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THE SET OF SOLUTIONS OF VOLTERRA AND URYSOHN INTEGRAL EQUATIONS IN BANACH SPACES

ANETA SIKORSKA-NOWAK

ABSTRACT. We prove two existence theorems of solutions of nonlinear integral equations of Urysohn type $x(t) = \varphi(t) + \lambda \int_0^\alpha f(t, s, x(s))\, ds$ and Volterra type $x(t) = \varphi(t) + \int_0^t f(t, s, x(s))\, ds$, $t \in I_\alpha = [0, \alpha]$, $\alpha, \lambda \in R_+$, with the Henstock-Kurzweil-Pettis integral. Moreover, we show that the set S of all solutions of the Volterra integral equation is compact and connected. The assumptions about the function f are really weak: scalar measurability and weak sequential continuity with respect to the third variable. Moreover, we suppose that the function f satisfies some conditions expressed in terms of the measure of weak noncompactness.

1. Introduction. There is a large number of papers treating differential or integral problems via classical Bochner or Pettis integrals in Banach spaces. In the last two decades, when studying such problems, integrals of highly oscillating functions were taken into account. Thus, on the real line, significant results were obtained using the Henstock-Kurzweil integral ([**6, 7, 14, 15, 27**]) and then, in the general case of Banach spaces, similar problems were investigated under Henstock integrability assumptions ([**26, 35, 36, 37**]) or imposing some Henstock-Kurzweil-Pettis integrability conditions ([**10, 11**]).

Let $(E, \|\cdot\|)$ be a real Banach space, E^* its dual space and $I_\alpha = [0, \alpha]$, $\alpha \in R_+$. Moreover, let $(C(I_\alpha, E), \omega)$ denote the space of all continuous functions from I_α to E endowed with the topology $\sigma(C(I_\alpha, E), C(I_\alpha, E)^*)$.

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In the present work we will prove an existence theorem for the Urysohn integral equation:

$$x(t) = \varphi(t) + \lambda \int_0^{\alpha} f(t, s, x(s))\, ds, \quad t \in I_\alpha,\ \alpha, \lambda \in R_+, \tag{1}$$

and for the Volterra integral equation:

$$x(t) = \varphi(t) + \int_0^{t} f(t, s, x(s))\, ds, \quad t \in I_\alpha,\ \alpha \in R_+, \tag{2}$$

where $f : I_\alpha \times I_\alpha \times E \to E$, $\varphi : I_\alpha \to E$ and $x : I_\alpha \to E$ are functions with values in E and integrals are taken in the sense of Henstock-Kurzweil-Pettis ([**10**]).

Moreover, we prove, that the set S of all solutions of the Volterra integral equation on I_β, $0 < \beta \leq \alpha$, is connected and compact in $(C(I_\beta, E), \omega)$. This problem was investigated by Cichoń and Kubiaczyk [**9**], Kubiaczyk [**25**], Schwabik [**37**], Szufla [**38**] and others.

We should mention that extensive work has been done in the study of the solutions of particular cases of (1) (see, for example, [**1, 2, 3, 12, 22, 23, 30, 31, 32**]).

A Kubiaczyk fixed point theorem [**24**] and the techniques of the theory of measure of weak noncompactness are used to prove the existence of solution of problems (1) and (2). Assumptions about the function f are really weak: scalar measurability and weak sequential continuity with respect to the third variable. By using these conditions, we define a completely continuous operator F over the Banach space $(C(I_\alpha, E), \omega)$, whose fixed points are solutions of (1). The fixed point theorem of Kubiaczyk [**24**] is used to prove the existence of a fixed point of the operator F.

Let us recall that a function $f : I_\alpha \to E$ is said to be *weakly continuous* if it is continuous from I_α to E endowed with its weak topology. A function $g : E \to E_1$, where E and E_1 are Banach spaces, is said to be *weakly-weakly sequentially continuous* if for each weakly convergent sequence (x_n) in E, the sequence $(g(x_n))$ is weakly convergent in E_1. When the sequence x_n tends weakly to x_0 in E, we will write $x_n \overset{\omega}{\to} x_0$.

Our fundamental tool is the measure of weak noncompactness developed by DeBlasi [**13**].

Let A be a bounded nonempty subset of E. The *measure of weak noncompactness* $\mu(A)$ is defined by

$$\mu(A) = \inf\{t > 0 : \text{ there exists } C \in K^\omega \text{ such that } A \subset C + tB_0\},$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball in E.

We will use the following properties of the measure of weak noncompactness μ (for bounded nonempty subsets A and B of E):

(i) if $A \subset B$, then $\mu(A) \leq \mu(B)$;

(ii) $\mu(A) = \mu(\bar{A})$, where $\bar{A}$ denotes the weak closure of A;

(iii) $\mu(A) = 0$ if and only if A is relatively weakly compact;

(iv) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$;

(v) $\mu(\lambda A) = |\lambda|\mu(A)$, $(\lambda \in R)$;

(vi) $\mu(A + B) \leq \mu(A) + \mu(B)$;

(vii) $\mu(\overline{\text{conv}}\,(A)) = \mu(\text{conv}\,A) = \mu(A)$, where $\text{conv}\,(A)$ denotes the convex extension of A.

It is necessary to remark that if μ has these properties, then the following lemma is true.

Lemma 1.1 [**29**]. *Let $H \subset C(I_\alpha, E)$ be a family of strongly equicontinuous functions. Let, for $t \in I_\alpha$, $H(t) = \{h(t) \in E,\ h \in H\}$. Then $\mu(H(I_\alpha)) = \sup_{t \in I_\alpha} \mu(H(t))$ and the function $t \to \mu(H(t))$ is continuous.*

Moreover, the following holds:

Lemma 1.2 ([**9**]). *Let (X, d) be a metric space, and let $f : X \to (E, \omega)$ be sequentially continuous. If $A \subset X$ is a connected subset in X, then $f(A)$ is the connected subset in (E, ω).*

In the proof of the main result we will apply the following fixed point theorem.

Theorem 1.3 ([**24**]). *Let X be a metrizable locally convex topological vector space. Let D be a closed convex subset of X, and let F be a weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication*

(3) $\overline{V} = \overline{\text{conv}}\,(\{x\} \cup F(V)) \Longrightarrow V$ *is relatively weakly compact,*

holds for every subset V of D, then F has a fixed point.

Let us introduce the following definitions:

Definition 1.4 [**33**]. Let $G : [a,b] \to E$, and let $A \subset [a,b]$. The function $g : A \to E$ is a *pseudoderivative* of G on A if for each x^* in E^* the real-valued function x^*G is differentiable almost everywhere on A and $(x^*G)' = x^*g$ almost everywhere on A.

Definition 1.5 [**18, 28**]. A family F of functions F is said to be *uniformly absolutely continuous* in the restricted sense on X or, in short, uniformly $AC_*(X)$ if, for every $\varepsilon > 0$, there is an $\eta > 0$ such that for every F in F and for every finite or infinite sequence of nonoverlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in X$ and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F, [a_i, b_i]) < \varepsilon$, where $\omega(F, [a_i, b_i])$ denotes the oscillation of F over $[a_i, b_i]$ (i.e., $\omega(F, [a_i, b_i]) = \sup\{|F(r) - F(s)| : r, s \in [a_i, b_i]\}$).

A family F of functions F is said to be *uniformly generalized absolutely continuous* in the restricted sense on $[a,b]$ or uniformly ACG_* on $[a,b]$ if $[a,b]$ is the union of a sequence of closed sets A_i such that on each A_i, the family F is uniformly $AC_*(A_i)$.

2. Henstock-Kurzweil-Pettis integral in Banach spaces. In this part we present the Henstock-Kurzweil-Pettis integral and we give properties of this integral.

Definition 2.1 [**18,28**]. Let δ be a positive function defined on the interval $[a,b]$. A tagged interval $(x, [c,d])$ consists of an interval $[c,d] \subseteq [a,b]$ and a point $x \in [c,d]$.

The tagged interval $(x, [c,d])$ is subordinate to δ if $[c,d] \subseteq (x - \delta(x), x + \delta(x))$.

Let $P = \{(s_i, [c_i, d_i]) : 1 \le i \le n,\ n \in N\}$ be such a collection in $[a, b]$. Then

(i) The points $\{s_i : 1 \le i \le n\}$ are called the tags of P.

(ii) The intervals $\{[c_i, d_i] : 1 \le i \le n\}$ are called the intervals of P.

(iii) If $\{(s_i, [c_i, d_i]) : 1 \le i \le n\}$ is subordinate to δ for each i, then we write P is sub δ.

(iv) If $[a, b] = \cup_{i=1}^{n} [c_i, d_i]$, then P is called a tagged partition of $[a, b]$.

(v) If P is a tagged partition of $[a, b]$ and if P is sub δ, then we write P is sub δ on $[a, b]$.

(vi) If $f : [a,\ b] \to E$ then $f(P) = \sum_{i=1}^{n} f(s_i)(d_i - c_i)$.

(vii) If F is defined on the subintervals of $[a, b]$, then

$$F(P) = \sum_{i=1}^{n} F([c_i, d_i]) = \sum_{i=1}^{n} [F(d_i) - F(c_i)].$$

If $F : [a, b] \to E$, then F can be treated as a function of intervals by defining $F([c, d]) = F(d) - F(c)$. For such a function, $F(P) = F(b) - F(a)$ if P is a tagged partition of $[a, b]$.

Definition 2.2 [**18, 28**]. A function $f : [a, b] \to R$ is *Henstock-Kurzweil integrable* on $[a, b]$ if there exists a real number L with the following property: for each $\varepsilon > 0$, there exists a positive function δ on $[a, b]$ such that $|f(P) - L| < \varepsilon$ whenever P is a tagged partition of $[a, b]$ that is subordinate to δ.

The function f is *Henstock-Kurzweil integrable on a measurable set* $A \subset [a, b]$ if $f\chi_A$ is Henstock-Kurzweil integrable on $[a, b]$. The number L is called the *Henstock-Kurzweil integral of* f. We will denote this integral by $(HK) \int_a^b f(t)\, dt$.

Definition 2.3 [**5**]. A function $f : [a, b] \to E$ is *Henstock-Kurzweil integrable* on $[a, b]$ ($f \in HK([a, b], E)$) if there exists a vector $z \in E$ with the following property: for every $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $\|f(P) - z\| < \varepsilon$ whenever P is a tagged partition of $[a, b]$ sub δ. The function f is Henstock-Kurzweil integrable on a measurable set $A \subset [a, b]$ if $f\chi_A$ is Henstock-Kurzweil integrable on $[a, b]$. The vector z is the *Henstock-Kurzweil integral of* f.

We remark that this definition includes the generalized Riemann integral defined by Gordon [19].

Definition 2.4 [**5**]. A function $f : [a,b] \to E$ is *HL integrable* on $[a,b]$ ($f \in HL([a,b],E)$) if there exists a function $F : [a,b] \to E$, defined on the subintervals of $[a,b]$, satisfying the following property: given $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that if $P = \{(s_i,[c_i,d_i]) : 1 \le i \le n\}$ is a tagged partition of $[a,b]$ sub δ, then

$$\sum_{i=1}^{n} \|f(s_i)(d_i - c_i) - F([c_i,d_i])\| < \varepsilon.$$

Remark 1. We note that by triangle inequality:

$$f \in HL([a,b],E) \text{ implies } f \in HK([a,b],E).$$

In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

Definition 2.5 [33]. The function $f : I_\alpha \to E$ is *Pettis integrable* (*P* integrable for short) if

(i) $\underset{x^* \in E^*}{\forall}\ x^* f$ is Lebesgue integrable on I_α,

(ii) $\underset{\substack{A \subset I_\alpha \\ A\text{-measurable}}}{\forall}\ \underset{g \in E}{\exists}\ \underset{x^* \in E^*}{\forall}\ x^* g = (L)\int_A x^* f(s)\, ds.$

Now we present a definition of the integral which is a generalization for both: Pettis and Henstock-Kurzweil integrals.

Definition 2.6 [**11**]. The function $f : I_\alpha \to E$ is *Henstock-Kurzweil-Pettis integrable* (HKP integrable for short) if there exists a function $g : I_\alpha \to E$ with the following properties:

(i) $\underset{x^* \in E^*}{\forall}\ x^* f$ is Henstock-Kurzweil integrable on I_α and

(ii) $\underset{t \in I_\alpha}{\forall}\ \underset{x^* \in E^*}{\forall}\ x^* g(t) = (HK)\int_0^t x^* f(s)\, ds.$

This function g will be called a *primitive of* f and by $g(\alpha) = \int_0^\alpha f(t)\,dt$ we will denote the *Henstock-Kurzweil-Pettis integral* of f on the interval I_α.

Remark 2. Each function which is HL integrable is integrable in the sense of Henstock-Kurzweil-Pettis. Our notion of integral is essentially more general than the previous ones (in Banach spaces):

(i) Pettis integral: by the definition of the Pettis integral and since each Lebesgue integrable function is HK integrable, a P integrable function is clearly HKP integrable.

(ii) Bochner, Riemann, and Riemann-Pettis integrals [**19**].

(iii) MsShane integral [**17, 20**].

(iv) Henstock-Kurzweil (HL) integral [**5**].

We present below an example of function which is HKP integrable but neither HL integrable nor P integrable.

Example. Let $f : [0,1] \to (L^\infty[0,1], \|\cdot\|_\infty)$, and let $f(t) = \chi_{[0,t]} + A(t) \cdot F'(t)$, where

$$\begin{aligned} F(t) &= t^2 \sin t^{-2}, \quad t \in (0,1], \\ F(0) &= 0, \\ \chi_{[0,t]}(\tau) &= \begin{cases} 1 & \tau \in [0,t], \\ 0 & \tau \notin [0,t], \end{cases} \quad t, \tau \in [0,1], \\ A(t)(\tau) &= 1 \quad \text{for } \tau, t \in [0,1]. \end{aligned}$$

Put $f_1(t) = \chi_{[0,t]}$, $f_2(t) = A(t) \cdot F'(t)$.

We will show that a function $f(t) = f_1(t) + f_2(t)$ is integrable in the sense of Henstock-Kurzweil-Pettis.

Observe that

$$x^*(f(t)) = x^*(f_1(t) + f_2(t)) = x^*(f_1(t)) + x^*(f_2(t)).$$

Moreover, the function $x^*(f_1(t))$ is Lebesgue integrable (in fact f_1 is Pettis integrable [**16**]), so is Henstock-Kurzweil integrable, and the function $x^*(f_2(t))$ is Henstock-Kurzweil integrable by Definition 2.2.

For each $x^* \in E^*$ the function $x^* f$ is not Lebesgue integrable because $x^* f_2$ is not Lebesgue integrable. So f is not Pettis integrable. Moreover, the function f_1 is not strongly measurable [**16**] and the function f_2 is strongly measurable. So their sum f is not strongly measurable. Then by [**5**, Theorem 9] f is not HL integrable.

In the sequel we will investigate some properties of the HKP integral which are important in the next part of our paper.

Theorem 2.7 [**11**]. *Let* $f : [a,b] \to E$ *be* HKP *integrable on* $[a,b]$, *and let* $F(x) = \int_a^x f(s)\,ds$, $x \in [a,b]$. *Then*

(i) *for each* x^* *in* E^*, *the function* $x^* f$ *is* HK *integrable on* $[a,b]$ *and*

$$(HK) \int_a^x x^*(f(s))\,ds = x^*(F(x)).$$

(ii) *the function* F *is weakly continuous on* $[a,b]$ *and* f *is a pseudo-derivative of* F *on* $[a,b]$.

Theorem 2.8 [**11**]. *Let* $f : [a,b] \to E$. *If* $f = 0$ *almost everywhere on* $[a,b]$, *then* f *is* HKP *integrable on* $[a,b]$ *and* $\int_a^b f(t)\,dt = 0$.

Theorem 2.9 [**11**] (Mean value theorem for the HKP integral). *If the function* $f : I_\alpha \to E$ *is* HKP *integrable, then*

$$\int_I f(t)\,dt \in |I| \cdot \overline{\text{conv}}\, f(I),$$

where I *is an arbitrary subinterval of* I_α *and* $|I|$ *is the length of* I.

Theorem 2.10 [**8**]. *Let* $f : I_\alpha \to E$, *and assume that* $f_n : I_\alpha \to E$, $n \in N$, *are* HKP *integrable on* I_α. *Let* F_n *be a primitive of* f_n. *If we assume that*:

(i) *for all* $x^* \in E^*$, $x^*(f_n(t)) \to x^*(f(t))$ *almost everywhere on* I_α,

(ii) *for each* $x^* \in E^*$, *the family* $G = \{x^* F_n : n = 1, 2, \dots)$ *is uniformly* ACG_* *on* I_α, (*i.e., weakly uniformly* ACG_* *on* I_α),

(iii) *for each* $x^* \in E^*$ *the set* G *is equicontinuous on* I_α, *then f is* HKP *integrable on I_α and $\int_0^t f_n(s)\,ds$ tends weakly in E to $\int_0^t f(s)\,ds$ for each $t \in I_\alpha$.*

3. Existence of a solution.

I. The Urysohn integral equation. Now we prove the existence theorem for problem (1) under the weakest assumptions on f, as it is known.

For $x \in C(I_\alpha, E)$; we denote the norm of x by $\|x\|_C = \sup\{\|x(t)\|,\ t \in I_\alpha\}$.

Put

$$C(\varphi, \alpha) = \{x \in C(I_\alpha, E) : x(0) = \varphi(0), \|x\|_C \le \|\varphi\|_C + \lambda\rho\},$$

where $\varphi \in C(I_\alpha, E)$, ρ, α are some positive numbers and $\lambda \in R_+$ is from (1). This set is closed and convex.

We define the operator $F : C(I_\alpha, E) \to C(I_\alpha, E)$ by

$$F(x)(t) = \varphi(t) + \lambda \int_0^\alpha f(t, s, x(s))\,ds,$$
$$\lambda \in R_+, \quad t \in I_\alpha, \quad x \in C(\varphi, \alpha),$$

where the integral is taken in the sense of Henstock-Kurzweil-Pettis.

Moreover, let $\Gamma = \{F(x) \in C(I_\alpha, E) : x \in C(\varphi, \alpha)\}$.

Theorem 3.1. *Assume that for each ACG_* function $x : I_\alpha \to E$, $f(t, \cdot, x(\cdot))$ is* HKP *integrable, for each $t \in I_\alpha$, $f(t, s, \cdot)$ is weakly-weakly sequentially continuous and there exists a measurable function $k : I_\alpha \times I_\alpha \to R_+$ such that $k(t, \cdot)$ is continuous and*

(4)
$$\mu(f(t, I, X)) \le \sup_{s \in I} k(t, s)\,\mu(X), \text{ for each bounded } X \subset E \text{ and } I \subset I_\alpha.$$

Moreover, let $\lambda r(K) < 1$, where $\lambda \in R_+$ and $r(K)$ is a spectral radius of the integral operator K defined by

$$K(u)(t) = \int_0^\alpha k(t, s) u(s)\,ds, \quad u \in C(\varphi, \alpha), \quad t \in I_\alpha.$$

Suppose that Γ is equicontinuous and uniformly ACG_* on I_α. Then there exists at least one solution of problem (1) *on I_β, for some $0 < \beta \le \alpha$ with continuous initial function φ.*

Proof. By equicontinuity of Γ, there exists a number β, $0 < \beta \le \alpha$, such that $\| \int_0^\beta f(t,s,x(s))\,ds\| \le \rho$ for fixed $\rho > 0$, $t \in I_\beta$ and $x \in C(\varphi,\alpha)$. Recall, that a continuous function $F(x) \in \Gamma$ defined on $[0,\alpha]$ is equicontinuous on $[0,\alpha]$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|F(x)(t) - F(x)(\tau)\| < \varepsilon$ for all $x \in C(\varphi,\alpha)$ whenever $|t-\tau| < \delta$ and $t,\tau \in [0,\alpha]$. Thus, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\| \int_\tau^t \int_0^z k(z,s,x(s))\,ds)\,dz\| < \varepsilon$ for all $x \in \widetilde{B}$ whenever $|t-\tau| < \delta$ and $t,\tau \in [0,a]$. As a result, there exists a number d, $0 < d \le a$, such that

$$\left\| \int_0^t f\Big(z, x(z), \int_0^z k(z,s,x(s))\,ds\Big)\,dz \right\| \le b, \quad t \in I_d \text{ and } x \in \widetilde{B}.$$

By our assumptions, the operator F is well defined and maps $C(\varphi,\beta)$ into $C(\varphi,\beta)$.

$$\begin{aligned} \|F(x)(t)\| &= \left\| \varphi(t) + \lambda \int_0^\beta f(t,s,x(s))\,ds \right\| \\ &\le \|\varphi(t)\|_C + \lambda \left\| \int_0^\beta f(t,s,x(s))\,ds \right\| \\ &\le \|\varphi(t)\|_C + \lambda\rho. \end{aligned}$$

Now we will show that the operator F is weakly sequentially continuous.

By [**29**, Lemma 9] a sequence $x_n(\cdot)$ is weakly convergent in $(C(I_\beta,E),\omega)$ to $x(\cdot)$ if and only if $x_n(t)$ tends weakly to $x(t)$ for each $t \in I_\beta$. Because $f(t,s,\cdot)$ is weakly-weakly sequentially continuous so if $x_n \stackrel{\omega}{\to} x$ in $(C(I_\beta,E),\omega)$, then $f(t,s,x_n(s)) \to\stackrel{\omega}{\to} f(t,s,x(s))$ in E for $t \in I_\beta$ and by Theorem 2.10, we have

$$\lim_{n\to\infty} \int_0^\beta f(t,s,x_n(s))\,ds = \int_0^\beta f(t,s,x(s))\,ds$$

weakly in E, for each $t \in I_\beta$. We see that $F(x_n)(t) \to F(x)(t)$ weakly in E for each $t \in I_\beta$ so [**29**, Lemma 9] guarantees that $F(x_n) \to F(x)$ in $(C(I_\beta,E),\omega)$.

Suppose that $V \subset C(\varphi,\beta)$ satisfies the condition $\overline{V} = \overline{\text{conv}}\,(\{x\} \cup F(V))$ for some $x \in C(\varphi,\beta)$. We will prove that V is relatively weakly compact in $C(\varphi,\beta)$ and so (3) is satisfied.

For $t \in I_\beta$, let $V(t) = \{v(t) \in E,\ v \in V\}$. Since $V \subset C(\varphi,\beta)$, $F(V) \subset \Gamma$. Then $V \subset \overline{V} = \overline{\text{conv}}\,(\{x\} \cup F(V))$ is equicontinuous. By Lemma 1.1, $t \mapsto v(t) = \mu(V(t))$ is continuous on I_β.

We divide the interval $[0,\beta]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = \beta$, where $t_i = (i\beta/m)$, $i = 0, 1, \dots, m-1$. Let $V([t_i, t_{i+1}]) = \{u(s) \in E : u \in V,\ t_i \le s \le t_{i+1}\} \subset E$, $i = 0, \dots, m-1$. By Lemma 1.1 and the continuity of v there exists an $s_i \in T_i = [t_i, t_{i+1}]$, such that

$$\mu(V(T_i)) = \sup\{\mu(V(s)) : s \in T_i\} =: v(s_i). \tag{5}$$

Let $f(t, T_i, V(T_i)) = \{f(t,s,u(s)) \in E : u \in V, s \in T_i\}$.

By the definition of the operator F and Theorem 2.9, we obtain:

$$\begin{aligned} F(u)(t) = \varphi(t) + \lambda \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f(t,s,u(s))\,ds \in \varphi(t) \\ + \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) \overline{\text{conv}}\,(f(t, T_i, V(T_i))), \end{aligned}$$

for each $u \in V$.

Therefore,

$$F(V)(t) \subset \varphi(t) + \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i)\, \overline{\text{conv}}\, f\,(t, T_i, V(T_i)).$$

Using (4), (5) and the properties of the measure of weak noncompactness μ, we obtain

$$\begin{aligned} \mu(F(V)(t)) &\le \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) \sup_{s \in T_i} k(t,s) \mu(V(T_i)) \\ &= \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i)\, k(t, p_i) v(s_i), \end{aligned}$$

where $s_i, p_i \in T_i$; hence,

$$\begin{aligned}\mu(F(V)(t)) &\leq \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) k(t, p_i) v(p_i) \\ &\quad + \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k(t, p_i)(v(s_i) - v(p_i))] \\ &= \lambda \sum_{i=0}^{m-1} (t_{i+1} - t_i) k(t, p_i)\, v(p_i) \\ &\quad + \lambda \frac{\beta}{m} \sum_{i=0}^{m-1} k(t, p_i)(v(s_i) - v(p_i)).\end{aligned}$$

Fix $\varepsilon > 0$. From the continuity of v we may choose m large enough so that $v(s_i) - v(p_i) < \varepsilon$ and so

$$\mu(F(V)(t)) < \lambda \int_0^\beta k(t, s) v(s)\, ds + \lambda\beta \sup_{p \in I_\beta} k(t, p) \varepsilon.$$

Since $\varepsilon \to 0$ and $\lambda\beta \sup_{p \in I_\beta} k(t, p)$ is bounded, we have $\lambda\beta \sup_{p \in I_\beta} k(t, p)\, \varepsilon \to 0$.

Therefore,

$$\mu(F(V)(t)) \leq \lambda \int_0^\beta k(t, s) v(s)\, ds \text{ for } t \in I_\beta. \tag{6}$$

Since $V = \overline{\mathrm{conv}}\,(\{x\} \cup F(V))$ we have $\mu(V(t)) \leq \mu(F(V)(t))$ and so, in view of (6), it follows that $v(t) \leq \lambda \int_0^\beta k(t, s) v(s)\, ds$ for $t \in I_\beta$.

Because this inequality holds for every $t \in I_\beta$ and $\lambda r(K) < 1$, so applying Gronwall's inequality, we get that $\mu(V(t)) = 0$ for $t \in I_\beta$. Hence, Arzela-Ascoli's theorem proves that the set V is relatively weakly compact. Consequently, by Theorem 1.3, F has a fixed point which is a solution of the problem (1).

II. The Volterra integral equation. Now, we consider the integral equation (2).

Put

$$C_1(\varphi,\alpha)=\{x\in C(I_\alpha,E):\|x\|_C\le\|\varphi\|_C+\sigma\},$$

where $\varphi\in C(I_\alpha,E)$ and σ,α are some positive numbers. This set is closed and convex.

Let $G:C(I_\alpha,E)\to C(I_\alpha,E)$ be defined by

$$G(x)(t)=\varphi(t)+\int_0^t f(t,s,x(s))\,ds,\quad t\in I_\alpha,\quad x\in C_1(\varphi,\alpha),$$

where the integral is taken in the sense of Henstock-Kurzweil-Pettis.

Moreover, let $\Lambda=\{G(x)\in C(I_\alpha,E):x\in C_1(\varphi,\alpha)\}$.

Theorem 3.2. *Assume that for each ACG_* function $x:I_\alpha\to E$, $f(t,\cdot,x(\cdot))$ is* HKP *integrable for each $t\in I_\alpha$. Let $f(t,s,\cdot)$ be weakly-weakly sequentially continuous, and there exists a measurable function $k_1:I_\alpha\times I_\alpha\to R_+$ such that $k_1(t,\cdot)$ is continuous and*

(7)

$$\mu(f(t,I,X))\le\sup_{s\in I}k_1(t,s)\mu(X)\ \text{for each bounded } X\subset E \text{ and } I\subset I_\alpha.$$

Moreover, let $r(K)<1$, where $r(K)$ is a spectral radius of the integral operator K defined by

$$K(u)(t)=\int_0^t k_1(t,s)u(s)\,ds,\quad t\in I_\alpha,\quad u\in C_1(\varphi,\alpha).$$

Suppose that Λ is equicontinuous and uniformly ACG_* on I_α. Then there exists a solution of problem (2) *for some $\beta, 0<\beta\le\alpha$ with the initial continuous function φ.*

Proof. By equicontinuity of Λ there exists some number β, $0<\beta\le\alpha$, such that

$$\left\|\int_0^t f(t,s,x(s))\,ds\right\|\le\sigma \text{ for fixed } \sigma>0,\quad t\in I_\beta,\quad x\in C_1(\varphi,\beta).$$

Let $V\subset C_1(\varphi,\beta)$ satisfy the condition $\overline{V}=\overline{\text{conv}}\,(\{x\}\cup G(V))$ for some $x\in C_1(\varphi,\beta)$. Analogously to the proof of Theorem 3.1, we divide the interval $[0,t]$, for fixed $t\in I_\beta$ into m parts:

$$0=t_0<t_1<\cdots<t_m=t,\ \text{where } t_i=\frac{it}{m},\quad i=0,1,\ldots,m,$$

and we prove that

$$v(t) \le \int_0^t k_1(t,s)v(s)\,ds, \quad t \in I_\beta.$$

Because this inequality holds for every $t \in I_\beta$ and $r(K) < 1$, so applying Gronwall's inequality, we get $\mu(V(t)) = 0$ for $t \in I_\beta$. Hence, Arzela-Ascoli's theorem proves that the set V is relatively weakly compact. Consequently, by Theorem 1.3, G has a fixed point which is a solution of the problem (2).

4. Compactness and connectedness. In this part we show that the set S of all solutions of the Volterra integral equation on I_β is compact and connected in $(C(I_\beta, E), \omega)$.

Theorem 4.1. *Under the assumptions of Theorem* 3.2 *the set S of all solutions of the Volterra integral equation* (2) *is compact and connected in $(C(I_\beta, E), \omega)$.*

Proof. Let S be a set of all solutions of the problem (2) on I_β. As $S = G(S)$, by repeating the above argument, with $V = S$ we can show that S is relatively compact in $(C(I_\beta, E), \omega)$. Since G is weakly continuous on $\overline{S(I_\beta)^\omega}$, S is weakly closed and consequently weakly compact.

Now we prove that S is connected.

For any $\eta > 0$, denote by S_η, the set of all functions $u : I_\beta \to E$ satisfying the following conditions:

$$u(0) = \varphi(0), u \in C_1(\varphi, \alpha), \tag{8}$$

$$\sup_{t \in I_\beta} \left\| u(t) - \varphi(t) - \int_0^t f(t,s,u(s))\,ds \right\| < \eta. \tag{9}$$

The set S_η is nonempty as $S \subset S_\eta$.

By equicontinuity of Λ we can choose ρ such that

$$\left\| \int_J f(t,s,x(s))\,ds \right\| \le \eta^* < \eta, \tag{10}$$

for any $x \in (C(I_\beta, E), \omega)$, $J \subset I_\beta$, $|J| < \rho$.

For any $\varepsilon \in (0,\beta)$, $0 < \beta \le \alpha$, let $v(\cdot,\varepsilon) : I_\beta \to E$ be defined by the formula:

$$v(t,\varepsilon) = \begin{cases} \varphi(t) & \text{for } 0 \le t \le \varepsilon \\ \varphi(t) + \int_0^{t-\varepsilon} f(t,s,v(s,\varepsilon))\,ds & \text{for } \varepsilon < t \le \beta. \end{cases}$$

Clearly $v(\cdot,\varepsilon)$ satisfies (8).

Furthermore, for $0 < \varepsilon \le \min(\rho,\beta) = d$, we have

$$\left\| v(t,\varepsilon) - \varphi(t) - \int_0^t f(t,s,v(s,\varepsilon))\,ds \right\| = \begin{cases} \left\| \int_0^t f(t,s,v(s,\varepsilon))\,ds \right\| & \text{for } 0 \le t \le \varepsilon \\ \left\| \int_{t-\varepsilon}^t f(t,s,v(s,\varepsilon))\,ds \right\| & \text{for } \varepsilon < t \le \beta. \end{cases} \le \eta^* < \eta;$$

thus, $v(\cdot,\varepsilon)$ satisfies (9).

Now, we will prove that S_η is connected. Define

$$v_\varepsilon(t) = \begin{cases} \varphi(t) & 0 \le t \le \varepsilon \\ G(v_\varepsilon)(t-\varepsilon) & \varepsilon < t \le \beta, \end{cases}$$

where $v_\varepsilon = v(\cdot,\varepsilon)$. We will show that the mapping $\varepsilon \to v_\varepsilon(\cdot)$ is sequentially continuous from $(0,\beta)$ into $(C(I_\beta,E),\omega)$.

Let $0 < \varepsilon < \delta \le \beta$ (when $\delta \le \varepsilon$ the argument is similar).

By the definition of $v_\varepsilon(t)$, for $t \in [0,\varepsilon]$, we have

$$|x^*(v_\varepsilon(t) - v_\delta(t))| = 0. \tag{11}$$

Next, if $t \in (\varepsilon,\delta]$, we have

$$\begin{aligned} |x^*(v_\varepsilon(t) - v_\delta(t))| &= \left| x^* \left[\int_0^{t-\varepsilon} f(t,s,v_\varepsilon(s))\,ds - \int_0^{t-\delta} f(t,s,v_\delta(s))\,ds \right] \right| \\ &= \left| x^* \int_{t-\delta}^{t-\varepsilon} f(t,s,v_\varepsilon(s))\,ds \right| \\ &= \|x^*\| \left\| \int_{t-\delta}^{t-\varepsilon} f(t,s,v_\varepsilon(s))\,ds \right\| \end{aligned}$$

Consequently,

$$(12)\qquad |x^*(v_\varepsilon(t)-v_\delta(t))| \le \left\| \int_{t-\delta}^{t-\varepsilon} f(t,s,v_\varepsilon(s))\,ds \right\| := A_\delta.$$

Because Λ is equicontinuous, so if $\delta \to \varepsilon$, then $A_\delta \to 0$.

Now, for $t \in (\delta, 2\delta]$, we have

$$\begin{aligned}
&|x^*(v_\varepsilon(t)-v_\delta(t))| \\
&\qquad = \left| x^*\left[\int_0^{t-\varepsilon} f(t,s,v_\varepsilon(s))\,ds - \int_0^{t-\delta} f(t,s,v_\delta(s))\,ds \right] \right| \\
&\qquad = |x^*(G(v_\varepsilon)(t-\varepsilon) - G(v_\delta)(t-\delta))| \\
&\qquad = |x^*[G(v_\varepsilon)(t-\varepsilon) - G(v_\varepsilon)(t-\delta)\ . \\
&\qquad\quad + G(v_\varepsilon)(t-\delta) - G(v_\delta)(t-\delta)]| \\
&\qquad \le |x^*(G(v_\varepsilon)(t-\varepsilon) - G(v_\varepsilon)(t-\delta))| \\
&\qquad\quad + |x^*(G(v_\varepsilon)(t-\delta) - G(v_\delta)(t-\delta))|\,.
\end{aligned}$$

So

$$(13)\qquad \begin{aligned} |x^*(v_\varepsilon(t)-v_\delta(t))| &\le \|x^*\|\,\|G(v_\varepsilon)(t-\varepsilon) - G(v_\varepsilon)(t-\delta)\| \\ &\quad + \|x^*\|\,\|G(v_\varepsilon)(t-\delta) - G(v_\delta)(t-\delta)\| \end{aligned}$$

Let (δ_n) be a sequence such that $\delta_n \to \epsilon$ $(\delta_n \ge \epsilon)$.

By (11) and (12), it follows that $v_{\delta_n}(t)$ converges weakly to $v_\varepsilon(t)$, uniformly for $t \in [0,\delta]$. So $G(v_{\delta_n})(t) \to G(v_\varepsilon)(t)$ weakly on $[0,\delta]$. Now, by (13), $v_{\delta_n}(t)$ tends to $v_\varepsilon(t)$ weakly for each $t \in [0, 2\delta]$.

By repeating the above argument and using induction, we obtain that the map $\varepsilon \to v_\varepsilon(\cdot)$ from $(0,\beta)$ into $(C(I_\beta,E),\omega)$ is sequentially continuous [**29**, Lemma 1.9]. Therefore, by Lemma 1.2 the set $V = \{v_\varepsilon(\cdot) : 0 < \varepsilon < \beta\}$ is connected in $(C(I_\beta,E),\omega)$.

Let $x \in S_\eta$. Choose $\varepsilon > 0$ such that $0 < \varepsilon < \beta$ and

$$\sup_{t\in I_\beta} \left\| x(t) - \varphi(t) - \int_0^t f(t,s,x(s))\,ds \right\| + \left\| \int_{I_\varepsilon} f(t,s,x(s))\,ds \right\| < \eta.$$

For any p, $0 \le p \le \beta$, let $y(\cdot,p) : I_\beta \to E$ be defined by the formula:

$$y(t,p) = \begin{cases} x(t) & \text{for } 0 \le t \le p, \\ x(p) + \frac{\varphi(t)-x(p)}{\varepsilon}(t-p) & \text{for } p < t \le \min(\beta, p+\varepsilon), \\ \varphi(t) + \int_p^{t-\varepsilon} f(t,s,y(s,p))\,ds & \text{for } \min(\beta,p+\varepsilon) < t < \beta, \\ v(t,\varepsilon) & \text{for } p = 0. \end{cases}$$

By repeating the above considerations, with $y(\cdot,p)$ in the place of $v(\cdot,\varepsilon)$, one can show that $y(\cdot,p) \in S_\eta$ for each $p \in [0,\beta]$ and that the mapping $p \to y(\cdot,p)$ from I_β into $(C(I_\beta,E),\omega)$ is sequentially continuous (for more details, see [**25, 38**]).

Consequently, by Lemma 1.2, the set $T_x = \{y(\cdot,p) : 0 \le p \le \beta\}$ is connected in $(C(I_\beta,E),\omega)$.

As $y(\cdot,0) = v(\cdot,\varepsilon) \in V \cap T_x$, the set $V \cup T_x$ is connected and therefore the set

$$W = \bigcup_{x \in S_\eta} T_x \cup V$$

is connected in $(C(I_\beta,E),\omega)$.

Moreover, $S_\eta \subset W$ because $x = y(\cdot,p) \in T_x$, for each $x \in S_\eta$. On the other hand, $W \subset S_\eta$, since $T_x \subset S_\eta$ and $V \subset S_\eta$. Finally $S_\eta = W$ is a connected subset of $(C(I_\beta,E),\omega)$.

Suppose that the set S is not connected. As S is weakly compact, there exist nonempty weakly compact sets W_1 and W_2 such that $S = W_1 \cup W_2$ and $S = W_1 \cup W_2$. Consequently, there exist two disjoint weakly open sets U_1, U_2 such that $W_1 \cap W_2 = \varnothing$, $W_2 \subset U_2$. Suppose that, for every $n \in N$, there exists a $u_n \in V_n \backslash U$, where $V_n = \overline{S_{1/n}^{\omega}}$ and $U = U_1 \cup U_2$.

Put $H = \overline{\{u_n : n \in N\}^{\omega}}$. Since $u_n - G(u_n) \to 0$ in $(C(I_\beta,E),\omega)$ as $n \to \infty$ and $H(t) \subset \{u_n(t) - G(u_n)(t) : u_n \in H\} + G(H)(t)$. By repeating the argument from the proof of Theorem 3.1, one can show that there exists a $u_0 \in H$ such that $u_0 = G(u_0)$, i.e., $u_0 \in S$.

Furthermore, $S \subset (C(I_\beta,E),\omega)\backslash U$, since U is weakly open and hence $u_0 \in S\backslash U$, a contradiction.

Therefore, there is an $m \in N$ such that $V_m \subset U$. Since $U_1 \cap V_m \ne \varnothing \ne U_2 \cap V_m$, V_m is not connected, a contradiction with the connectedness of each V_n. Consequently, S is connected in $(C(I_\beta,E),\omega)$.

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Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland
Email address: anetas@amu.edu.pl