ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 4, 2010

BOUNDEDNESS OF LITTLEWOOD-PALEY OPERATORS IN GENERALIZED ORLICZ-CAMPANATO SPACES

SONGYAN ZHANG AND XIANGXING TAO

ABSTRACT. In this paper the Littlewood-Paley operators, including the g-function $g(f)$, Lusin area function $S(f)$ and Stein's function $g_\lambda^*(f)$, are all considered as the operators in generalized Orlicz-Campanato spaces $\mathcal{L}^{\Phi,\phi}$. It is proved that the image of a function in $\mathcal{L}^{\Phi,\phi}$ under one of these operators is either equal to infinity almost everywhere or is still in $\mathcal{L}^{\Phi,\phi}$. Our results extend and improve the boundedness of the Littlewood-Paley operators in BMO spaces and Campanato spaces.

1. Introduction and main results. Let $\mathbf{R}^n$ be the n-dimensional Euclidean space, and let f be a locally integrable function in $\mathbf{R}^n$. Define the Poisson integral u of f on the upper half space $\mathbf{R}_+^{n+1} = \{(x,t) : x \in r^n, t > 0\}$ by

$$u(x,t) = \int_{\mathbf{R}^n} f(z)P(x-z,t)\,dz,$$

where $P(x,t) = c_n t(t^2+|x|^2)^{-(n+1)/2}$ is the Poisson kernel. We consider the Littlewood-Paley g-function $g(f)$, Lusin area function $S(f)$ and the Stein's function $g_\lambda^*(f)$ as follows:

$$g(f)(x) = \left\{ \int_0^\infty t|\nabla u(x,t)|^2\,dt \right\}^{1/2},$$

$$S(f)(x) = \left\{ \int_{\Gamma(x)} t^{1-n}|\nabla u(z,t)|^2\,dz\,dt \right\}^{1/2},$$

2010 AMS *Mathematics subject classification.* Primary 42B25, 42B35.
Keywords and phrases. Generalized Orlicz-Campanato space, Littlewood-Paley operator, Poisson integral, N-function, doubling condition.
Research of the second author supported by National Natural Science Foundation of China under Grant #10771110, and sponsored by SRF for ROCS, SEM. Research of the first author supported by Scientific Research Fund of Zhejiang Provincial Education Department under Grant #20040962, and sponsored by the NSF of Ningbo City under Grant #2009A610084.
The second author is the corresponding author.
Received by the editors on January 16, 2007, and in revised form on March 14, 2008.

DOI:10.1216/RMJ-2010-40-4-1355

$$g_\lambda^*(f)(x) = \left\{ \int_{\mathbf{R}_+^{n+1}} \left(\frac{t}{t + |z - x|} \right)^{\lambda n} t^{1-n} |\nabla u(z,t)|^2 \, dz \, dt \right\}^{1/2},$$

respectively, where $\lambda > 2$ and $\nabla u = ((\partial u/\partial x_1), \ldots, (\partial u/\partial x_n), (\partial u/\partial t))$, and $\Gamma(x) = \{(z,t) \in \mathbf{R}_+^{n+1} : |z - x| < t\}$ is the cone with vertex $x \in \mathbf{R}^n$. It's easy to see that

$$|\nabla u(x,t)| \le c_n \int_{\mathbf{R}^n} \frac{|f(z)|}{(t + |z - x|)^{n+1}} \, dz.$$

The above Littlewood-Paley operators are important classical operators in harmonic analysis. In 1984, Wang [**9**] proved that, for a BMO function $f(x)$, the Littlewood-Paley g-function $g(f)$ is either equal to infinity almost everywhere or is still in the BMO spaces. Soon after, such kind of results was generalized to Lusin's area functions, Stein's g_λ^* functions and so on; see [**4**], for example. Because of application to partial differential equations, more interest is focused on the boundedness of Littlewood-Paley operators in Campanato-type spaces. In [**6, 7, 9, 11**], and the references therein, the authors have shown some boundedness of Littlewood-Paley operators in Lipschitz function spaces $\mathrm{Lip}_\alpha(\mathbf{R}^n)$ and classical Campanato spaces $L^{p,\alpha}(\mathbf{R}^n)$, respectively. In this paper, we will establish boundedness in a generalized Orlicz-Campanato space $\mathcal{L}^{\Phi,\phi}(\mathbf{R}^n)$ for Littlewood-Paley operators; our theorems will extend and improve the above earlier results such as in [**4, 6, 9, 10, 11**].

In order to state our results exactly, we first recall some related notations and definitions about the Orlicz-Campanato space.

The N-function $\Phi(s)$ is given by $\Phi(s) = \int_0^s \varphi(t)\, dt$, $s \ge 0$, where $\varphi(t)$ is a nondecreasing right continuous function defined on $[0, +\infty)$ with $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$. The complementary N-function is given by $\Psi(s) = \int_0^s \rho(t)\, dt$, $s \ge 0$, where $\rho(t) = \sup\{s : \varphi(s) \le t\}$. It's clear that an N-function is a convex function. The N-functions Φ and Ψ are said to satisfy the $\triangle_2$ condition in $(0, \infty)$ if positive constants C_1 and C_2 exist such that for all $s > 0$,

$$\Phi(2s) \le C_1 \Phi(s), \qquad \Psi(2s) \le C_2 \Psi(s).$$

We introduce another positive increasing function ϕ defined on $(0, +\infty)$ which satisfies the following doubling condition: there exists a constant $1 \leq D < 2^n$ such that

$$\phi(2r) \leq D\phi(r) \tag{1.1}$$

for any $r > 0$. It's not difficult to see that there exists a constant C_3 such that, for any $0 < t \leq s < \infty$,

$$\frac{\phi(s)}{s^n} \leq C_3 \frac{\phi(t)}{t^n}. \tag{1.2}$$

We say a weight function $\omega \in A_p$, the Muckenhoupt's class, $1 < p < \infty$, if ω is a positive function and there exists a constant C_p such that

$$\left\{\frac{1}{|B|}\int_B \omega(x)\,dx\right\}\left\{\frac{1}{|B|}\int_B \omega(x)^{-1/(p-1)}dx\right\}^{p-1} \leq C_p$$

for any ball $B \subset \mathbf{R}^n$; and we say $\omega \in A_1$ if

$$\frac{1}{|B|}\int_B \omega(x)\,dx \leq C_1 \operatorname{ess}\inf_{x\in B} \omega(x)$$

for any ball $B \subset \mathbf{R}^n$. It's clear that $A_{p_1} \subset A_{p_2}$ if $1 \leq p_1 < p_2 < \infty$. We also remark that, if $0 < \gamma < 1$, M is the Hardy-Littlewood operator, and f is a locally integral function, then $(Mf)^\gamma \in A_1$. For more properties of A_p weight, one can see [**2**].

Definition 1.1. Let Φ be an N-function satisfying the Δ_2 condition, and let ω be a weight function in $\mathbf{R}^n$. Then the weighted Orlicz space is defined as follows

$$L^\Phi(\omega) = \left\{f : \int_{\mathbf{R}^n} \Phi(|f(x)|)\omega(x)\,dx < \infty\right\},$$

with the Luxemberg norm defined by

$$\|f\|_{L^\Phi(\omega)} = \inf\left\{\lambda > 0 : \int_{\mathbf{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)\omega(x)\,dx \leq 1\right\}.$$

Definition 1.2. Let Φ be the N-function satisfying the Δ_2 condition, and let ϕ be a positive increasing function satisfying the doubling condition (1.1) with $1 \le D < 2^n$. We define the generalized Orlicz-Campanato space as follows

$$\mathcal{L}^{\Phi,\phi}(\mathbf{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) : \|f\|_{\mathcal{L}^{\Phi,\phi}} < \infty\},$$

where

$$\|f\|_{\mathcal{L}^{\Phi,\phi}} = \sup_{\substack{y\in\mathbf{R}^n,\\ r>0}} \frac{1}{\phi(r)} \int_{B(y,r)} \Phi(|f(x) - f_{B(y,r)}|)\, dx,$$

and $f_{B(y,r)} = |B(y,r)|^{-1} \int_{B(y,r)} f(x)\, dx$ is the integral mean of function f over $B(y,r)$, and $B(y,r)$ always denotes the ball in $\mathbf{R}^n$ with center y and radius r.

In particular, if one takes $\Phi(t) = t^p$ for $1 \le p < \infty$, and $\phi(r) = r^\alpha$ for $0 \le \alpha < n$, then $\mathcal{L}^{\Phi,\phi}(\mathbf{R}^n)$ becomes the classical Campanato space $L^{p,\alpha}(\mathbf{R}^n)$, which was introduced by Campanato [**1**].

Assume that Tf is one of the Littlewood-Paley operators $g(f)$, $S(f)$ and $g^*_\lambda(f)$; our main results can be stated as follows.

Theorem 1.3. *If an N-function Φ and its complementary N-function Ψ both satisfy the Δ_2 condition, the positive increasing function ϕ satisfies the doubling condition* (1.1) *with $1 \le D < 2^n$, and if $f \in \mathcal{L}^{\Phi,\phi}(\mathbf{R}^n)$, then either $T(f)(x) = \infty$ for almost every $x \in \mathbf{R}^n$, or $T(f)(x) < \infty$ for almost every $x \in \mathbf{R}^n$. In latter case, moreover, there exists a positive constant C independent of f such that $T(f) \in \mathcal{L}^{\Phi,\phi}(\mathbf{R}^n)$, and*

$$\|T(f)\|_{\mathcal{L}^{\Phi,\phi}} \le C\|f\|_{\mathcal{L}^{\Phi,\phi}}. \tag{1.3}$$

Throughout this paper, the letter C always denotes an absolute positive constant and may have a different value in each line. If B is a ball in $\mathbf{R}^n$, we denote by dB the ball with the same center and d times radius as the ball B. The notation $t \simeq r$ means that $c|t| \le |r| \le C|t|$ with some positive constants c and C.

2. Some propositions. From now on, we always assume that the N-function Φ and its complementary N-function Ψ both satisfy the Δ_2 condition. We will use the following basic properties of the N-function Φ.

Proposition 2.1 [**2, 3**]. *Define the lower index* $q_\Phi = \lim_{\lambda\to 0^+} (\log h(\lambda)/(\log \lambda))$, *and upper index* $p_\Phi = \lim_{\lambda\to+\infty}(\log h(\lambda)/(\log \lambda)$, *where* $h(\lambda) = \sup_{t>0}[\Phi(\lambda t)/\Phi(t)]$. *Then we have*

$$1 < q_\Phi \le p_\Phi < \infty.$$

Proposition 2.2 [**5**]. *There exist constants* α_0 *and* β_0 *such that*

$$1 \le \beta_0 \le \frac{s\varphi(s)}{\Phi(s)} \le \alpha_0 < \infty \tag{2.1}$$

holds for all $s > 0$.

Proposition 2.3. *For any* $k \ge 1$, *we have*

$$k^{\beta_0}\Phi(s) \le \Phi(ks) \le k^{\alpha_0}\Phi(s), \text{ for all } s \ge 0, \tag{2.2}$$

$$k^{1/\alpha_0}\Phi^{-1}(u) \le \Phi^{-1}(ku) \le k^{1/\beta_0}\Phi^{-1}(u), \text{ for all } u \ge 0. \tag{2.3}$$

Proof. From inequality (2.1), we can see for any $t > 0$,

$$\frac{1}{t}\beta_0 \le \frac{\varphi(t)}{\Phi(t)} \le \frac{1}{t}\alpha_0;$$

then by integrating the above inequality in t from s to ks, we can get the inequality (2.2). Let $u = \Phi(s)$, i.e., $s = \Phi^{-1}(u)$, and use k in place of k^{α_0} and k^{β_0} respectively. Then from (2.2) we can obtain

$$\Phi(k^{1/\alpha_0}s) \le ku \le \Phi(k^{1/\beta_0}s),$$

which implies that the inequality (2.3) obviously. $\square$

Proposition 2.4 [**8**]. *Assume* Φ *and its complementary* N*-function* Ψ *both satisfy the* Δ_2 *condition, and let* q_Φ *be the lower index of* Φ*. Then* $\omega \in A_{q_\Phi}$*, if and only if there exists a constant* C *such that*

$$\int_{\mathbf{R}^n} \Phi(Tf(x))\omega(x)\,dx \le C\int_{\mathbf{R}^n} \Phi(|f(x)|)\omega(x)\,dx$$

for any $f \in L^\Phi(\omega)$.

Proposition 2.5. *Assume* $f \in \mathcal{L}^{\Phi,\phi}(\mathbf{R}^n)$*, the* N*-function* Φ *satisfies the* Δ_2 *condition, and that the positive increasing function* ϕ *satisfies the doubling condition* (1.1) *with* $1 \le D < 2^n$*. Let* B_0 *be the ball centered at* x_0 *with radius* r_0*. Then, for any* $\delta > 0$*, we have*

$$\int_{\mathbf{R}^n} \frac{|f(x) - f_{B_0}|}{{r_0}^{n+\delta} + |x - x_0|^{n+\delta}}\,dx \le C{r_0}^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right); \tag{2.4}$$

and for any $t > (1/8)r_0$,

$$\int_{\mathbf{R}^n} \frac{|f(x) - f_{B_0}|}{t^{n+\delta} + |x - x_0|^{n+\delta}}\,dx \le Ct^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right). \tag{2.5}$$

Proof. One could first write

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{|f(x) - f_{B_0}|}{{r_0}^{n+\delta} + |x - x_0|^{n+\delta}}\,dx =& \int_{B_0} \frac{|f(x) - f_{B_0}|}{{r_0}^{n+\delta} + |x - x_0|^{n+\delta}}\,dx \\ &+ \sum_{k=1}^{+\infty} \int_{2^kB_0\setminus 2^{k-1}B_0} \frac{|f(x) - f_{B_0}|}{{r_0}^{n+\delta} + |x - x_0|^{n+\delta}}dx \\ =:&\, I_1 + I_2.\end{aligned}$$

It's not difficult to see from Jensen's inequality and Proposition 2.3 that

$$\begin{aligned}I_1 &\le {r_0}^{-n-\delta}\int_{B_0} |f(x) - f_{B_0}|\,dx \\ &\le C{r_0}^{-\delta}\left[\Phi^{-1}\circ\Phi\left(\frac{1}{|B_0|}\int_{B_0} |f(x) - f_{B_0}|\,dx\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\leq C{r_0}^{-\delta}\left[\Phi^{-1}\left(\frac{\phi(r_0)}{r_0^n}\cdot\frac{1}{\phi(r_0)}\int_{B_0}\Phi(|f(x)-f_{B_0}|)\,dx\right)\right]\\
&\leq C{r_0}^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\leq \sum_{k=1}^{+\infty}\frac{1}{(2^{k-1}r_0)^{n+\delta}}\int_{2^kB_0}|f(x)-f_{B_0}|\,dx\\
&\leq \sum_{k=1}^{+\infty}\frac{1}{(2^{k-1}r_0)^{n+\delta}}\left[\int_{2^kB_0}|f(x)-f_{2^kB_0}|\,dx\right.\\
&\qquad\qquad\left.+\int_{2^kB_0}|f_{2^kB_0}-f_{B_0}|\,dx\right]\\
&=: J_1+J_2.
\end{aligned}
$$

Using condition (1.1) of ϕ, Jensen's inequality and Proposition 2.3 again, we obtain

$$
\begin{aligned}
J_1 &\leq C{r_0}^{-\delta}\sum_{k=1}^{+\infty}2^{-k\delta}\left[\Phi^{-1}\circ\Phi\left(\frac{1}{|2^kB_0|}\int_{2^kB_0}|f(x)-f_{2^kB_0}|\,dx\right)\right]\\
&\leq C{r_0}^{-\delta}\sum_{k=1}^{+\infty}2^{-k\delta}\left[\Phi^{-1}\left(\frac{\phi(2^kr_0)}{(2^kr_0)^n}\right.\right.\\
&\qquad\qquad\left.\left.\cdot\frac{1}{\phi(2^kr_0)}\int_{2^kB_0}\Phi(|f(x)-f_{2^kB_0}|)\,dx\right)\right]\\
&\leq C{r_0}^{-\delta}\sum_{k=1}^{+\infty}2^{-k\delta}\left[\Phi^{-1}\left(\frac{D^k\phi(r_0)}{(2^kr_0)^n}\|f\|_{\mathcal{L}^{\Phi,\phi}}\right)\right]\\
&\leq C{r_0}^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right).
\end{aligned}
$$

To estimate the term J_2, we observe that

(2.6)

$$
\begin{aligned}
|f_{2^kB_0}-f_{B_0}| &\leq \sum_{i=0}^{k-1}|f_{2^{i+1}B_0}-f_{2^iB_0}|\\
&\leq \sum_{i=0}^{k-1}\Phi^{-1}\left[\frac{1}{|2^iB_0|}\int_{2^iB_0}\Phi\left(|f-f_{2^{i+1}B_0}|\right)\,dy\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{k-1} \Phi^{-1}\Bigg[\frac{\phi(2^{i+1}r_0)}{|2^iB_0|}\cdot \\
&\qquad\qquad \frac{1}{\phi(2^{i+1}r_0)}\int_{2^{i+1}B_0} \cdot\Phi\left(|f-f_{2^{i+1}B_0}|\right)\,dy\Bigg] \\
&\leq C\sum_{i=0}^{k-1} \Phi^{-1}\left[\frac{D^{i+1}\phi(r_0)}{(2^ir_0)^n}\|f\|_{\mathcal{L}^{\Phi,\phi}}\right].
\end{aligned}
$$

Noting $2^{-n}D<1$, we deduce from Proposition 2.3 that

$$
\begin{aligned}
&\Phi^{-1}\left[D(2^{-n}D)^i {r_0}^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right] \\
&\qquad\qquad \leq D^{1/\beta_0}(2^{-n}D)^{i/\alpha_0}\Phi^{-1}({r_0}^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}).
\end{aligned}
$$

Therefore, the last summation on the righthand side of inequality (2.6) is bounded by $C\Phi^{-1}({r_0}^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}})$. Thus, we can obtain that

$$
\begin{aligned}
J_2 &\leq C{r_0}^{-\delta}\sum_{k=1}^{+\infty} 2^{-k\delta}\Phi^{-1}\left(Cr_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right) \\
&\leq C{r_0}^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right).
\end{aligned}
$$

Combining the estimates of I_1, J_1 and J_2, we have deduced the inequality (2.4).

To prove (2.5), let E be the ball concentric with B_0 and having radius t. We first consider the case $t>r_0$, and let $k\geq 0$ be the integer which satisfies $2^kr_0\leq t<2^{k+1}r_0$. We write

$$
\begin{aligned}
\int_{\mathbf{R}^n}\frac{|f(x)-f_{B_0}|}{t^{n+\delta}+|x-x_0|^{n+\delta}}\,dx &\leq \int_{\mathbf{R}^n}\frac{|f(x)-f_E|}{t^{n+\delta}+|x-x_0|^{n+\delta}}\,dx \\
&\quad + Ct^{-\delta}|f_E-f_{2^kB_0}| + Ct^{-\delta}|f_{2^kB_0}-f_{B_0}| \\
&=: K_1+K_2+K_3.
\end{aligned}
$$

From inequality (2.4) and property (1.2) of ϕ, we can get

$$
\begin{aligned}
K_1 &\leq Ct^{-\delta}\Phi^{-1}\left(t^{-n}\phi(t)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right) \\
&\leq Ct^{-\delta}\Phi^{-1}\left(C_3r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right) \\
&\leq Ct^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right).
\end{aligned}
$$

Using similar arguments as for J_1 and J_2 above, respectively, we obtain

$$\begin{aligned}
K_2 &\le Ct^{-\delta}\frac{1}{|2^kB_0|}\int_{2^kB_0}|f(x)-f_E|\,dx\\
&\le Ct^{-\delta}\frac{|E|}{|2^kB_0|}\left[\Phi^{-1}\circ\Phi\left(\frac{1}{|E|}\int_E|f(x)-f_E|\,dx\right)\right]\\
&\le Ct^{-\delta}\frac{|E|}{|2^kB_0|}\left[\Phi^{-1}\left(C\frac{\phi(t)}{t^n}\cdot\frac{1}{\phi(t)}\int_E\Phi(|f(x)-f_E|)\,dx\right)\right]\\
&\le Ct^{-\delta}\frac{t^n}{(2^kr_0)^n}\left[\Phi^{-1}\left(\frac{\phi(t)}{t^n}\|f\|_{\mathcal{L}^{\Phi,\phi}}\right)\right]\\
&\le Ct^{-\delta}\frac{(2^{k+1}r_0)^n}{(2^kr_0)^n}\left[\Phi^{-1}\left(C_3r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right)\right]\\
&\le Ct^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right),
\end{aligned}$$

and by (2.6),

$$K_3 \le Ct^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right).$$

Now, combining the estimates of K_1, K_2 and K_3, we have shown inequality (2.5) for $t > r_0$.

In the case $(1/8)r_0 < t \le r_0$, we have $t \simeq r_0$. It's easy to see that

(2.7)
$$\int_{\mathbf{R}^n}\frac{|f(x)-f_{B_0}|\,dx}{t^{n+\delta}+|x-x_0|^{n+\delta}} \le \int_{\mathbf{R}^n}\frac{|f(x)-f_E|dx}{t^{n+\delta}+|x-x_0|^{n+\delta}} + Ct^{-\delta}|f_E-f_{B_0}|.$$

Using inequality (2.4), we can get that the integral on the righthand side of above inequality is bounded by

$$Ct^{-\delta}\Phi^{-1}\left(t^{-n}\phi(t)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right) \simeq Ct^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right),$$

and the last term on the righthand side of inequality (2.7) is equal to

$$\begin{aligned}
&Ct^{-\delta}\Phi^{-1}\circ\Phi\left(\frac{1}{|E|}\int_E|f-f_{B_0}|\,dx\right)\\
&\qquad\le Ct^{-\delta}\Phi^{-1}\left(C\frac{\phi(r_0)}{t^n}\cdot\frac{1}{\phi(r_0)}\int_{B_0}\Phi(|f-f_{B_0}|)\,dx\right)\\
&\qquad\le Ct^{-\delta}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right),
\end{aligned}$$

where we have used the Jensen inequality and inequality (2.3). Hence, we get (2.5) and finish the proof of the proposition. □

Proposition 2.6. *Let B_0 be a ball centered at x_0 with radius r_0, and $f_3(x) = (f(x) - f_{B_0})\chi_{B_0^c}(x)$, $u_3(x,y) = (f_3 * P_y)(x)$. If there is a point $x' \in (1/2)B_0$ such that $g(f_3)(x') < \infty$, then there is a constant C such that for every $x \in (1/2)B_0$, $g(f_3)(x) < \infty$ and*

$$\Phi(|g(f_3)(x) - g(f_3)(x')|) \leq C r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}. \tag{2.8}$$

Proof. For any fixed $x \in (1/2)B_0$, we have

$$\begin{aligned} g(f_3)(x) = \left\{ \int_0^\infty t|\nabla u_3(x,t)|^2\,dt \right\}^{1/2} &\leq \left\{ \int_0^{r_0} t|\nabla u_3(x,t)|^2\,dt \right\}^{1/2} \\ &+ \left\{ \int_{r_0}^\infty t|\nabla u_3(x,t)|^2\,dt \right\}^{1/2} =: H_1 + H_2. \end{aligned}$$

First we note that, for $x, x_0 \in (1/2)B_0$ and $y \in B_0^c$, one has $|x - x_0| \leq (r_0/2) \leq |y - x_0|/2$, and so

$$|y - x| \geq |y - x_0| - |x - x_0| \geq \frac{1}{2}|y - x_0| \geq \frac{1}{4}(r_0 + |y - x_0|). \tag{2.9}$$

Thus by (2.9) and Proposition 2.5, we can get

$$\begin{aligned} H_1 &\leq C\left\{ \int_0^{r_0} t\left[\int_{{B_0}^c} \frac{|f(y) - f_{B_0}|}{(t + |y - x|)^{n+1}}\,dy \right]^2 dt \right\}^{1/2} \\ &\leq C\left\{ \int_0^{r_0} t\left[\int_{{B_0}^c} \frac{|f(y) - f_{B_0}|}{(r_0 + |y - x_0|)^{n+1}}\,dy \right]^2 dt \right\}^{1/2} \\ &\leq C\left\{ \int_0^{r_0} t\left[r_0^{-1}\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}} \right) \right]^2\,dt \right\}^{1/2} \\ &\leq C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} H_2 &\leq \left\{ \int_{r_0}^\infty t|\nabla u_3(x',t)|^2\,dt \right\}^{1/2} \\ &\quad + \left\{ \int_{r_0}^\infty t|\nabla u_3(x,t) - \nabla u_3(x',t)|^2\,dt \right\}^{1/2} \\ &\leq g(f_3)(x') + F, \end{aligned}$$

where

$$\begin{aligned} F &= \left\{ \int_{r_0}^{\infty} t|\nabla u_3(x,t) - \nabla u_3(x',t)|^2 \, dt \right\}^{1/2} \\ &\leq \left\{ \int_{r_0}^{\infty} t \left[\int_{{B_0}^c} |\nabla P(x-y,t) \right.\right. \\ &\qquad\qquad \left.\left. - \nabla P(x'-y,t)||f(y) - f_{B_0}| \, dy \right]^2 dt \right\}^{1/2}. \end{aligned}$$

Since for $x, x' \in (1/2)B_0$ and $y \in {B_0}^c$, it's easy to see that $|y-x| \simeq |y-x_0| \simeq |y-x'|$. The mean value theorem implies

$$|\nabla P(x-y,t) - \nabla P(x'-y,t)| \leq C \frac{r_0}{(t+|y-x_0|)^{n+2}}. \tag{2.10}$$

Hence, by (2.10) and Proposition 2.5, we can deduce that

$$\begin{aligned} F &\leq C \left\{ \int_{r_0}^{\infty} t \left[\int_{{B_0}^c} \frac{r_0|f(y) - f_{B_0}|}{(t+|y-x_0|)^{n+2}} \, dy \right]^2 dt \right\}^{1/2} \\ &\leq C r_0 \left\{ \int_{r_0}^{\infty} t \left[t^{-2} \Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right) \right]^2 dt \right\}^{1/2} \\ &\leq C \Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right). \end{aligned}$$

Combining the estimates for H_1, H_2 and F, we get

$$g(f_3)(x) \leq g(f_3)(x') + C\Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right).$$

This and condition $g(f_3)(x') < \infty$ imply the inequality (2.8). The proposition is proved. □

Proposition 2.7. *With the same notations of Proposition* 2.6, *if there is a point* $x' \in (1/4)B_0$ *such that* $S(f_3)(x') < \infty$, *then there is a constant* C *such that for every* $x \in (1/4)B_0$, $S(f_3)(x) < \infty$ *and*

$$\Phi(|S(f_3)(x) - S(f_3)(x')|) \leq C r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}}. \tag{2.11}$$

Proof. Fix $x \in (1/4)B_0$. Set $\Gamma^-(x) = \{(z,t) \in \Gamma(x) : t \leq r_0\}$ and $\Gamma^+(x) = \{(z,t) \in \Gamma(x) : t > r_0\}$. Then we can write

$$S(f_3)(x) \leq S^-(f_3)(x) + S^+(f_3)(x),$$

where

$$S^-(f_3)(x) = \left\{ \int_{\Gamma^-(x)} t^{1-n} |\nabla u_3(z,t)|^2 \, dz \, dt \right\}^{1/2},$$

$$S^+(f_3)(x) = \left\{ \int_{\Gamma^+(x)} t^{1-n} |\nabla u_3(z,t)|^2 \, dz \, dt \right\}^{1/2}.$$

For $(z,t) \in \Gamma^-(x)$, $x, x_0 \in (1/2)B_0$ and $y \in {B_0}^c$, one can see that

$$|y - x_0| \leq |y - z| + |z - x| + |x - x_0| \leq |y - z| + t + \frac{|y - x_0|}{2},$$

and so

$$\frac{1}{4}(r_0 + |y - x_0|) \leq \frac{1}{2}|y - x_0| \leq (t + |y - z|). \tag{2.12}$$

Then by (2.12) and Proposition 2.5, we obtain

$$\begin{aligned}
S^-(f_3)(x) &\leq C \left\{ \int_{\Gamma^-(x)} t^{1-n} \left[\int_{{B_0}^c} \frac{|f(y) - f_{B_0}|}{(t + |y - z|)^{n+1}} \, dy \right]^2 dz \, dt \right\}^{1/2} \\
&\leq C \left\{ \int_{\Gamma^-(x)} t^{1-n} \left[\int_{{B_0}^c} \frac{|f(y) - f_{B_0}|}{(r_0 + |y - x_0|)^{n+1}} \, dy \right]^2 dz \, dt \right\}^{1/2} \\
&\leq C \left\{ \int_{\Gamma^-(x)} t^{1-n} \left[r_0^{-1} \Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right) \right]^2 \, dz \, dt \right\}^{1/2} \\
&\leq C \Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right) \\
&\quad \cdot r_0^{-1} \left\{ \int_0^{r_0} t^{1-n} \left(\int_{|z-x|<t} dz \right) dt \right\}^{1/2} \\
&\leq C \Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
S^+(f_3)(x) &= \left\{ \int_{\Gamma^+(0)} t^{1-n} |\nabla u_3(x + z, t)|^2 \, dz \, dt \right\}^{1/2} \\
&\leq \left\{ \int_{\Gamma^+(0)} t^{1-n} |\nabla u_3(x' + z, t)|^2 \, dz \, dt \right\}^{1/2} \\
&\quad + \left\{ \int_{\Gamma^+(0)} t^{1-n} |\nabla u_3(x + z, t) - \nabla u_3(x' + z, t)|^2 \, dz \, dt \right\}^{1/2} \\
&\leq S(f_3)(x') + M,
\end{aligned}$$

where

$$\begin{aligned}M^2 &= \int_{\Gamma^+(0)} t^{1-n} |\nabla u_3(x+z,t) - \nabla u_3(x'+z,t)|^2 \, dz \, dt \\ &\le \int_{\Gamma^+(0)} t^{1-n} \bigg[\int_{{B_0}^c} |\nabla P(x+z-y,t) \\ &\qquad\qquad - \nabla P(x'+z-y,t)| |f(y) - f_{B_0}| \, dy \bigg]^2 \, dz \, dt.\end{aligned}$$

Similarly, by the mean value theorem, we can get

$$\begin{aligned}&|\nabla P(x+z-y,t) - \nabla P(x'+z-y,t)| \\ &\qquad \le C|x-x'| \bigg(\sum_{j=1}^{n+1} (t + |x+z-y+\theta_j(x-x')|)^{-2(n+2)} \bigg)^{1/2}\end{aligned}$$

for some constants θ_j, $0 < \theta_j < 1$. Since, for $(z,t) \in \Gamma^+(0)$, $x, x' \in (1/4)B_0$ and $y \in {B_0}^c$, we have

$$\begin{aligned}|y - x_0| &\le |x+z-y+\theta_j(x-x')| + |x - x_0| + |z| + |x - x'| \\ &\le |x+z-y+\theta_j(x-x')| + \frac{|y-x_0|}{4} + t + \frac{|y-x_0|}{2},\end{aligned}$$

or

$$\frac{1}{5}(|y-x_0| + t) \le |x+z-y+\theta_j(x-x')| + t,$$

and so

$$|\nabla P(x+z-y,t) - \nabla P(x'+z-y,t)| \le C \frac{r_0}{(t + |y-x_0|)^{n+2}}. \tag{2.13}$$

By (2.13) and Proposition 2.5, we obtain

$$\begin{aligned}M &\le \bigg\{ \int_{\Gamma^+(0)} t^{1-n} \bigg[\int_{{B_0}^c} \frac{r_0 |f(y) - f_{B_0}|}{(t + |y-x_0|)^{n+2}} \, dy \bigg]^2 \, dz \, dt \bigg\}^{1/2} \\ &\le C \bigg\{ \int_{\Gamma^+(0)} t^{1-n} \left[r_0 t^{-2} \Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right) \right]^2 \, dz dt \bigg\}^{1/2} \\ &\le C \Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right) r_0 \bigg\{ \int_{r_0}^{+\infty} t^{-n-3} \bigg(\int_{|z|<t} dz \bigg) \, dt \bigg\}^{1/2} \\ &\le C \Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}^{\Phi,\phi}} \right).\end{aligned}$$

Now, combining the estimates for $S^+(f_3)(x)$, $S^-(f_3)(x)$ and M, we then have

$$S(f_3)(x) \le S(f_3)(x') + C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right),$$

which implies the inequality (2.11) because of $S(f_3)(x') < \infty$. The proof of the proposition is finished. □

Proposition 2.8. *With the same notations as Proposition* 2.6, *if there is a point $x' \in (1/4)B_0$ such that $g^*_\lambda(f_3)(x') < \infty$, then there is a constant C such that for every $x \in (1/4)B_0$, $g^*_\lambda(f_3)(x) < \infty$ and*

$$\Phi(|g^*_\lambda(f_3)(x) - g^*_\lambda(f_3)(x')|) \le Cr_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}. \tag{2.14}$$

Proof. Recalling $B_0 = B(x_0, r_0)$, for any fixed $x \in (1/4)B_0$, we write

$$\begin{aligned} g^*_\lambda(f_3)(x) &\le \left\{\int_0^{r_0}\int_{\mathbf{R}^n}\left(\frac{t}{t+|z-x|}\right)^{\lambda n} t^{1-n}|\nabla u_3(z,t)|^2\,dz\,dt\right\}^{1/2} \\ &\quad + \left\{\int_{r_0}^{\infty}\int_{\mathbf{R}^n}\left(\frac{t}{t+|z-x|}\right)^{\lambda n} t^{1-n}|\nabla u_3(z,t)|^2\,dz\,dt\right\}^{1/2} \\ &=: G^-(f_3)(x) + G^+(f_3)(x). \end{aligned}$$

We will estimate the two terms $G^-(f_3)(x)$ and $G^+(f_3)(x)$, respectively. First, using the integral representation of ∇u_3 and the Minkowski inequality, we have

$$\begin{aligned} &G^-(f_3)(x) \\ &\le C\bigg\{\int_0^{r_0}\int_{\mathbf{R}^n}\left(\frac{t}{t+|z-x|}\right)^{\lambda n} \\ &\qquad \times \left[\int_{\mathbf{R}^n}\frac{|f_3(y)|}{(t+|y-z|)^{n+1}}\,dy\right]^2 dz\frac{dt}{t^{n-1}}\bigg\}^{1/2} \\ &\le C\bigg\{\int_0^{r_0}\bigg(\int_{\mathbf{R}^n}\left[\int_{\mathbf{R}^n}\frac{1}{(t+|y-z|)^{2n+2}}\frac{t^{\lambda n}}{(t+|z-x|)^{\lambda n}}\,dz\right]^{1/2} \\ &\qquad\qquad \cdot |f_3(y)|\,dy\bigg)^2\frac{dt}{t^{n-1}}\bigg\}^{1/2}. \end{aligned}$$

Next, we denote the inner integral on the righthand side of the inequality above by

$$E(x,y,t)=\int_{\mathbf{R}^n}\frac{1}{(t+|y-z|)^{2n+2}}\frac{t^{\lambda n}}{(t+|z-x|)^{\lambda n}}\,dz,$$

and note that, for $x\in(1/4)B_0$ and $y\in B_0^c$,

$$|x-y|\ge|y-x_0|-|x-x_0|\ge\frac{3}{4}|y-x_0|\ge\frac{1}{4}(|y-x_0|+r_0). \tag{2.15}$$

(a1) If $|z-x|\ge(1/2)|x-y|$, then $t+|z-x|\ge(1/8)(|y-x_0|+r_0)$ and

$$\begin{aligned}E(x,y,t)&\le\frac{Ct^{\lambda n}}{(r_0+|y-x_0|)^{\lambda n}}\int_{\mathbf{R}^n}\frac{dz}{(t+|y-z|)^{2n+2}}\\&\le\frac{Ct^{\lambda n-n-2}}{(r_0+|y-x_0|)^{\lambda n}}.\end{aligned}$$

Thus, by Proposition 2.5, we have

$$\begin{aligned}G^-(f_3)(x)&\le C\bigg\{\int_0^{r_0}\bigg(\int_{\mathbf{R}^n}\bigg[\frac{Ct^{\lambda n-n-2}}{(r_0+|y-x_0|)^{\lambda n}}\bigg]^{1/2}\\&\qquad\qquad\cdot|f_3(y)|\,dy\bigg)^2\frac{dt}{t^{n-1}}\bigg\}^{1/2}\\&\le C\int_{\mathbf{R}^n}\frac{|f_3(y)|dy}{(r_0+|y-x_0|)^{\lambda n/2}}\bigg\{\int_0^{r_0}t^{\lambda n-2n-1}\,dt\bigg\}^{1/2}\\&\le C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}}^{\Phi,\phi}\right),\end{aligned}$$

where we have used the assumption that $\lambda>2$.

(a2) If $|z-x|\le(1/2)|x-y|$, then

$$|y-z|\ge|y-x|-|x-z|\ge\frac{1}{2}|x-y|\ge\frac{1}{8}(|y-x_0|+r_0),$$

and so

$$\begin{aligned}E(x,y,t)&\le\frac{Ct^{\lambda n}}{(r_0+|y-x_0|)^{2n+2}}\int_{\mathbf{R}^n}\frac{dz}{(t+|x-z|)^{\lambda n}}\\&\le\frac{Ct^{n}}{(r_0+|y-x_0|)^{2n+2}}.\end{aligned}$$

This and Proposition 2.5 imply that

$$\begin{aligned} G^-(f_3)(x) &\le C\int_{\mathbf{R}^n}\frac{|f_3(y)|dy}{(r_0+|y-x_0|)^{n+1}}\left\{\int_0^{r_0} t\,dt\right\}^{1/2} \\ &\le C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right). \end{aligned}$$

Thus, in any case, we obtain that

$$G^-(f_3)(x) \le C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right) \tag{2.16}$$

with the constant C independent of $x \in (1/4)B_0$ and r_0.

To estimate $G^+(f_3)(x)$, we observe that

$$\begin{aligned} G^+(f_3)(x) &= \left\{\int_{r_0}^\infty\int_{\mathbf{R}^n}|\nabla u_3(z+x,t)|^2\frac{t^{\lambda n}}{(t+|z|)^{\lambda n}}\,dz\frac{dt}{t^{n-1}}\right\}^{1/2} \\ &\le G^+(f_3)(x') \\ &\quad + \left\{\int_{r_0}^\infty\int_{\mathbf{R}^n}|\nabla u_3(z+x,t)\right. \\ &\qquad\qquad \left. - \nabla u_3(z+x',t)|^2\frac{t^{\lambda n}}{(t+|z|)^{\lambda n}}\,dz\frac{dt}{t^{n-1}}\right\}^{1/2} \\ &=: G^+(f_3)(x') + N. \end{aligned}$$

As for N, by the integral representation of ∇u_3, we have

$$\begin{aligned} &|\nabla u_3(z+x,t) - \nabla u_3(z+x',t)| \\ &\quad \le \int_{\mathbf{R}^n}\left|\frac{1}{(t+|y-z-x|)^{n+1}} - \frac{1}{(t+|y-z-x'|)^{n+1}}\right||f_3(y)|\,dy. \end{aligned}$$

Since $t+|y-z-x| \simeq t+|y-z-x'|$, whenever $|x-x'| \le (r_0/4) \le (t/4)$, we get from the mean value theorem that

$$|\nabla u_3(z+x,t) - \nabla u_3(z+x',t)| \le Cr_0\int_{\mathbf{R}^n}\frac{|f_3(y)|dy}{(t+|y-z-x|)^{n+2}}.$$

This implies
(2.17)

$$N^2 \le Cr_0^2 \int_{r_0}^{\infty} \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} \frac{|f_3(y)|dy}{(t+|y-z-x|)^{n+2}} \right|^2 \frac{t^{\lambda n}}{(t+|z|)^{\lambda n}}\, dz \frac{dt}{t^{n-1}}$$
$$\le Cr_0^2 \int_{r_0}^{\infty} \left(\int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} \frac{t^{\lambda n} dz}{(t+|y-z-x|)^{2n+4}(t+|z|)^{\lambda n}} \right|^{1/2} \cdot |f_3(y)|\, dy \right)^2 \frac{dt}{t^{n-1}},$$

by Minkowski's inequality. To estimate the inner integral on the righthand side of the last inequality above, we need to consider the following two cases:

(b1) If $|z| \ge (1/2)|y-x|$, then we have $|z| \ge (1/8)(r_0+|y-x_0|)$ by inequality (2.15); and so, if we take $0<\varepsilon<\min\{1,((\lambda-2)n/2)\}$, then

$$\int_{\mathbf{R}^n} \frac{t^{\lambda n} dz}{(t+|y-z-x|)^{2n+4}(t+|z|)^{\lambda n}}$$
$$\le \frac{Ct^{\lambda n}}{t^{\lambda n-2n-2\varepsilon}(r_0+|y-x_0|)^{2n+2\varepsilon}} \int_{\mathbf{R}^n} \frac{dz}{(t+|y-z-x|)^{2n+4}}$$
$$\le \frac{Ct^{n+2\varepsilon-4}}{(r_0+|y-x_0|)^{2n+2\varepsilon}}.$$

Moreover, by (2.17) and Proposition 2.5, we deduce that
(2.18)

$$N \le Cr_0 \left\{ \int_{r_0}^{\infty} \left(\int_{\mathbf{R}^n} \left| \frac{t^{n+2\varepsilon-4}}{(r_0+|y-x_0|)^{2n+2\varepsilon}} \right|^{1/2} |f_3(y)|\, dy \right)^2 \frac{dt}{t^{n-1}} \right\}^{1/2}$$
$$\le Cr_0 \int_{\mathbf{R}^n} \frac{|f_3(y)|dy}{(r_0+|y-x_0|)^{n+\varepsilon}} \left\{ \int_{r_0}^{\infty} t^{2\varepsilon-3}\, dt \right\}^{1/2}$$
$$\le C\Phi^{-1} \left(r_0^{-n} \phi(r_0) \|f\|_{\mathcal{L}}^{\Phi,\phi} \right).$$

(b2) If $|z| \le (1/2)|y-x|$, then we have $|y-z-x| \ge |y-x|-|z| \ge (1/2)|y-x| \ge (1/8)(r_0+|y-x_0|)$ by the inequality (2.15); and so, if we let $0 < \varepsilon < 1$, then we have

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{t^{\lambda n} dz}{(t+|y-z-x|)^{2n+4}(t+|z|)^{\lambda n}} &\le \frac{C}{t^{4-2\varepsilon}(r_0+|y-x_0|)^{2n+2\varepsilon}} \int_{\mathbf{R}^n} \frac{t^{\lambda n} dz}{(t+|z|)^{\lambda n}} \\ &\le \frac{Ct^{n+2\varepsilon-4}}{(r_0+|y-x_0|)^{2n+2\varepsilon}}.\end{aligned}$$

This also implies the estimate (2.18) for N.

Now we have deduced that, in any case,

$$G^+(f_3)(x) \le G^+(f_3)(x') + C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right) \tag{2.19}$$

with the constant C independent of $x, x' \in (1/4)B_0$ and r_0.

From estimates (2.16) and (2.19), we obtain that, for $x \in (1/4)B_0$,

$$\begin{aligned} g_\lambda^*(f_3)(x) &\le G^+(f_3)(x') + C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right) \\ &\le g_\lambda^*(f_3)(x') + C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right), \end{aligned} \tag{2.20}$$

which yields $g_\lambda^*(f_3)(x) < \infty$ for any $x \in (1/4)B_0$ and $f \in \mathcal{L}^{\Phi,\phi}$, by the assumption that $g_\lambda^*(f_3)(x') < \infty$.

Further, since $g_\lambda^*(f_3)(x) < \infty$, we can repeat the above procedure to get that, for $x, x' \in (1/4)B_0$,

$$g_\lambda^*(f_3)(x') \le g_\lambda^*(f_3)(x) + C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right). \tag{2.21}$$

Combining inequalities (2.20) and (2.21), we obtain the desired inequality (2.14). This completes the proof of the proposition. □

3. The proof of Theorem 1.3. Let $f \in \mathcal{L}^{\Phi,\phi}$ and $T(f)(x)$ denote one of the three Littlewood-Paley functions $g(f)(x)$, $S(f)(x)$ and $g_\lambda^*(f)(x)$. Suppose $|(1/4)B_0 \cap \{x \in \mathbf{R}^n : T(f)(x) < \infty\}| > 0$ for a ball $B_0 = B(x_0, r_0)$ in $\mathbf{R}^n$ with radius r_0 large enough. We decompose the function $f(x)$ as follows

$$\begin{aligned} f(x) &= f_{B_0} + (f(x) - f_{B_0})\chi_{B_0}(x) + (f(x) - f_{B_0})\chi_{B_0{}^c}(x) \\ &= f_1(x) + f_2(x) + f_3(x). \end{aligned} \tag{3.1}$$

Obviously, $T(f_1)(x) \equiv 0$ for any $x \in \mathbf{R}^n$. Let $0 < \gamma < 1$ and $\chi(x) = \chi_{B_0}(x)$ be the characteristic function of B_0; we note that $(M\chi(x))^\gamma \leq 1$ and $(M\chi(x))^\gamma \in A_1 \subset A_{q_\Phi}$ since $q_\Phi > 1$ by Proposition 2.1. Hence by Proposition 2.4 and (1.2), we can get that

$$
\begin{aligned}
\int_{B_0} \Phi(|T(f_2)(x)|)\,dx &= \int_{\mathbf{R}^n} \Phi(|T(f_2)(x)|)(\chi(x))^\gamma\,dx \\
&\leq C\int_{\mathbf{R}^n} \Phi(|T(f_2)(x)|)(M\chi(x))^\gamma\,dx \\
&\leq C\int_{\mathbf{R}^n} \Phi(|f_2(x)|)(M\chi(x))^\gamma\,dx \\
&\leq C\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}} < \infty
\end{aligned} \tag{3.2}
$$

This follows $T(f_2)(x) < \infty$, almost everywhere $x \in B_0$.

Noting $T(f_3)(x) \leq T(f_2)(x) + T(f)(x)$, and so

$$
\left|\frac{1}{4}B_0 \cap \{x \in \mathbf{R}^n : T(f_3)(x) < \infty\}\right| > 0,
$$

we can take $x' \in (1/4)B_0$ such that $T(f_3)(x') < \infty$. Then by Proposition 2.6, Proposition 2.7 and Proposition 2.8, we know

$$
T(f_3)(x) \leq T(f_3)(x') + C\Phi^{-1}\left(r_0^{-n}\phi(r_0)\|f\|_{\mathcal{L}^{\Phi,\phi}}\right) < \infty, \tag{3.3}
$$

for any $x \in (1/4)B_0$. Also since $T(f)(x) \leq T(f_2)(x) + T(f_3)(x)$, we have $T(f)(x) < \infty$ for almost every $x \in (1/4)B_0$. Moreover, by the arbitrariness of the radius of $B_0 = B(x_0, r_0)$, we get that $T(f_3)(x) < \infty$ and $T(f)(x) < \infty$ for almost everywhere $x \in \mathbf{R}^n$.

Now we take any ball $B = B_r$ in $\mathbf{R}^n$ with radius r. Then we can choose a point $x' \in (1/4)B = B_{(1/4)r}$ such that $T(f_3)(x') < \infty$. Again decompose the function $f(x)$ into three parts,

$$
\begin{aligned}
f(x) &= f_B + (f(x) - f_B)\chi_B(x) + (f(x) - f_B)\chi_{B^c}(x) \\
&= f_1(x) + f_2(x) + f_3(x).
\end{aligned} \tag{3.4}
$$

Applying Proposition 2.6, Proposition 2.7 and Proposition 2.8, and inequality (3.2) with $r_0 = r$, we then obtain that

$$\begin{aligned}\int_B \Phi(|T(f)(x) - T(f_3)(x')|)\,dx &\\ \leq \int_B \Phi(|T(f_2)(x)| + |T(f_3)(x) - T(f_3)(x')|)\,dx &\\ \leq C\int_B \Phi(|T(f_2)(x)|)\,dx + C\int_B \Phi(|T(f_3)(x) - T(f_3)(x')|)\,dx &\\ \leq C\phi(r)\|f\|_{\mathcal{L}^{\Phi,\phi}}. &\end{aligned}$$

This yields inequality (1.3). The proof of Theorem 1.3 is complete.

Acknowledgments. The authors would like to express their thanks to the referee for his/her very careful reading and valuable comments.

REFERENCES

1. S. Campanato, *Proprietá di hölderianitá di alcune di funzioni*, Ann. Scuola Norm. Sup. Pisa **17** (1963), 175–188.

2. R. Kerman and A. Torchinsky, *Integral inequalities with weights for the Hardy maximal function*, Studia Math. **71** (1982), 277–284.

3. V. Kokilashvili and M. Krbec, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific, Singapore, 1991.

4. D.S. Kurtz, *Littlewood-Paley operators on* BMO, Proc. Amer. Math. Soc. **99** (1987), 657–666.

5. Qihong Lu and Xiangxing Tao, *Characterization of maximal operators in Orlicz-Morrey spaces of homogeneous type*, Appl. Math. J. Chinese Univ. **21** (2006), 52–58.

6. Shanzhen Lu, Changmei Tan and Kozo Yabuta, *Littlewood-Paley operators on the generalized Lipschitz spaces*, Georgian Math. J. **3** (1996), 69–80.

7. Yongzhong Sun, *On the existence and boundedness of square function operators on Campanato spaces*, Nagoya Math. J. **173** (2004), 139–151.

8. Changmei Tan, *The Littlewood-Paley operators on Orlicz spaces with weights*, Math. Acta Sci. **24** (2004), 81–87.

9. Shilin Wang, *Boundedness of the Littlewood-Paley g-functions on* $Lip_\alpha(\mathbf{R}^n)$ $(0 < \alpha < 1)$, Illinois Math. J. **33** (1989), 531–541.

10. Silei Wang, *Some properties of g-functions*, Sci. China **10** (1984), 890–899.

11. K. Yabuta, *Boundedness of Littlewood-Paley operators*, Math. Japonica **43** (1996), 143–150.

School of Economics and Management, Zhejiang University of Science and Technology, Hangzhou, 310023, P.R. China
Email address: syzh201@163.com

Department of Mathematics, School of Science, Zhejiang University of Science and Technology, Hangzhou, 310023, P.R. China
Email address: xxtao@hotmail.com