m is even, say $m = 2l$. Since

$$r = \frac{m}{2^{n+1}} = \frac{l}{2^n}, \quad \text{where} \quad 0 \leq l \leq 2^n,$$

by the hypothesis of induction, we have that $T(f^r g^{1-r}) = T(f)^r T(g)^{1-r}$.

We next consider the case where $m = 2l + 1$. Since

$$r = \frac{2l+1}{2^{n+1}} = \left(\frac{l}{2^n} + \frac{l+1}{2^n}\right)\frac{1}{2},$$

we can write

$$f^r = \left(f^{l/2^n} f^{(l+1)/2^n}\right)^{1/2}$$

and

$$g^{1-r} = \left(g^{1-l/2^n} g^{1-(l+1)/2^n}\right)^{1/2}.$$

Note that $l + 1 \leq 2^n$ since $2l + 1 = m \leq 2^{n+1}$. By hypothesis of induction, we have that

$$\begin{aligned}
T(f^r g^{1-r}) &= T\Big(\left(f^{l/2^n} g^{1-l/2^n}\right)^{1/2} \left(f^{(l+1)/2^n} g^{1-(l+1)/2^n}\right)^{1/2} \Big) \\
&= T\left(f^{l/2^n} g^{1-l/2^n}\right)^{1/2} T\left(f^{(l+1)/2^n} g^{1-(l+1)/2^n}\right)^{1/2} \\
&= \left(T(f)^{l/2^n} T(g)^{1-l/2^n}\right)^{1/2} \left(T(f)^{(l+1)/2^n} T(g)^{1-(l+1)/2^n}\right)^{1/2} \\
&= T(f)^{(2l+1)/2^{n+1}} T(g)^{1-(2l+1)/2^{n+1}} \\
&= T(f)^r T(g)^{1-r}
\end{aligned}$$

holds. By induction, this completes the proof. □

Lemma 3.8. *The equation*

$$T(f^p g^{1-p}) = T(f)^p T(g)^{1-p}$$

holds for every f, $g \in C^+(X)$ *and* $p \in \mathbf{R}$.

Proof. We first consider the case where $0 \leq p \leq 1$. There exists a sequence of dyadic rational numbers p_n such that p_n converges to p as $n \to \infty$. Note that T is continuous by Lemma 3.5. It follows from Lemma 3.7 that $T(f^p g^{1-p}) = T(f)^p T(g)^{1-p}$ holds for every f, $g \in C^+(X)$.

Next we consider the general case. Pick $p \in \mathbf{R}$ arbitrarily, and put $p = m + r$, where m is an integer and $0 \leq r < 1$. It follows from (a) of Lemma 3.6 that

$$\begin{aligned} T(f^p g^{1-p}) &= T(f^m f^r g^{1-r} g^{-m}) \\ &= T(f^m g^{-m}) T(f^r g^{1-r}) T(1)^{-1} \\ &= T(f^m) T(g^{-m}) T(1)^{-1} T(f^r g^{1-r}) T(1)^{-1} \\ &= T(f^m) T(g^{-m}) T(f)^r T(g)^{1-r} T(1)^{-2} \end{aligned}$$

since $0 \leq r < 1$. By (c) of Lemma 3.6, we obtain that

$$T(f^m) = T(f)^m T(1)^{1-m}$$

and

$$T(g^{-m}) = T(g)^{-m} T(1)^{1+m},$$

and consequently, $T(f^m)T(g^{-m}) = T(f)^m T(g)^{-m} T(1)^2$. It follows that

$$\begin{aligned} T(f^p g^{1-p}) &= T(f)^m T(g)^{-m} T(1)^2 T(f)^r T(g)^{1-r} T(1)^{-2} \\ &= T(f)^{m+r} T(g)^{1-(m+r)} = T(f)^p T(g)^{1-p} \end{aligned}$$

holds for every f, $g \in C^+(X)$. $\square$

Notation. Let K be a compact Hausdorff space. For $u \in C^+(K)$, put

$$m_u = \min\{u(x) : x \in K\} \quad \text{and} \quad M_u = \max\{u(x) : x \in K\}.$$

Lemma 3.9. *Suppose that $T(1) = 1$ and $T(fg) = T(f)T(g)$ holds for every $f, g \in C^+(X)$. Let $\phi \in C^+(X)$ and $\varphi \in C^+(Y)$ be such that $T(\phi) = \varphi$.*

(a) *If $1 \le \phi \le 3$ with $\|\phi\| = 3$, then*

$$m_\varphi = \frac{1}{3} \le \varphi \le 1, \quad \text{or} \quad 1 \le \varphi \le 3 = M_\varphi.$$

(b) *If $1 \le \varphi \le 3$ with $\|\varphi\| = 3$, then*

$$m_\phi = \frac{1}{3} \le \phi \le 1, \quad \text{or} \quad 1 \le \phi \le 3 = M_\phi.$$

Proof. Since $T(1) = 1$, it follows from (3.3.6) that $\delta(T(f), 1) = \delta(f, 1)$ holds for every $f \in C^+(X)$. This implies that

$$\delta(\varphi, 1) = \delta(T(\phi), 1) = \delta(\phi, 1).$$

(a) If $1 \le \phi \le 3$ with $\|\phi\| = 3$, then we see that

$$\Delta(\phi, 1) = \|\phi - 1\| = 2 \quad \text{and} \quad \Delta(1, \phi) = \left\| \frac{1}{\phi} - 1 \right\| = \frac{2}{3}.$$

It follows that

$$\begin{aligned} \gamma(\Delta(\varphi, 1), \Delta(1, \varphi)) &= \delta(\varphi, 1) = \delta(\phi, 1) \\ &= \gamma(\Delta(\phi, 1), \Delta(1, \phi)) \\ &= \gamma\left(2, \frac{2}{3}\right), \end{aligned}$$

and so

$$\gamma(\Delta(\varphi, 1), \Delta(1, \varphi)) = \gamma\left(2, \frac{2}{3}\right). \tag{3.3.12}$$

If $\gamma = \gamma_+$, then we have

$$\begin{aligned} \|\varphi - 1\| + \left\| \frac{1}{\varphi} - 1 \right\| &= \gamma_+(\Delta(\varphi, 1), \Delta(1, \varphi)) \\ &= \gamma_+\left(2, \frac{2}{3}\right) = \frac{8}{3}. \end{aligned}$$

If $\gamma = \gamma_\times$, then we have

$$\|\varphi - 1\| \left\| \frac{1}{\varphi} - 1 \right\| = \gamma_\times(\Delta(\varphi, 1), \Delta(1, \varphi))$$
$$= \gamma_\times\left(2, \frac{2}{3}\right) = \frac{4}{3}.$$

In any case, we see that $1/3 \leq \varphi \leq 3$.

Secondly, we prove that $m_\varphi < 1$ implies $M_\varphi \leq 1$. Suppose, on the contrary, that $m_\varphi < 1 < M_\varphi$. Let n be a positive integer. Since T is assumed to preserve multiplication, $T(\phi^n) = \varphi^n$ holds. So, for a sufficiently large n with ${M_\varphi}^n > 2$ and $1/{m_\varphi}^n > 2$, we have that

$${M_\varphi}^n - 1 = \|\varphi^n - 1\| = \Delta(\varphi^n, 1)$$

and

$$\frac{1}{{m_\varphi}^n} - 1 = \left\| \frac{1}{\varphi^n} - 1 \right\| = \Delta(1, \varphi^n).$$

This implies that

$$\gamma\left({M_\varphi}^n - 1, \frac{1}{{m_\varphi}^n} - 1\right) = \gamma(\Delta(\varphi^n, 1), \Delta(1, \varphi^n))$$
$$= \delta(\varphi^n, 1) = \delta(T(\phi^n), 1) = \delta(\phi^n, 1)$$
$$= \gamma\left(\|\phi^n - 1\|, \left\| \frac{1}{\phi^n} - 1 \right\|\right).$$

Since $1 \leq \phi \leq 3$ with $\|\phi\| = 3$, we see that

$$\|\phi^n - 1\| = 3^n - 1 \quad \text{and} \quad \left\| \frac{1}{\phi^n} - 1 \right\| = 1 - \frac{1}{3^n}.$$

It follows that

$$\gamma\left({M_\varphi}^n - 1, \frac{1}{{m_\varphi}^n} - 1\right) = \gamma\left(3^n - 1, 1 - \frac{1}{3^n}\right). \tag{3.3.13}$$

If $\gamma = \gamma_+$, then we have

$$({M_\varphi}^n - 1) + \left(\frac{1}{{m_\varphi}^n} - 1\right) = (3^n - 1) + \left(1 - \frac{1}{3^n}\right)$$

that is,

$$\left(\frac{M_\varphi}{3}\right)^n + \left(\frac{1}{3m_\varphi}\right)^n - \frac{2}{3^n} = 1 - \frac{1}{9^n}. \tag{3.3.14}$$

Letting $n \to \infty$, we have that $M_\varphi = 3$, or $m_\varphi = 1/3$. If $M_\varphi = 3$, it would follow from (3.3.14) that

$$\frac{1}{{m_\varphi}^n} - 2 = -\frac{1}{3^n},$$

which would be a contradiction since $1/{m_\varphi}^n > 2$. Therefore, we have $m_\varphi = 1/3$. From (3.3.14), we obtain

$${M_\varphi}^n - 2 = -\frac{1}{3^n},$$

which is also impossible since ${M_\varphi}^n > 2$. We now arrive at a contradiction. This proves that $m_\varphi < 1$ implies $M_\varphi \leq 1$ if $\gamma = \gamma_+$.

If $\gamma = \gamma_\times$, then it follows from (3.3.13) that

$$({M_\varphi}^n - 1)\left(\frac{1}{{m_\varphi}^n} - 1\right) = (3^n - 1)\left(1 - \frac{1}{3^n}\right) = \frac{1}{3^n}(3^n - 1)^2$$

for every $n \geq 1$. This is a contradiction by Lemma 3.4. We thus conclude that $m_\varphi < 1$ implies $M_\varphi \leq 1$ even if $\gamma = \gamma_\times$.

Since $1/3 \leq \varphi \leq 3$, we now get that

$$\frac{1}{3} \leq m_\varphi \leq \varphi \leq M_\varphi \leq 1, \quad \text{or} \quad 1 \leq m_\varphi \leq \varphi \leq M_\varphi \leq 3.$$

Finally, we prove that $m_\varphi = 1/3$ if $m_\varphi < 1$, and that $M_\varphi = 3$ if $1 \leq m_\varphi$. In fact, if $m_\varphi < 1$, then $m_\varphi \leq M_\varphi \leq 1$ as proved above. So, we obtain

$$\Delta(\varphi, 1) = \|\varphi - 1\| = 1 - m_\varphi$$

and

$$\Delta(1,\varphi) = \left\| \frac{1}{\varphi} - 1 \right\| = \frac{1}{m_\varphi} - 1.$$

It follows from (3.3.12) that

$$\gamma\left(1 - m_\varphi, \frac{1}{m_\varphi} - 1\right) = \gamma(\Delta(\varphi, 1), \Delta(1, \varphi)) = \gamma\left(2, \frac{2}{3}\right).$$

If $\gamma = \gamma_+$, then we have that

$$\frac{1}{m_\varphi} - m_\varphi = 1 - m_\varphi + \left(\frac{1}{m_\varphi} - 1\right) = 2 + \frac{2}{3} = \frac{8}{3}.$$

Since $0 < m_\varphi$, we have that $m_\varphi = 1/3$.

If $\gamma = \gamma_\times$, then we have that

$$(1 - m_\varphi)\left(\frac{1}{m_\varphi} - 1\right) = 2 \times \frac{2}{3} = \frac{4}{3}.$$

Since $m_\varphi < 1$, we obtain $m_\varphi = 1/3$. This proves that $m_\varphi = 1/3$ if $m_\varphi < 1$.

Suppose that $1 \le m_\varphi$. Then we have $1 \le m_\varphi \le M_\varphi \le 3$ as proved above. It follows that

$$\Delta(\varphi, 1) = \|\varphi - 1\| = M_\varphi - 1 \quad \text{and} \quad \Delta(1, \varphi) = \left\| \frac{1}{\varphi} - 1 \right\| = 1 - \frac{1}{M_\varphi}.$$

In a way similar to the above, we see that $M_\varphi = 3$ if $1 \le m_\varphi$.

(b) Suppose that $1 \le \varphi \le 3$ with $\|\varphi\| = 3$. By interchanging ϕ with φ, we see that the proof of (a) works well. So, we get

$$m_\phi = \frac{1}{3} \le \phi \le 1, \quad \text{or} \quad 1 \le \phi \le 3 = M_\phi.$$

This completes the proof. □

Lemma 3.10. *Suppose that $T(1) = 1$ and $T(fg) = T(f)T(g)$ holds for every f, $g \in C^+(X)$.*

(a) *If* $1 \le m_{T(3)}$*, then* $T(3) = 3$ *on* Y*.*

(b) *If* $m_{T(3)} < 1$*, then* $T(3) = 1/3$ *on* Y*.*

Proof. Put $T(3) = \varphi \in C^+(Y)$. By (a) of Lemma 3.9 we see that

$$m_\varphi = \frac{1}{3} \le \varphi \le 1, \quad \text{or} \quad 1 \le \varphi \le 3 = M_\varphi. \tag{3.3.15}$$

(a) We show that $\varphi = 3$ on Y whenever $1 \le m_\varphi$. Suppose, on the contrary, that $\varphi(y_0) \ne 3$ for some $y_0 \in Y$, while $1 \le m_\varphi$. Set $\varepsilon_0 = \varphi(y_0)$ and ${\varepsilon_0}' = 1/(3 - \varepsilon_0)$. Then $1 \le \varepsilon_0 < 3$ and $1/2 \le {\varepsilon_0}'$ since $1 \le \varphi \le 3 = M_\varphi$. If we define $\zeta\colon [1,3] \to [1,3]$ by

$$\zeta(t) = \begin{cases} 3{\varepsilon_0}^{-1} t & \text{if } 1 \le t \le \varepsilon_0 \\ 2{\varepsilon_0}'(t-3) + 1 & \text{if } \varepsilon_0 < t \le 3 \end{cases},$$

then ζ is onto. Set $\widetilde{\varphi} = \zeta \circ \varphi \in C^+(Y)$. By a simple calculation we see that $1 \le \widetilde{\varphi} \le 3 = M_{\widetilde{\varphi}}$, $\|\varphi/\widetilde{\varphi}\| = 3$ and $\|\widetilde{\varphi}/\varphi\| = 3{\varepsilon_0}'$. Therefore, we have that

$$\Delta(\varphi^n, \widetilde{\varphi}^n) = \left\| \frac{\varphi^n}{\widetilde{\varphi}^n} - 1 \right\| = 3^n - 1$$

and that

$$\Delta(\widetilde{\varphi}^n, \varphi^n) = \left\| \frac{\widetilde{\varphi}^n}{\varphi^n} - 1 \right\| = (3{\varepsilon_0}')^n - 1$$

holds for a sufficiently large n, and hence

$$\begin{aligned} \delta(\varphi^n, \widetilde{\varphi}^n) &= \gamma(\Delta(\varphi^n, \widetilde{\varphi}^n), \Delta(\widetilde{\varphi}^n, \varphi^n)) \\ &= \gamma\,(3^n - 1, (3{\varepsilon_0}')^n - 1) \end{aligned} \tag{3.3.16}$$

holds for a sufficiently large n.

Since T is surjective, there exists a $\phi \in C^+(X)$ such that $T(\phi) = \widetilde{\varphi}$. By (b) of Lemma 3.9, we see that

$$\frac{1}{3} = m_\phi \le \phi \le 1, \quad \text{or} \quad 1 \le \phi \le 3 = M_\phi.$$

Since T preserves multiplication, it follows from (3.3.16) that

$$\begin{aligned} \delta(3^n, \phi^n) &= \delta(T(3^n), T(\phi^n)) \\ &= \delta(T(3)^n, T(\phi)^n) = \delta(\varphi^n, \tilde{\varphi}^n) \\ &= \gamma\,(3^n - 1, (3{\varepsilon_0}')^n - 1) \end{aligned} \tag{3.3.17}$$

holds for sufficiently large n. Suppose that $1/3 = m_\phi \leq \phi \leq 1$. In this case, we have that

$$\varDelta(3^n, \phi^n) = \left\| \frac{3^n}{\phi^n} - 1 \right\| = 9^n - 1 \tag{3.3.18}$$

and

$$\varDelta(\phi^n, 3^n) = \left\| \frac{\phi^n}{3^n} - 1 \right\| = 1 - \frac{1}{9^n} \tag{3.3.19}$$

for every n.

We first consider the case where $\gamma = \gamma_+$. Recall that

$$\delta(3^n, \phi^n) = \gamma(\varDelta(3^n, \phi^n), \varDelta(\phi^n, 3^n))$$

for every n. Since $\gamma = \gamma_+$, it follows from (3.3.18) and (3.3.19) that

$$\begin{aligned} \delta(3^n, \phi^n) &= \gamma_+(\varDelta(3^n, \phi^n), \varDelta(\phi^n, 3^n)) \\ &= (9^n - 1) + \left(1 - \frac{1}{9^n}\right) \\ &= 9^n - \frac{1}{9^n} \end{aligned} \tag{3.3.20}$$

for every n. By (3.3.17) and (3.3.20), we get

$$\begin{aligned} 9^n - \frac{1}{9^n} &= \delta(3^n, \phi^n) \\ &= \gamma_+\,(3^n - 1, (3{\varepsilon_0}')^n - 1) \\ &= (3^n - 1) + (3{\varepsilon_0}')^n - 1 \\ &= 3^n + (3{\varepsilon_0}')^n - 2, \end{aligned}$$

and hence

$$1-\frac{1}{9^{2n}}=\frac{1}{3^n}+\left(\frac{{\varepsilon_0}'}{3}\right)^n-\frac{2}{9^n}$$

for a sufficiently large n, which is impossible since ${\varepsilon_0}' \geq 1/2$. This shows that $1 \leq \phi \leq 3 = M_\phi$. In this case, we have that

$$\varDelta(3^n,\phi^n)=\left\|\frac{3^n}{\phi^n}-1\right\|=3^n-1 \tag{3.3.21}$$

and that

$$\varDelta(\phi^n,3^n)=\left\|\frac{\phi^n}{3^n}-1\right\|<1 \tag{3.3.22}$$

for every n. It follows from (3.3.17)–(3.3.19) that

$$\begin{aligned}(3^n-1)+1 &> \gamma_+(\varDelta(3^n,\phi^n),\varDelta(\phi^n,3^n))\\ &= \delta_+(3^n,\phi^n)=\gamma_+\left(3^n-1,(3{\varepsilon_0}')^n-1\right)\\ &= 3^n+(3{\varepsilon_0}')^n-2,\end{aligned}$$

which implies that $3^n > 3^n + (3{\varepsilon_0}')^n - 2$ for a sufficiently large n. This is also impossible since ${\varepsilon_0}' \geq 1/2$. We now arrived at a contradiction. This proves that if $\gamma = \gamma_+$, then $\varphi = 3$ on Y whenever $1 \leq m_\varphi$.

Next, we consider the case where $\gamma = \gamma_\times$. Recall that

$$\frac{1}{3}=m_\phi \leq \phi \leq 1, \quad \text{or} \quad 1 \leq \phi \leq 3 = M_\phi.$$

Suppose that $1/3 = m_\phi \leq \phi \leq 1$. It follows from (3.3.17)–(3.3.19) that

$$\begin{aligned}(9^n-1)\left(1-\frac{1}{9^n}\right) &= \gamma_\times(\varDelta(3^n,\phi^n),\varDelta(\phi^n,3^n))\\ &= \delta_\times(3^n,\phi^n)=\gamma_\times\left(3^n-1,(3{\varepsilon_0}')^n-1\right)\\ &= (3^n-1)\left\{(3{\varepsilon_0}')^n-1\right\},\end{aligned}$$

which proves that

$$\left(1+\frac{1}{3^n}\right)\left(1-\frac{1}{9^n}\right)=({\varepsilon_0}')^n-\frac{1}{3^n}$$

holds for sufficiently large n. Letting $n \to \infty$, we have $\varepsilon_0{}' = 1$. By simple calculation, we see that this is impossible. So, we must have $1 \leq \phi \leq 3 = M_\phi$. In this case, we have that

$$\delta_\times(3^n, \phi^n) = \left\| \frac{3^n}{\phi^n} - 1 \right\| \left\| \frac{\phi^n}{3^n} - 1 \right\| \leq 3^n - 1. \tag{3.3.23}$$

It follows from (3.3.17) and (3.3.23) that

$$(3^n - 1)\left\{(3\varepsilon_0{}')^n - 1\right\} \leq 3^n - 1, \quad \text{and so} \quad (3\varepsilon_0{}')^n - 1 \leq 1$$

for sufficiently large n, which is impossible since $\varepsilon_0{}' \geq 1/2$. We now arrive at a contradiction. So, we have proved that if $\gamma = \gamma_\times$, then $\varphi = 3$ on Y whenever $1 \leq m_\varphi$.

From the above, we have proved that $\varphi = 3$ on Y whenever $1 \leq m_\phi$.

(b) We need to prove that $\varphi = 1/3$ on Y whenever $m_\varphi < 1$. To prove this, suppose that $m_\varphi < 1$. Put $S = 1/T$. Then we see that S is a surjection from $C^+(X)$ onto $C^+(Y)$ such that $S(1) = 1$, $S(fg) = S(f)S(g)$ and $\delta(S(f), S(g)) = \delta(f, g)$ holds for every f, $g \in C^+(X)$. Since $m_\varphi = 1/3 \leq \varphi = T(3) \leq 1$ by (3.3.15), we see that $1 \leq S(3) \leq 3$, and so $1 \leq m_{S(3)}$. It follows from (a) of Lemma 3.10 that $S(3) = 3$ on Y, which proves that $\varphi = T(3) = 1/3$ on Y whenever $m_\varphi < 1$. $\square$

Lemma 3.11. *Suppose that $T(\alpha) = \alpha$ holds for every $\alpha \in \mathbf{R}^+$. If $T(fg) = T(f)T(g)$ holds for every f, $g \in C^+(X)$, then*

$$\Delta(f, g) = \Delta(T(f), T(f))$$

for all f, $g \in C^+(X)$.

Proof. We first prove that $\Delta(h, 1) = \Delta(T(h), 1)$, that is, $\|h - 1\| = \|T(h) - 1\|$ holds for every $h \in C^+(X)$. Suppose, on the contrary, that there exists an $h_0 \in C^+(X)$ such that $\|h_0 - 1\| \neq \|T(h_0) - 1\|$. Then we see that $m_{h_0} \neq m_{T(h_0)}$, or $M_{h_0} \neq M_{T(h_0)}$. If $m_{h_0} \neq m_{T(h_0)}$, then pick $\alpha_0 > 0$ so that $\alpha_0 h_0 \leq 1$ and $\alpha_0 T(h_0) \leq 1$. Without loss of generality, we may assume $m_{h_0} < m_{T(h_0)}$. In this case, we have that

$$\begin{aligned}
\Delta(\alpha_0 h_0, 1) &= \|\alpha_0 h_0 - 1\| = 1 - \alpha_0 m_{h_0} \\
&> 1 - \alpha_0 m_{T(h_0)} = \|\alpha_0 T(h_0) - 1\| \\
&= \Delta(\alpha_0 T(h_0), 1)
\end{aligned}$$

and that

$$\begin{aligned}\Delta(1, \alpha_0 h_0) &= \left\| \frac{1}{\alpha_0 h_0} - 1 \right\| = \frac{1}{\alpha_0 m_{h_0}} - 1 \\ &> \frac{1}{\alpha_0 m_{T(h_0)}} - 1 = \left\| \frac{1}{\alpha_0 T(h_0)} - 1 \right\| \\ &= \Delta(1, \alpha_0 T(h_0)).\end{aligned}$$

This implies that

$$\begin{aligned}\delta(\alpha_0 h_0, 1) &= \gamma(\Delta(\alpha_0 h_0, 1), \Delta(1, \alpha_0 h_0)) \\ &> \gamma(\Delta(\alpha_0 T(h_0), 1), \Delta(1, \alpha_0 T(h_0))) \\ &= \delta(\alpha_0 T(h_0), 1).\end{aligned}$$

On the other hand, since T preserves multiplication and since $T(\alpha) = \alpha$ for every $\alpha \in \mathbf{R}^+$, we get that

$$\delta(\alpha_0 h_0, 1) = \delta(T(\alpha_0 h_0), T(1)) = \delta(\alpha_0 T(h_0), 1),$$

a contradiction.

If $M_{h_0} \neq M_{T(h_0)}$, then pick $\alpha_1 > 0$ so that $1 \leq \alpha_1 h_0$ and $1 \leq \alpha_1 T(h_0)$. By a quite similar argument to the above, we will arrive at a contradiction. We thus conclude that $\|h - 1\| = \|T(h) - 1\|$ for every $h \in C^+(X)$.

Finally, we will prove that $\Delta(f, g) = \Delta(T(f), T(g))$ for every f, $g \in C^+(X)$. Since T preserves multiplication with $T(1) = 1$, we see that $T(1/g) = 1/T(g)$. It now follows that

$$\begin{aligned}\Delta(f, g) &= \left\| \frac{f}{g} - 1 \right\| = \left\| T\left(\frac{f}{g}\right) - 1 \right\| \\ &= \left\| \frac{T(f)}{T(g)} - 1 \right\| = \Delta(T(f), T(g))\end{aligned}$$

holds for every f, $g \in C^+(X)$, and the proof is complete. □

Lemma 3.12. *Let T be a map from $C^+(X)$ into $C^+(Y)$ such that $T(3) = 3$ and $T(f^p) = T(f)^p$ holds for every $f \in C^+(X)$ and $p \in \mathbf{R}$. Then $T(\alpha) = \alpha$ for every $\alpha \in \mathbf{R}^+$.*

Proof. Since $\alpha = 3^{\log_3 \alpha}$, we have that $T(\alpha) = T(3)^{\log_3 \alpha} = 3^{\log_3 \alpha} = \alpha$, and the proof is complete. $\square$

Proof of Theorem 3.3. Put $\widetilde{T} = T/T(1)$. We see that $\widetilde{T}: C^+(X) \to C^+(Y)$ is a surjection such that

$$\delta(f,g) = \delta(\widetilde{T}(f), \widetilde{T}(g))$$

holds for every f, $g \in C^+(X)$. By Lemma 3.8, we have that

$$\widetilde{T}(f^p g^{1-p}) = \widetilde{T}(f)^p \ \widetilde{T}(g)^{1-p}$$

holds for every f, $g \in C^+(X)$ and $p \in \mathbf{R}$. In particular, since $\widetilde{T}(1) = 1$, we have

$$\widetilde{T}(f^p) = \widetilde{T}(f)^p$$

holds for every $f \in C^+(X)$ and $p \in \mathbf{R}$, and hence

$$\widetilde{T}(fg) = \widetilde{T}((f^2)^{1/2}(g^2)^{1/2}) = \widetilde{T}(f^2)^{1/2} \ \widetilde{T}(g^2)^{1/2} = \widetilde{T}(f) \ \widetilde{T}(g)$$

holds for every $f, g \in C^+(X)$. By Lemma 3.10, we see that $\widetilde{T}(3) = 3$, or $1/3$. Suppose that $\widetilde{T}(3) = 3$. By Lemma 3.12, $\widetilde{T}(\alpha) = \alpha$ holds for every $\alpha \in \mathbf{R}^+$. By Lemma 3.11,

$$\varDelta(f,g) = \varDelta(\widetilde{T}(f), \widetilde{T}(g)) = \varDelta(T(f), T(g))$$

holds for every f, $g \in C^+(X)$. It follows from Corollary 3.2 that there exist a $w \in C^+(Y)$ and a homeomorphism $\Phi: Y \to X$ such that $T(f) = w(f \circ \Phi)$ holds for every $f \in C^+(X)$.

Suppose that $\widetilde{T}(3) = 1/3$. Put $S = 1/T$ and $\widetilde{S} = S/S(1)$. Then $\widetilde{S}$ is a surjection from $C^+(X)$ onto $C^+(Y)$ such that $\delta(\widetilde{S}(f), \widetilde{S}(g)) = \delta(f,g)$ for every f, $g \in C^+(X)$. Since $\widetilde{S}(3) = T(1)/T(3) = 1/\widetilde{T}(3) = 3$, it follows from the above argument that there exist a $w \in C^+(Y)$ and a homeomorphism $\Phi: Y \to X$ such that $1/T(f) = S(f) = w(f \circ \Phi)$ holds for every $f \in C^+(X)$. This completes the proof. $\square$

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REFERENCES

1. O. Hatori, T. Miura, R. Shindo and H. Takagi, *Generalizations of spectrally multiplicative surjections between uniform algebras*, Rend. Circ. Mat. Palermo **59** (2010), 161–183.

2. S. Mazur and S. Ulam, *Sur les transformationes isometrique d'espaces vectoriels norms*, C.R. Acad. Sci. Paris **194** (1932), 946–948.

3. T. Miura, D. Honma and R. Shindo, *Divisibly norm-preserving maps between commutative Banach algebras*, Rocky Mountain J. Math. **41** (2011), 1677–1699.

4. L. Molnár, *Thompson isometries of the space of invertible positive operators*, Proc. Amer. Math. Soc. **137** (2009), 3849–3859.

5. J. Väisälä, *A proof of the Mazur-Ulam theorem*, Amer. Math. Monthly **110–7** (2003), 633–635.

6. A. Vogt, *Maps which preserve equality of distance*, Stud. Math, **45** (1973), 43–48.

Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181 Japan
Email address: hatori@m.sc.niigata-u.ac.jp

Yoyogi Seminar, Shibuya 151-8559, Japan
Email address: top-of-the-pops@cronos.ocn.ne.jp

Department of Applied Mathematics and Physics, Graduate School of Science and Engineering, Yamagata University, Yonezawa 992-8510 Japan
Email address: miura@yz.yamagata-u.ac.jp

Toho University, Yamagata University (Professor Emeritus), Funabashi 273-0866 Japan
Email address: sin-ei@emperor.yz.yamagata-u.ac.jp