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HAUSDORFF DIMENSION OF THE IMAGE OF ITERATED ADDITIVE PROCESSES

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ABSTRACT. An iterated additive process X_t in $\mathbf{R}^d$ is a random field defined by

$$X_t = X^1_{t_1} + X^2_{t_2} + \cdots + X^N_{t_N}, \quad t = (t_1, t_2, \dots, t_N) \in \mathbf{R}^N_+,$$

where the $X^j_{t_j}$ are independent additive processes in $\mathbf{R}^d$. We compute the Hausdorff dimension of the image of an arbitrary iterated additive process.

1. Introduction. A process Y_t, $t \in \mathbf{R}_+$, with independent increments, rcll paths and values in $\mathbf{R}^d$ is called additive if Y_t is continuous in probability and $Y_0 = 0$. An iterated additive process X_t in $\mathbf{R}^d$ is a random field defined by

$$X_t = X^1_{t_1} + X^2_{t_2} + \cdots + X^N_{t_N}, \quad t = (t_1, t_2, \dots, t_N) \in \mathbf{R}^N_+$$

where the $X^j_{t_j}$ are independent additive processes in $\mathbf{R}^d$.

When all the $X^j_{t_j}$ above are Lévy processes, X_t is called an additive Lévy process (cf. Khoshnevisan, et al. [**2**]). Let $X(G) = \{X_t : t \in G\}$ for $G \in \mathcal{B}(\mathbf{R}^N_+)$, and denote by $\dim_H$ the Hausdorff dimension. We are interested in computing $\dim_H X(G)$. This paper was motivated by the author's desire to find the solution to an unsolved problem. The author raised the following question in a previous note:

Question. *Given an arbitrary additive process Y_t, can one find a closed-form expression of $\dim_H Y([0,t])$ for $t > 0$? The closed-form expression can be given in terms of the expected sojourn times, the characteristics of Y_t, or any data containing the information of Y_t.*

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($\dim_H Y([0,t])$ does not depend on t for Lévy processes but does for a general additive process.) In this paper, we give the answer to the above question in a rather general way. It turns out that the potential-theoretic approach of Khoshnevisan, et al. [**2**] can be developed further to compute the Hausdorff dimension of the image of an arbitrary iterated additive process.

Recall that the characteristic function for an additive process Y_t takes on the form

$$Ee^{i\xi\cdot Y_t} = e^{-\Psi_t(\xi)}, \quad \xi \in \mathbf{R}^d,$$

where for each fixed t, $\Psi_t(\xi)$ is a Lévy exponent. We mention that $\Psi_t(\xi)$ is jointly continuous in t and ξ with $\Psi_0(\xi) = 0$. In particular, $\operatorname{Re}\Psi_t(\xi)$ is a nondecreasing, nonnegative function of t for each fixed ξ. Let $(X;\ \Psi^1, \dots, \Psi^N)$ be an iterated additive process; that is, $\Psi^j_{t_j}(\xi)$ is the characteristic exponent of $X^j_{t_j}$, for $1 \le j \le N$. Let $\mathcal{P}(G)$ denote the collection of probability measures on G. Let

$$S^\alpha_t = S^1_{t_1} + S^2_{t_2} + \cdots + S^d_{t_d}$$

be the standard d-parameter additive α-stable Lévy process in $\mathbf{R}^d$ for $\alpha \in (0,1)$; that is, the S^j are independent standard α-stable Lévy processes in $\mathbf{R}^d$ with the common Lévy exponent $|\xi|^\alpha$. For any probability measure μ in $\mathbf{R}^N_+$ and $\xi \in \mathbf{R}^d$, define

$$\text{(1.1)} \quad Q_\mu(\xi) = \int_{\mathbf{R}^N_+}\int_{\mathbf{R}^N_+} e^{-\sum_{j=1}^N \operatorname{sgn}(s_j-t_j)[\Psi^j_{s_j}(\operatorname{sgn}(s_j-t_j)\xi) - \Psi^j_{t_j}(\operatorname{sgn}(s_j-t_j)\xi)]}\mu(ds)\mu(dt).$$

We will see that $0 \le Q_\mu(\xi) \le 1$ in a moment. The main result of this paper is our characterization of the following intersection probability.

Theorem 1.1. *For all $\beta \in (0,d)$ and $S^{1-\beta/d}$ independent of X,*

$$\text{(1.2)} \quad P\left\{X(G)\bigcap S^{1-\beta/d}((0,\infty)^d) \neq \varnothing\right\} > 0 \Longleftrightarrow \int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi)d\xi < \infty$$

for some $\mu \in \mathcal{P}(G)$, where $|x| = (x\cdot x)^{1/2}$.

An immediate consequence of Theorem 1.1 is the following dimension result.

Theorem 1.2. *Let $(X;\ \Psi^1, \dots, \Psi^N)$ be any iterated additive process in $\mathbf{R}^d$, and let G be any Borel set in $\mathbf{R}^N_+$. Then*

(1.3)
$$\dim_H X(G) = \sup\left\{\beta \in (0,d) : \inf_{\mu\in\mathcal{P}(G)} \int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi)\, d\xi < \infty\right\} \ a.s.$$

Theorem 1.2 implies that $\dim_H X(G)$ is deterministic for any Borel set G of $\mathbf{R}^N_+$. If we let $N=1$ and $G=[0,t]\subset\mathbf{R}_+$ in (1.3), we obtain a closed-form formula for $\dim_H Y([0,t])$ for any additive process Y.

A more difficult question arose naturally to the author's mind. Suppose that G in Theorem 1.2 has a nonempty interior. Is it possible to determine a probability measure μ_e^G on $\mathbf{R}^N_+$ such that

$$\dim_H X(G) = \sup\left\{\beta \in (0,d) : \int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_{\mu_e^G}(\xi)\, d\xi < \infty\right\} \text{ a.s.?}$$

In the case of additive Lévy processes, this has been known: $\mu_e^G = \kappa$, where $\kappa(dt) = e^{-\sum_{j=1}^N t_j} dt$. It can be obtained by a different but related argument. However, the above-mentioned argument did not do quite as well for a general iterated additive process, even in the cases when $G=[0,t]^N$ and $G=\mathbf{R}^N_+$. The author believes that, due to the nature of time-nonhomogeneity, it is highly unlikely to advance beyond the results of this paper with the method of [**2**]. One case (somewhat nonessential) is possible. If X is symmetric and the function $Ee^{i\xi\cdot(X_t-X_s)}$ (which is real and positive in this case) is comparable to that of some symmetric additive Lévy process, then we can choose μ_e^G to be κ.

We point out that our proof of Theorem 1.1 is very close to the proof of Theorem 2.2 of Khoshnevisan and Xiao [**1**] for the direction $\Longrightarrow$, and to the proof of Theorem 2.1 of Khoshnevisan, et al. [**2**] for the direction $\Longleftarrow$.

2. Proof of the lower bound. The direction $\Longleftarrow$ in (1.2) is referred to as the "lower bound" in the literature. The reader can find that this part of the proof in this section does not need the assumption of independent increments. The argument works for any kind of process Y. One can use the characteristic function $Ee^{i\xi\cdot Y_t}$

instead of the exponent "Ψ," which does not make sense to Y outside the class of additive processes.

Let Y be an additive process, and let $0 \le s \le t$. By the hypothesis of independent increments,

$$Ee^{i\xi\cdot Y_t} = Ee^{i\xi\cdot(Y_t - Y_s) + i\xi\cdot Y_s} = Ee^{i\xi\cdot(Y_t - Y_s)}Ee^{i\xi\cdot Y_s}.$$

Thus,

$$Ee^{i\xi\cdot(Y_t - Y_s)} = e^{-[\Psi_t(\xi) - \Psi_s(\xi)]}.$$

It follows from calculations that, for any iterated additive process $(X;\ \Psi^1, \dots, \Psi^N)$ and any $s,\ t \in \mathbf{R}_+^N$,

$$Ee^{i\xi\cdot(X_t - X_s)} = e^{-\sum_{j=1}^N \operatorname{sgn}(t_j - s_j)[\Psi^j_{t_j}(\operatorname{sgn}(t_j - s_j)\xi) - \Psi^j_{s_j}(\operatorname{sgn}(t_j - s_j)\xi)]}.$$

Lemma 2.1.

$$Q_\mu(\xi) = E\left|\int_{\mathbf{R}_+^N} e^{i\xi\cdot X_t}\mu(dt)\right|^2 = E\int_{\mathbf{R}_+^N}\int_{\mathbf{R}_+^N} e^{i\xi\cdot(X_s - X_t)}\mu(ds)\mu(dt).$$

Proof. This lemma follows simply from the Fubini theorem and the fact that

$$Ee^{i\xi\cdot(X_s - X_t)} = e^{-\sum_{j=1}^N \operatorname{sgn}(s_j - t_j)[\Psi^j_{s_j}(\operatorname{sgn}(s_j - t_j)\xi) - \Psi^j_{t_j}(\operatorname{sgn}(s_j - t_j)\xi)]}. \qquad \square$$

Lemma 2.1 implies that $0 \le Q_\mu(\xi) \le 1$ since $|\int_{\mathbf{R}_+^N} e^{i\xi\cdot X_t}\mu(dt)| \le 1$.

Introduce the μ-occupation measure O_μ by

$$\int_{\mathbf{R}^d} f(x)O_\mu(dx) = \int_{\mathbf{R}_+^N} f(X_t)\mu(dt),$$

where $f : \mathbf{R}^d \to \mathbf{R}_+$ is a measurable function. In particular,

$$O_\mu(E) = \int_{\mathbf{R}_+^N} 1(X_t \in E)\mu(dt), \quad E \in \mathcal{B}(\mathbf{R}^d).$$

Note that O_μ is a random probability measure on $\mathbf{R}^d$ and, if μ is supported on F, then O_μ is supported on $X(F)$. Recall that the Fourier transform of a probability measure ν on $\mathbf{R}^d$ is defined by

$$\widehat{\nu}(\xi) = \int_{\mathbf{R}^d} e^{i\xi\cdot x}\nu(dx), \quad \xi \in \mathbf{R}^d.$$

Lemma 2.2.

$$Q_\mu(\xi) = E|\widehat{O}_\mu(\xi)|^2.$$

Proof. From the very definition of O_μ, we have a frequently used identity

$$\int_{\mathbf{R}^d} e^{i\xi\cdot x} O_\mu(dx) = \int_{\mathbf{R}_+^N} e^{i\xi\cdot X_t}\mu(dt). \tag{2.1}$$

By (2.1),

$$|\widehat{O}_\mu(\xi)|^2 = \int_{\mathbf{R}_+^N}\int_{\mathbf{R}_+^N} e^{i\xi\cdot(X_t - X_s)}\mu(dt)\mu(ds).$$

Finally, use Lemma 2.1 to finish. □

In the following, $\|f\|^2_{L^2(\mathbf{R}^d)} = \int_{\mathbf{R}^d} |f(\xi)|^2 d\xi$ for any complex-valued function f.

Lemma 2.3.

$$E\|\widehat{O}_\mu\|^2_{L^2(\mathbf{R}^d)} = \int_{\mathbf{R}^d} Q_\mu(\xi)\, d\xi.$$

Proof.

$$\|\widehat{O}_\mu\|^2_{L^2(\mathbf{R}^d)} = \int_{\mathbf{R}^d} |\widehat{O}_\mu(\xi)|^2 d\xi.$$

Use the Fubini theorem and Lemma 2.2 to finish. □

Let λ_d denote Lebesgue measure in $\mathbf{R}^d$.

Proposition 2.4. *Let F be any Borel set in $\mathbf{R}_+^N$, and let μ be any probability measure on F. Then*

$$\int_{\mathbf{R}^d} Q_\mu(\xi)\, d\xi < \infty \Longrightarrow E\{\lambda_d(X(F))\} > 0. \tag{2.2}$$

Proof. By Lemma 2.3, $E\|\widehat{O}_\mu\|^2_{L^2(\mathbf{R}^d)} < \infty$. Thus, $\|\widehat{O}_\mu\|^2_{L^2(\mathbf{R}^d)} < \infty$ almost surely. By Plancherel's theorem, the measure O_μ is absolutely continuous with respect to λ_d almost surely, and there exists a measurable version $L_\mu(x)$ of the density (also called the μ-local time) of O_μ satisfying

$$\|L_\mu\|^2_{L^2(\mathbf{R}^d)} = (2\pi)^{-d}\|\widehat{O}_\mu\|^2_{L^2(\mathbf{R}^d)} \text{ almost surely.}$$

Note that $O_\mu(A) = \int_A L_\mu(x)\, dx$, $A \in \mathcal{B}(\mathbf{R}^d)$. Since μ is supported on F, O_μ is supported on $X(F)$, so is L_μ. Also note that $O_\mu(\mathbf{R}^d) = 1$. Thus, by the Cauchy-Schwarz inequality,

$$\begin{aligned} 1 = (O_\mu(\mathbf{R}^d))^2 &= \left(\int_{\mathbf{R}^d} 1_{X(F)}(x) L_\mu(x)\, dx\right)^2 \\ &\le \int_{\mathbf{R}^d} 1^2_{X(F)}(x) dx \int_{\mathbf{R}^d} L^2_\mu(x)\, dx \\ &= \lambda_d(X(F))(2\pi)^{-d}\|\widehat{O}_\mu\|^2_{L^2(\mathbf{R}^d)} \text{ almost surely.} \end{aligned}$$

Applying the inequality $E(\zeta^{-1})E\zeta \ge 1$ valid for all positive random variables ζ and the fact that $E\|\widehat{O}_\mu\|^2_{L^2(\mathbf{R}^d)} = \int_{\mathbf{R}^d} Q_\mu(\xi)\, d\xi$ by Lemma 2.3, we obtain

$$E\{\lambda_d(X(F))\} \ge (2\pi)^d \left[\int_{\mathbf{R}^d} Q_\mu(\xi)\, d\xi\right]^{-1} > 0. \qquad \square$$

Proof of the direction $\Longleftarrow$ of (1.2). First, note that

$$\int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi)\, d\xi < \infty \Longleftrightarrow \int_{\mathbf{R}^d} \left(\frac{1}{1+|\xi|^{1-\beta/d}}\right)^d Q_\mu(\xi)\, d\xi < \infty, \tag{2.3}$$

since $Q_\mu(\xi) \le 1$. Now we apply Proposition 2.4 to the case where the iterated additive process is $X - S^{1-\beta/d}$, the set is $G \times (0,\infty)^d$, and the measure is $\mu \times \kappa$, where $\kappa(dt) = e^{-\sum_{j=1}^d t_j} dt$, $t = (t_1, \dots, t_d) \in \mathbf{R}_+^d$ to find that

$$E\{\lambda_d(X(G) - S^{1-\beta/d}((0,\infty)^d))\} > 0,$$

since the quantity $Q_\mu(\xi)$ in Proposition 2.4 in this case equals

$$\left(\frac{1}{1+|\xi|^{1-\beta/d}}\right)^d Q_\mu(\xi).$$

Finally, since for every $t \in (0,\infty)^N$ the distribution of $S_t^{1-\beta/d}$ is mutually absolutely continuous with respect to λ_d and since $S^{1-\beta/d}$ and X are independent, by [**1**, Lemma 4.1], we have

$$\begin{aligned} (2.4) \quad & E\{\lambda_d(X(G) - S^{1-\beta/d}((0,\infty)^d))\} > 0 \\ & \qquad \iff P\left\{X(G) \bigcap S^{1-\beta/d}((0,\infty)^d) \neq \varnothing\right\} > 0. \end{aligned}$$

Thus,

$$\int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi) d\xi < \infty \Longrightarrow P\left\{X(G) \bigcap S^{1-\beta/d}((0,\infty)^d) \neq \varnothing\right\} > 0. \square$$

3. Proof of the upper bound. By (2.4), to prove

$$P\left\{X(G) \bigcap S^{1-\beta/d}((0,\infty)^d) \neq \varnothing\right\} > 0 \Longrightarrow \int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi)\, d\xi < \infty,$$

we only need to establish

(3.1)
$$E\{\lambda_d(X(G) - S^{1-\beta/d}((0,\infty)^d))\} > 0 \Longrightarrow \int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi)\, d\xi < \infty$$

for some $\mu \in \mathcal{P}(G)$.

To work toward this direction, we need an appropriate sigma-finite measure as defined below, which gives the information of the Lebesgue measure of the image set of the process naturally via the Fubini theorem. Let us get started with a simple lemma. Let Z be any random variable taking values in $\mathbf{R}^d$. Suppose that $f,\ \hat{f} \in L^1$. Define the function $Pf(x) = Ef(Z+x)$, $x \in \mathbf{R}^d$. Here, Pf is a symbol. Pf can always be expressed in terms of the characteristic function of Z.

Lemma 3.1.

$$Pf(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{ix\cdot\xi} \hat{f}(-\xi) E e^{iZ\cdot\xi} d\xi, \quad x \in \mathbf{R}^d.$$

Proof. Let Q be the distribution of $-Z$. Then,

$$Pf(\xi) = Ef(Z+\xi) = Ef(-(-Z)+\xi) = \int_{\mathbf{R}^d} f(\xi-y)Q(dy) = f \star Q(\xi).$$

Thus,

$$\widehat{Pf}(\xi) = \widehat{f \star Q}(\xi) = \hat{f}(\xi)\widehat{Q}(\xi).$$

By the inversion formula and the fact that $|\widehat{Q}(\xi)| \le 1$,

$$\begin{aligned} Pf(x) &= (2\pi)^{-d} \int_{\mathbf{R}^d} e^{-ix\cdot\xi} \hat{f}(\xi)\widehat{Q}(\xi)\, d\xi \\ &= (2\pi)^{-d} \int_{\mathbf{R}^d} e^{ix\cdot\xi} \hat{f}(-\xi)\widehat{Q}(-\xi)\, d\xi \\ &= \int_{\mathbf{R}^d} e^{ix\cdot\xi} \hat{f}(-\xi) E e^{iZ\cdot\xi} d\xi, \end{aligned}$$

since $\widehat{Q}(-\xi) = Ee^{iZ\cdot\xi}$ is the characteristic function of Z. □

Let X be an N-parameter iterated additive process in $\mathbf{R}^d$. For any $s,\ t \in \mathbf{R}_+^N$ and f with $f,\ \hat{f} \in L^1$, we define

$$P_{s,t}f(x) = Ef(X_t - X_s + x), \quad x \in \mathbf{R}^d.$$

Note that the order of $s,\ t$ in $P_{s,t}f$ matters. It follows from Lemma 3.1 that

Lemma 3.2.

$$\begin{aligned}P_{s,t}f(x) &= (2\pi)^{-d}\int_{\mathbf{R}^d} e^{ix\cdot\xi}\hat{f}(-\xi)\\ &\quad\times e^{-\sum_{j=1}^{N}\operatorname{sgn}(t_j-s_j)[\Psi^j_{t_j}(\operatorname{sgn}(t_j-s_j)\xi)-\Psi^j_{s_j}(\operatorname{sgn}(t_j-s_j)\xi)]}d\xi\\ &= (2\pi)^{-d}\int_{\mathbf{R}^d} e^{ix\cdot\xi}\hat{f}(-\xi)Ee^{i\xi\cdot(X_t-X_s)}d\xi, \quad x\in\mathbf{R}^d.\end{aligned}$$

Since X has rcll paths for every parameter t_j of $(t_1,\dots,t_N)\in\mathbf{R}^N_+$, for each $x\in\mathbf{R}^d$ there is a probability measure P^x on Ω, the law that X starts from x. ($P^0=P$.) Hence, we have a sigma-finite measure P_{λ_d} on Ω given by $P_{\lambda_d}(\bullet)=\int_{\mathbf{R}^d}P^x(\bullet)\,dx$. Let E_{λ_d} denote the expectation operator with respect to P_{λ_d} for real-valued random variables. P_{λ_d} has some favorable features. For example, for any nonnegative function f, by Fubini's theorem, a change of variable and the fact that the Lebesgue measure is translation-invariant,

$$E_{\lambda_d}f(X_t)=\int_{\mathbf{R}^d}E[f(x+X_t)]\,dx=\int_{\mathbf{R}^d}f(x)\,dx.$$

In particular, for all $t\in\mathbf{R}^N_+$ and Borel sets A in $\mathbf{R}^d$, $P_{\lambda_d}(X_t\in A)=\lambda_d(A)$. That is, under P_{λ_d}, the distribution of X_t is λ_d for all $t\in\mathbf{R}^N_+$.

For the sake of completeness, we reproduce but selectively the material between the definition of E_{λ_d} and Lemma 3.4 in Section 3 of Khoshnevisan, et al. [**2**]. Let A be a subset of $\{1,\dots,N\}$, and define the partial order $\preceq_A$ on $\mathbf{R}^N_+$ by

$$s\preceq_A t \iff s_i\le t_i,\quad i\in A \text{ and } s_i\ge t_i,\ i\notin A.$$

A worthy computational observation is that if $t\succeq_A s\succeq_A s'$ then X_t-X_s and $X_s-X_{s'}$ are independent. Define $\mathcal{F}^A_t$ to be the sigma-field generated by $\{X_s;\ s\preceq_A t\}$, which is P^x-complete for all $x\in\mathbf{R}^d$, and $\mathcal{F}^A$ is right-continuous under $\preceq_A$.

Let f be a nonnegative function in L^1. Then, for any $t\in\mathbf{R}^N_+$, $E_{\lambda_d}f(X_t)<\infty$. Thus, the conditional expectation $E_{\lambda_d}[f(X_t)|\mathcal{G}]$ of $f(X_t)$ given a sub sigma-field $\mathcal{G}$ of $\mathcal{F}^A_t$ exists under P_{λ_d}. In this paper, only Propositions 3.3 and 3.5 below require the assumption of independent increments.

Proposition 3.3. *For any $s \preceq_A t$, $P_{s,t}f(X_s)$ is a version of $E_{\lambda_d}[f(X_t)|\mathcal{F}_s^A]$, i.e.,*

$$E_{\lambda_d}[f(X_t)|\mathcal{F}_s^A] = P_{s,t}f(X_s), \quad P_{\lambda_d} \text{ almost surely.}$$

Here, the P_{λ_d}-null set is independent of t.

Proof. Let $g, h_1, \dots, h_m$ be bounded measurable nonnegative functions, and assume that $s \succeq_A s_i$, $1 \leq i \leq m$. Since $X_t - X_s$ and $X_s - X_{s_i}$ are independent,

$$\begin{aligned}
&E_{\lambda_d}[f(X_t)g(X_s)\prod_{i=1}^m h_i(X_{s_i})] \\
&\quad = \int_{\mathbf{R}^d} E\left[f(X_t + x)g(X_s + x)\prod_{i=1}^m h_i(X_{s_i} + x)\right] dx \\
&\quad = \int_{\mathbf{R}^d} E\left[f(X_t - X_s + y)g(y)\prod_{i=1}^m h_i(X_{s_i} - X_s + y)\right] dy \\
&\quad = \int_{\mathbf{R}^d} E[f(X_t - X_s + y)]g(y)E\left[\prod_{i=1}^m h_i(X_{s_i} - X_s + y)\right] dy.
\end{aligned}$$

Since the distribution of X_s is λ_d under P_{λ_d}, the proposition follows. □

For nonnegative f in L^1 and probability measure μ on $\mathbf{R}_+^N$, define the nonnegative random variable

$$O_\mu f = \int_{\mathbf{R}_+^N} f(X_s)\mu(ds).$$

We have

$$\begin{aligned}
E_{\lambda_d}O_\mu f &= E_{\lambda_d}\int_{\mathbf{R}_+^N} f(X_s)\mu(ds) = \int_{\mathbf{R}_+^N} E_{\lambda_d}f(X_s)\mu(ds) \\
&= \int_{\mathbf{R}^d} f(x)\,dx \int_{\mathbf{R}_+^N}\mu(ds) = \int_{\mathbf{R}^d} f(x)\,dx < \infty.
\end{aligned}$$

Thus, we can define an N-parameter process

$$\mathcal{M}_t^{A,f,\mu} = E_{\lambda_d}[O_\mu f | \mathcal{F}_t^A], \quad t \in \mathbf{R}_+^N$$

with $E_{\lambda_d}\mathcal{M}_t^{A,f,\mu} = E_{\lambda_d} O_\mu f = \int_{\mathbf{R}^d} f(x)\,dx$ for all $t \in \mathbf{R}_+^N$. □

Lemma 3.4. *For all $s \in \mathbf{R}_+^N$,*

$$\mathcal{M}_s^{A,f,\mu} \geq \int_{t \succeq_A s} P_{s,t} f(X_s)\mu(dt), \quad P_{\lambda_d} \text{ almost surely.}$$

Here, the P_{λ_d}-null set depends on s.

Proof. By Proposition 3.3 and Fubini's theorem,

$$\begin{aligned}
\mathcal{M}_s^{A,f,\mu} &= E_{\lambda_d}\left[\int_{\mathbf{R}_+^N} f(X_t)\mu(dt) | \mathcal{F}_s^A\right] \\
&\geq E_{\lambda_d}\left[\int_{t \succeq_A s} f(X_t)\mu(dt) | \mathcal{F}_s^A\right] \\
&= \int_{t \succeq_A s} E_{\lambda_d}[f(X_t) | \mathcal{F}_s^A]\mu(dt) \\
&= \int_{t \succeq_A s} P_{s,t} f(X_s)\mu(dt), \quad P_{\lambda_d} \text{ almost surely,}
\end{aligned}$$

since the P_{λ_d}-null set is independent of t. □

The theory of *multi-parameter* L^2-maximal inequalities is the very technical foundation of the approach developed by Khoshnevisan, et al. [**2**] and others. The following P_{λ_d}-type result has been carried out based on Lévy processes, but obviously it applies to additive processes without any trouble. We take it for granted. For further information about their arguments, see [**2**, Lemma 4.2]. Let $\mathbf{Q}$ denote the rational field as always.

Proposition 3.5. *Under any partial order A, for all nonnegative random variables Z,*

$$E_{\lambda_d} \sup_{t \in \mathbf{Q}_+^N} \left[E_{\lambda_d}[Z | \mathcal{F}_t^A]\right]^2 \leq 4^N E_{\lambda_d} Z^2.$$

Lemma 3.6. *If $f \in L^1 \cap L^2$ and $\hat{f} \in L^1$, then*

$$E_{\lambda_d} \sup_{t \in \mathbf{Q}_+^N} (\mathcal{M}_t^{A,f,\mu})^2 \le 4^N (2\pi)^{-d} \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 Q_\mu(\xi)\, d\xi.$$

Proof. By Proposition 3.5, this lemma is nothing but about computing $E_{\lambda_d}(O_\mu f)^2$. We show that

$$E_{\lambda_d}(O_\mu f)^2 = (2\pi)^{-d} \int_{\mathbf{R}}^d |\hat{f}(\xi)|^2 Q_\mu(\xi)\, d\xi. \tag{3.2}$$

To obtain (3.2), we prove a preliminary result: Suppose that g is another nonnegative function also satisfying $g \in L^1 \cap L^2$ and $\widehat{g} \in L^1$. Then, for any $s,\ t \in \mathbf{R}_+^N$,

$$E_{\lambda_d}[f(X_s)g(X_t)] = (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{f}(\xi)\overline{\widehat{g}(\xi)} E e^{i\xi\cdot(X_t - X_s)} d\xi. \tag{3.3}$$

First of all, as in the proof of Proposition 3.3, we find that

$$E_{\lambda_d}[f(X_s)g(X_t)] = \int_{\mathbf{R}^d} P_{s,t} g(y) f(y)\, dy.$$

By Lemma 3.2 and Fubini's theorem, thanks to $f,\ \widehat{g} \in L^2$ (by Plancherel's theorem, $\widehat{g} \in L^2$),

$$\begin{aligned}
&\int_{\mathbf{R}^d} P_{s,t} g(y) f(y)\, dy \\
&\qquad = \int_{\mathbf{R}^d} \left((2\pi)^{-d} \int_{\mathbf{R}^d} e^{iy\cdot\xi} \widehat{g}(-\xi) E e^{i\xi\cdot(X_t - X_s)} d\xi \right) f(y)\, dy \\
&\qquad = (2\pi)^{-d} \int_{\mathbf{R}^d} \hat{f}(\xi)\overline{\widehat{g}(\xi)} E e^{i\xi\cdot(X_t - X_s)} d\xi.
\end{aligned}$$

Equation (3.3) is proved. Recall that

$$E e^{i\xi\cdot(X_t - X_s)} = e^{-\sum_{j=1}^N \operatorname{sgn}(t_j - s_j)[\Psi_{t_j}^j(\operatorname{sgn}(t_j - s_j)\xi) - \Psi_{s_j}^j(\operatorname{sgn}(t_j - s_j)\xi)]}. \tag{3.4}$$

Now, by (3.3), (3.4) and Fubini's theorem (by Plancherel's theorem, $\hat{f} \in L^2$),

$$\begin{aligned} E_{\lambda_d}(O_\mu f)^2 &= E_{\lambda_d}\Big[\int_{\mathbf{R}_+^N}\int_{\mathbf{R}_+^N} f(X_s)f(X_t)\mu(ds)\mu(dt)\Big] \\ &= \int_{\mathbf{R}_+^N}\int_{\mathbf{R}_+^N} E_{\lambda_d}[f(X_s)f(X_t)]\mu(ds)\mu(dt) \\ &= \int_{\mathbf{R}_+^N}\int_{\mathbf{R}_+^N}\Big((2\pi)^{-d}\int_{\mathbf{R}^d}\hat{f}(\xi)\overline{\hat{f}(\xi)}Ee^{i\xi\cdot(X_t-X_s)}d\xi\Big)\mu(ds)\mu(dt) \\ &= (2\pi)^{-d}\int_{\mathbf{R}^d}|\hat{f}(\xi)|^2 Q_\mu(\xi)\,d\xi. \end{aligned}$$

Equation (3.2) follows and so does the lemma. □

Proof of the direction $\Longrightarrow$ *of* (1.2). As was stated at the very beginning of this section, this direction follows from (3.1). Let $B(x,r) \subset \mathbf{R}^d$ be the closed ball of radius r with center at x. To prove (3.1), it suffices to show that

$$E\{\lambda_d(X(G)-S^{1-\beta/d}([0,l]^d))\} > 0 \Longrightarrow \int_{\mathbf{R}^d}|\xi|^{\beta-d}Q_\mu(\xi)\,d\xi < \infty \tag{3.5}$$

for some $\mu \in \mathcal{P}(G)$, where G is compact and $l \in (0,\infty)$. Thus, there exists a large $R \in (0,\infty)$ such that

$$E\left\{\lambda_d\left([X(G) - S^{1-\beta/d}([0,l]^d)]\bigcap B(0,R)\right)\right\} > 0.$$

Let $Z = X - S^{1-\beta/d}$, which is an $(N+d)$-parameter iterated additive process in $\mathbf{R}^d$. Therefore, we have that

$$E\left\{\lambda_d\left(Z(G\times[0,l]^d)\bigcap B(0,R)\right)\right\} \in (0,\infty). \tag{3.6}$$

Let G^δ be the closed δ-enlargement of G for $\delta > 0$, that is, the smallest compact set such that for each point $s = (s_1,\dots,s_N) \in G$, $[s_1, s_1+\delta]\times\cdots\times[s_N, s_N+\delta] \subset G^\delta$. Let P_{λ_d} and E_{λ_d} be the sigma-finite measure and the corresponding expectation operator, respectively, with respect to Z. By the definition of P_{λ_d}, also thanks to Fubini's theorem and the fact that $-B(0,r)=B(0,r)$,

$$\begin{aligned}
&P_{\lambda_d}\left\{Z(G^\delta\times[0,l]^d)\bigcap B(0,\delta)\neq\varnothing,\ Z_0\in B(0,R)\right\}\\
&\qquad=\int_{B(0,R)}P\left\{[x+Z(G^\delta\times[0,l]^d)]\bigcap B(0,\delta)\neq\varnothing\right\}\,dx\\
&\qquad=\int_{B(0,R)}P\left\{x\in B(0,\delta)-Z(G^\delta\times[0,l]^d)\right\}\,dx\\
&\qquad=\int_{B(0,R)}P\left\{x\in Z(G^\delta\times[0,l]^d)-B(0,\delta)\right\}\,dx\\
&\qquad=E\left\{\lambda_d\left([Z(G^\delta\times[0,l]^d)-B(0,\delta)]\bigcap B(0,R)\right)\right\}\\
&\qquad=E\left\{\lambda_d\left([Z(G^\delta\times[0,l]^d)+B(0,\delta)]\bigcap B(0,R)\right)\right\}\\
&\qquad\longrightarrow E\left\{\lambda_d\left(\overline{Z(G\times[0,l]^d)}\bigcap B(0,R)\right)\right\}
\end{aligned}$$

downwards as $\delta\to 0$. Thus, by (3.6) for all $\delta>0$,

$$P_{\lambda_d}\left\{Z(G^\delta\times[0,l]^d)\bigcap B(0,\delta)\neq\varnothing,\ Z_0\in B(0,R)\right\}\in(0,\infty). \tag{3.7}$$

We add a cemetery point $\Delta\notin\mathbf{R}_+^N$ to $\mathbf{R}_+^N$ to construct a measurable map T^δ (random variable) from Ω to $\mathbf{Q}_+^N\cup\{\Delta\}$. T^δ is defined as follows. $T^\delta\neq\Delta$ if and only if $T^\delta\in\mathbf{Q}_+^N\cap(0,\infty)^N\cap G^\delta$ and there exists some $t\in(0,l]^d\cap\mathbf{Q}_+^d$ such that $|Z_{(T^\delta,t)}|\leq\delta$. This can always be done. We have

$$\begin{aligned}
&P_{\lambda_d}\left\{T^\delta\neq\Delta,\ Z_0\in B(0,R)\right\}\\
&\quad=P_{\lambda_d}\left\{Z(G^\delta\times[0,l]^d)\bigcap B(0,\delta)\neq\varnothing,\ Z_0\in B(0,R)\right\}\in(0,\infty).
\end{aligned}\tag{3.8}$$

There is therefore a probability measure μ^δ in $\mathbf{R}_+^N$ supported on G^δ given by

$$\mu^\delta(\bullet)=\frac{P_{\lambda_d}\{T^\delta\in\bullet,\ T^\delta\neq\Delta,\ Z_0\in B(0,R)\}}{P_{\lambda_d}\{T^\delta\neq\Delta,\ Z_0\in B(0,R)\}}. \tag{3.9}$$

For $\varepsilon>0$, define $f_\varepsilon(x)=(2\pi\varepsilon^2)^{-d/2}e^{-|x|^2/2\varepsilon^2}$, $x\in\mathbf{R}^d$. Let $u,\ v\in\mathbf{R}_+^N$ and $s,\ t\in\mathbf{R}_+^d$. In this paper, we only need to consider the partial orders on $\mathbf{R}_+^N$. Each partial order π on $\mathbf{R}_+^N$ corresponds to a partial order A on $\mathbf{R}_+^{N+d}$ by $(u,s) \preceq_A (v,t) \leftrightarrow u \preceq_\pi v$, $s_i \le t_i$, $1 \le i \le d$. Note that

$$\widehat{f_\varepsilon}(\xi) = e^{-\varepsilon^2|\xi|^2/2}, \qquad |\widehat{f_\varepsilon}(\xi)|^2 = e^{-\varepsilon^2|\xi|^2}$$

and that the Lévy exponent of $-S^{1-\beta/d}$ is

$$(|\xi|^{1-\beta/d}, \dots, |\xi|^{1-\beta/d}).$$

Let $\nu = \mu \times \kappa$, $\mathbf{s} = (u,s)$ and $\mathbf{t} = (v,t)$. By Lemma 3.2 along with interchanging order of integration due to the term $e^{-\varepsilon^2|\xi|^2/2}$, for any partial order A,

$$\begin{aligned}
&\int_{\mathbf{t} \succeq_A \mathbf{s}} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(0)\nu(d\mathbf{t}) \\
&\qquad = (2\pi)^{-d} e^{-\sum_{i=1}^d s_i} \int_{v \succeq_\pi u} \int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} \\
&\qquad \times e^{-\sum_{j=1}^N \operatorname{sgn}(v_j - u_j)[\Psi^j_{v_j}(\operatorname{sgn}(v_j-u_j)\xi) - \Psi^j_{u_j}(\operatorname{sgn}(v_j-u_j)\xi)]} \\
&\qquad \times \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi \mu(dv).
\end{aligned}$$

Clearly, f_ε is Lipschitz continuous. Let $D(\varepsilon)$ be the Lipschitz constant of f_ε. By the definition of $P_{\mathbf{s},\mathbf{t}} f_\varepsilon$,

$$D(\varepsilon)\delta + \inf_{|z|\le\delta} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(z) \ge P_{\mathbf{s},\mathbf{t}} f_\varepsilon(0).$$

Since ν is a probability measure,

$$D(\varepsilon)\delta + \int_{\mathbf{t} \succeq_A \mathbf{s}} \inf_{|z|\le\delta} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(z)\nu(d\mathbf{t}) \ge \int_{\mathbf{t} \succeq_A \mathbf{s}} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(0)\nu(d\mathbf{t}).$$

If $|Z_\mathbf{s}| \le \delta$, then $P_{\mathbf{s},\mathbf{t}} f_\varepsilon(Z_\mathbf{s}) \ge \inf_{|z|\le\delta} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(z)$. Note that $\inf_{|z|\le\delta} \times P_{\mathbf{s},\mathbf{t}} f_\varepsilon(z)$ is a function of $\mathbf{t}$ independent of ω for each fixed $\mathbf{s}$. Thus,

$$\begin{aligned}
&\int_{\mathbf{t} \succeq_A \mathbf{s}} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(Z_\mathbf{s})\nu(d\mathbf{t}) \cdot 1_{\{|Z_\mathbf{s}|\le\delta\}} \\
&\qquad \ge \int_{\mathbf{t} \succeq_A \mathbf{s}} \inf_{|z|\le\delta} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(z)\nu(d\mathbf{t}) \cdot 1_{\{|Z_\mathbf{s}|\le\delta\}} \\
&\qquad \ge \left[\int_{\mathbf{t} \succeq_A \mathbf{s}} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(0)\nu(d\mathbf{t}) - D(\varepsilon)\delta\right] \cdot 1_{\{|Z_\mathbf{s}|\le\delta\}}.
\end{aligned}$$

By Lemma 3.4,
(3.10)
$$\mathcal{M}_{\mathbf{s}}^{A,f_\varepsilon,\nu} \geq \int_{\mathbf{t}\succeq_A \mathbf{s}} P_{\mathbf{s},\mathbf{t}} f_\varepsilon(Z_{\mathbf{s}})\nu(d\mathbf{t}) \cdot 1_{\{|Z_{\mathbf{s}}|\leq\delta\}}, \quad P_{\lambda_d} \text{ almost surely.}$$

Here, the P_{λ_d}-null set in (3.10) depends on $\mathbf{s}$. Thus, if $\mathbf{s}$ is random and, if we wish (3.10) to hold uniformly in ω, one way is to require $\mathbf{s}$ to take rational points only. It follows from the definition of T^δ that

$$\begin{aligned}\sup_{\theta\in\mathbf{Q}_+^{N+d}} \mathcal{M}_\theta^{A,f_\varepsilon,\nu} \geq \Bigg\{ &(2\pi)^{-d} e^{-dl} \int_{v\succeq_\pi T^\delta} \int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} \\ &\times e^{-\sum_{j=1}^N \operatorname{sgn}(v_j - T_j^\delta)[\Psi_{v_j}^j(\operatorname{sgn}(v_j-T_j^\delta)\xi) - \Psi_{T_j^\delta}^j(\operatorname{sgn}(v_j-T_j^\delta)\xi)]} \\ &\times \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi\mu(dv) - D(\varepsilon)\,\delta\Bigg\} \\ &\cdot 1_{\{T^\delta\neq\Delta,\ Z_0\in B(0,R)\}}, \quad P_{\lambda_d} \text{ almost surely.}\end{aligned}$$

We rewrite the preceding as

$$\begin{aligned}D(\varepsilon)\delta\cdot 1_{\{T^\delta\neq\Delta,\ Z_0\in B(0,R)\}} &+ \sup_{\theta\in\mathbf{Q}_+^{N+d}} \mathcal{M}_\theta^{A,f_\varepsilon,\nu} \\ \geq\ &(2\pi)^{-d} e^{-dl} \int_{v\succeq_\pi T^\delta} \int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} \\ &\times e^{-\sum_{j=1}^N \operatorname{sgn}(v_j - T_j^\delta)[\Psi_{v_j}^j(\operatorname{sgn}(v_j-T_j^\delta)\xi) - \Psi_{T_j^\delta}^j(\operatorname{sgn}(v_j-T_j^\delta)\xi)]} \\ &\times \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi\mu(dv) \\ &\cdot 1_{\{T^\delta\neq\Delta,\ Z_0\in B(0,R)\}}, \quad P_{\lambda_d} \text{almost surely.}\end{aligned}$$

Recall the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^p x_i\right)^2 \leq p\sum_{i=1}^p x_i^2 \tag{3.11}$$

for any p real numbers x_i, $i=1,\ldots p$. We denote $\mu^\delta\times\kappa$ by ν^δ. Thus, by (3.11),

$$
\begin{aligned}
&2D^2(\varepsilon)\delta^2 \cdot 1_{\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\}} + 2 \sup_{\theta \in \mathbf{Q}_+^{N+d}} (\mathcal{M}_\theta^{A, f_\varepsilon, \nu^\delta})^2 \\
&\qquad \geq (2\pi)^{-2d} e^{-2dl} \bigg\{ \int_{v \succeq_\pi T^\delta} \int_{\mathbf{R}^d} e^{-\varepsilon^2 |\xi|^2/2} \\
&\qquad \times e^{-\sum_{j=1}^N \operatorname{sgn}(v_j - T_j^\delta)[\Psi_{v_j}^j(\operatorname{sgn}(v_j - T_j^\delta)\xi) - \Psi_{T_j^\delta}^j(\operatorname{sgn}(v_j - T_j^\delta)\xi)]} \\
&\qquad\quad \times \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi \mu^\delta(dv) \bigg\}^2 \\
&\qquad \cdot 1_{\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\}}, \quad P_{\lambda_d} \text{ almost surely.}
\end{aligned}
$$

Taking E_{λ_d}- expectation on both sides of the above inequality yields

$$
\begin{aligned}
&2D^2(\varepsilon)\delta^2 \cdot P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\} + 2E_{\lambda_d} \sup_{\theta \in \mathbf{Q}_+^{N+d}} (\mathcal{M}_\theta^{A, f_\varepsilon, \nu^\delta})^2 \\
&\qquad \geq (2\pi)^{-2d} e^{-2dl} \int_{\mathbf{R}_+^N} \bigg\{ \int_{v \succeq_\pi u} \int_{\mathbf{R}^d} e^{-\varepsilon^2 |\xi|^2/2} \\
&\qquad \times e^{-\sum_{j=1}^N \operatorname{sgn}(v_j - u_j)[\Psi_{v_j}^j(\operatorname{sgn}(v_j - u_j)\xi) - \Psi_{u_j}^j(\operatorname{sgn}(v_j - u_j)\xi)]} \\
&\qquad \times \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi \mu^\delta(dv) \bigg\}^2 \mu^\delta(du) \\
&\qquad\quad \cdot P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\}.
\end{aligned}
$$

By Jensen's inequality (noticing that μ^δ is a probability measure),

$$
\begin{aligned}
&2D^2(\varepsilon)\delta^2 \cdot P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\} + 2E_{\lambda_d} \sup_{\theta \in \mathbf{Q}_+^{N+d}} (\mathcal{M}_\theta^{A, f_\varepsilon, \nu^\delta})^2 \\
&\qquad \geq (2\pi)^{-2d} e^{-2dl} \bigg\{ \int_{\mathbf{R}_+^N} \int_{v \succeq_\pi u} \int_{\mathbf{R}^d} e^{-\varepsilon^2 |\xi|^2/2} \\
&\qquad \times e^{-\sum_{j=1}^N \operatorname{sgn}(v_j - u_j)[\Psi_{v_j}^j(\operatorname{sgn}(v_j - u_j)\xi) - \Psi_{u_j}^j(\operatorname{sgn}(v_j - u_j)\xi)]} \\
&\qquad \times \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi \mu^\delta(dv) \mu^\delta(du) \bigg\}^2 \\
&\qquad\quad \cdot P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\}.
\end{aligned}
$$

Summing up the above over the 2^N partial orders A in conjunction with (3.11) and noting that for any $u \in \mathbf{R}_+^N$, $\sum_\pi \int_{v \succeq_\pi u} (\bullet)\mu^\delta(dv) = \int_{\mathbf{R}_+^N} (\bullet)\mu^\delta(dv)$ show that

$$\begin{aligned}
k_1 D^2(\varepsilon)\delta^2 \cdot P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\}& \\
+ k_2 \sum_A E_{\lambda_d} \sup_{\theta \in \mathbf{Q}_+^{N+d}} (\mathcal{M}_\theta^{A, f_\varepsilon, \nu^\delta})^2& \\
\geq (2\pi)^{-2d} e^{-2dl} \bigg\{ \int_{\mathbf{R}_+^N} \int_{\mathbf{R}_+^N} \int_{\mathbf{R}^d} e^{-\varepsilon^2 |\xi|^2/2}& \\
\times e^{-\sum_{j=1}^N \operatorname{sgn}(v_j - u_j)[\Psi_{v_j}^j(\operatorname{sgn}(v_j-u_j)\xi) - \Psi_{u_j}^j(\operatorname{sgn}(v_j-u_j)\xi)]}& \\
\times \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi \mu^\delta(dv) \mu^\delta(du) \bigg\}^2& \\
\cdot P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\},&
\end{aligned}$$

where k_1 and k_2 are two integral constants. Once more, thanks to the term $e^{-\varepsilon^2|\xi|^2/2}$, an interchange of order of integration yields

$$\begin{aligned}
k_1 D^2(\varepsilon)\delta^2 \cdot P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\}& \\
+ k_2 \sum_A E_{\lambda_d} \sup_{\theta \in \mathbf{Q}_+^{N+d}} (\mathcal{M}_\theta^{A, f_\varepsilon, \nu^\delta})^2& \\
\geq (2\pi)^{-2d} e^{-2dl} \left(\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} Q_{\mu^\delta}(\xi) \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi \right)^2& \\
\cdot P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\}.&
\end{aligned}$$

By Lemma 3.6, for any A,

$$\begin{aligned}
E_{\lambda_d} \sup_{\theta \in \mathbf{Q}_+^{N+d}} (\mathcal{M}_\theta^{A, f_\varepsilon, \nu^\delta})^2& \\
\leq 4^{N+d} (2\pi)^{-d} \int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2} Q_{\mu^\delta}(\xi) \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi.&
\end{aligned}$$

Since $Q_{\mu^\delta}(\xi)(1+|\xi|^{1-\beta/d})^{-d} \in [0,1]$,

$$\begin{aligned}
\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2} Q_{\mu^\delta}(\xi) \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi& \\
\leq \int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} Q_{\mu^\delta}(\xi) \frac{1}{(1+|\xi|^{1-\beta/d})^d}\, d\xi.&
\end{aligned}$$

We can now conclude that

$$
\begin{aligned}
(3.12)\quad & c_1\delta^2 D^2(\varepsilon)P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\} \\
&\qquad + c_2 \int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} Q_{\mu^\delta}(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi \\
&\geq c_3\left(\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} Q_{\mu^\delta}(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi\right)^2 \\
&\quad\times P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0 \in B(0,R)\},
\end{aligned}
$$

where $c_1,\ c_2,\ c_3 \in (0,\infty)$ are some constants completely independent of δ and ε.

Choose any sequence $\delta_k \downarrow 0$ as $k \to \infty$ where $k = 1,2,\ldots$. Since G^{δ_1} is bounded, there exists a probability measure μ such that along some subsequence $\delta_m \to 0$, $\mu^{\delta_m} \to \mu$ weakly. To see that μ is supported on G, we notice that G, as well as each G^δ, is compact and that $G \subset G^{\delta_{m+1}} \subset G^{\delta_m}$. Taking the indicator function 1_{G^δ} and noting that μ^{δ_m} is supported on G^{δ_m}, we can easily find a contradiction if μ has a positive mass on a compact set B with $B \cap G = \varnothing$. Next we write

$$
\begin{aligned}
&\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} Q_{\mu^{\delta_m}}(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi \\
&\qquad = \int_{\mathbf{R}^n_+}\int_{\mathbf{R}^n_+} f(s,t)\mu^{\delta_m}(ds)\mu^{\delta_m}(dt),
\end{aligned}
$$

where

$$
\begin{aligned}
f(s,t) = &\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2} e^{-\sum_{j=1}^N \operatorname{sgn}(s_j-t_j)[\Psi^j_{s_j}(\operatorname{sgn}(s_j-t_j)\xi) - \Psi^j_{t_j}(\operatorname{sgn}(s_j-t_j)\xi)]} \\
&\qquad \times \frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi.
\end{aligned}
$$

Quite clearly, $f(s,t)$ is a bounded function. Since all the exponents $\Psi^j_t(\xi)$ are jointly continuous in t and ξ, we see that $f(s,t)$ is also a continuous function. According to the approximation argument (literally the definition) from simple functions to bounded continuous functions in the weak convergence for probability measures, it also holds that $\mu^{\delta_m} \to \mu$ weakly in the double space sense:

$$
\int_{\mathbf{R}^n_+}\int_{\mathbf{R}^n_+} f(s,t)\mu^{\delta_m}(ds)\mu^{\delta_m}(dt) \longrightarrow \int_{\mathbf{R}^n_+}\int_{\mathbf{R}^n_+} f(s,t)\mu(ds)\mu(dt).
$$

In other words,

$$\lim_{m\to\infty}\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2}Q_{\mu^{\delta_m}}(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi = \int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2}Q_{\mu}(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi > 0. \tag{3.13}$$

(The integral to the right in (3.13) must be strictly positive for any probability measure μ because otherwise $\int_{\mathbf{R}^d} Q_\mu(\xi)\,d\xi = 0$ would imply $\widehat{O}_\mu = 0$.) Now rewrite (3.12) as

$$\begin{aligned} &c_1\delta^2 D^2(\varepsilon)P_{\lambda_d}\{T^\delta \neq \Delta,\ Z_0\in B(0,R)\} \\ &\quad\times\left(\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2}Q_{\mu^\delta}(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi\right)^{-2} \\ &\quad + c_2\left(\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2}Q_{\mu^\delta}(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi\right)^{-1} \\ &\quad\ge c_3 P_{\lambda_d}\{T^\delta\neq\Delta,\ Z_0\in B(0,R)\}. \end{aligned} \tag{3.14}$$

Recall that [(3.8)]

$$P_{\lambda_d}\left\{T^{\delta_m}\neq\Delta,\ Z_0\in B(0,R)\right\} \longrightarrow E\left\{\lambda_d\left(\overline{Z(G\times[0,l]^d)}\bigcap B(0,R)\right)\right\}\in(0,\infty) \tag{3.15}$$

downwards as $m\to\infty$. It follows from (3.13), (3.14) and (3.15) that

$$c_2\left(\int_{\mathbf{R}^d} e^{-\varepsilon^2|\xi|^2/2}Q_{\mu}(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi\right)^{-1} \ge c_3 E\left\{\lambda_d\left(\overline{Z(G\times[0,l]^d)}\bigcap B(0,R)\right)\right\}. \tag{3.16}$$

Finally, let $\varepsilon\to 0$ in (3.16) to obtain

$$\int_{\mathbf{R}^d} Q_\mu(\xi)\frac{1}{(1+|\xi|^{1-\beta/d})^d}\,d\xi < \infty,$$

which is the same as saying that

$$\int_{\mathbf{R}^d} |\xi|^{\beta-d}Q_\mu(\xi)\,d\xi < \infty. \qquad \square$$

Proof of Theorem 1.2. Let $\mathcal{C}_\beta$ denote the Riesz capacity. By [**2**, Theorem 7.2], for all $\beta \in (0, d)$,

$$E\mathcal{C}_\beta(X(G)) > 0 \Longleftrightarrow P\left\{X(G)\bigcap S^{1-\beta/d}((0,\infty)^d) \neq \varnothing\right\} > 0. \tag{3.17}$$

Then, by Theorem 1.1,

$$E\mathcal{C}_\beta(X(G)) > 0 \Longleftrightarrow \int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi)\, d\xi < \infty \tag{3.18}$$

for some $\mu \in \mathcal{P}(G)$. Thanks to the Frostman theorem, it remains to show that $\mathcal{C}_\beta(X(G)) > 0$ is a trivial event. Let $\mathcal{E}_\beta$ denote the Riesz energy. By Plancherel's theorem, given any $\beta \in (0, d)$, there is a constant $c_{d,\beta} \in (0, \infty)$ such that, for all probability measures ν in $\mathbf{R}^d$,

$$\mathcal{E}_\beta(\nu) = c_{d,\beta} \int_{\mathbf{R}^d} |\widehat{\nu}(\xi)|^2 |\xi|^{\beta-d} d\xi. \tag{3.19}$$

Suppose that, for some $\mu \in \mathcal{P}(G)$, $\int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi) d\xi < \infty$. By Lemma 2.2, $Q_\mu(\xi) = E|\widehat{O}_\mu(\xi)|^2$. Recall that O_μ is a probability measure supported on $X(G)$. It follows from (3.19) that

$$E\mathcal{E}_\beta(O_\mu) = c_{d,\beta} \int_{\mathbf{R}^d} |\xi|^{\beta-d} E|\widehat{O}_\mu(\xi)|^2 d\xi = c_{d,\beta} \int_{\mathbf{R}^d} |\xi|^{\beta-d} Q_\mu(\xi)\, d\xi < \infty.$$

Therefore, $\mathcal{E}_\beta(O_\mu) < \infty$ almost surely and, subsequently, $\mathcal{C}_\beta(X(G)) > 0$ almost surely. □

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