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ON THE NUMBER OF p-GONAL COVERINGS OF RIEMANN SURFACES

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ABSTRACT. A classical Castelnuovo-Severi theorem says that, for $g > (p-1)^2$, a cyclic p-gonal covering of a Riemann surface of genus g is unique. Here we deal with the case $g \leq (p-1)^2$ for a prime p. We find some bounds for the number of such coverings in terms of g and p, and we derive from them bounds depending only on p and even an absolute bound equal to 30. We also show that a Riemann surface of genus $g \geq 2$ having less than $3(g-1)$ automorphisms admits at most one cyclic p-gonal covering.

1. Introduction. A compact Riemann surface X of genus $g \geq 2$ which can be realized as a p-sheeted ramified covering of the Riemann sphere is called p-gonal. If there is an automorphism φ of X of order p, which permutes the sheets, then X is said to be *cyclic p-gonal* and the corresponding covering is a *cyclic p-gonal covering*. In this way such coverings are in a one-to-one correspondence with subgroups of order p of the group of conformal automorphisms of X, for which the orbit space is the Riemann sphere. Here we shall deal with the classification of such coverings in terms of these groups, to which we shall refer as *p-gonality automorphism groups*. From a Castelnuovo-Severi theorem [**2**] it follows that, for $g > (p-1)^2$, such a group is unique. On the other hand, González-Diez [**4**] showed (see also [**5**] for an alternative proof) that for a prime p a cyclic p-gonal Riemann surface of genus $g \leq (p-1)^2$ has one conjugacy class of p-gonality automorphism groups. Here we find some bounds for the number of such groups in terms of g and p, and we derive from them bounds depending only on p and even an absolute bound equal to 30. We also show that a Riemann surface of genus g having less than $3(g-1)$ automorphisms admits at most one cyclic p-gonal covering.

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We shall use combinatorial methods based on the Riemann uniformization theorem and combinatorial theory of Fuchsian groups as in [**6**], where the reader can find necessary notions and facts. The principal tool is a theorem of Macbeath on fixed points from [**7**].

2. On fixed points of automorphism of Riemann surfaces. We start with the following obvious consequence of the Riemann Hurwitz ramification formula

Lemma 2.1. *A Riemann surface* $X = \mathcal{H}/\Gamma$ *is cyclic* p*-gonal for a prime* p *if and only if there exists a Fuchsian group with signature* $(0; p, \overset{r}{\ldots}, p)$*, where* $r = 2(g+p-1)/(p-1)$*, containing* Γ *as a normal subgroup of index* p.

Observe that a p-gonality automorphism of a Riemann surface of genus g has $2(g+p-1)/(p-1)$ fixed points. On the other hand there is a theorem of Macbeath [**7**] concerning fixed points of automorphisms of Riemann surfaces which we shall use in the sequel.

Theorem 2.2. *Let* $X = \mathcal{H}/\Gamma$ *be a Riemann surface with automorphism group* $G = \Lambda/\Gamma$*, and let* $x_1, \ldots, x_r$ *be a set of elliptic canonical generators of* Λ *whose periods are* $m_1, \ldots, m_r$ *respectively. Let* $\theta : \Lambda \to G$ *be the canonical projection. Then the number* $\mathrm{F}(g)$ *of points of* X *fixed by* g *is given by the formula*

$$\mathrm{F}(g) = |\mathrm{N}_G(\langle g \rangle)| \sum 1/m_i,$$

where N *denotes the normalizer and the sum is taken over those* i *for which* g *is conjugate to a power of* $\theta(x_i)$.

3. On p**-gonality automorphisms of Riemann surfaces.** As we mentioned before a cyclic p-gonal Riemann surface of genus $g > (p-1)^2$ admits exactly one cyclic p-gonal ramified covering. Here we deal with the case of arbitrary g for a prime p.

Theorem 3.1. *A Riemann surface of genus* $g \geq 2$ *admits at most* $6(g-1)(p-1)/(g+p-1)(p-6)$, $16(g-1)/(g+4)$ *and* $28(g-1)/(g+2)$

cyclic p-gonal ramified coverings for a prime p, $p \geq 7$, $p = 5$ and $p = 3$, respectively.

Proof. Since there is a one-to-one correspondence between cyclic p-gonal ramified coverings and p-gonality automorphism groups, it is enough to show that the values from Theorem 2.2 are bound above the number of such groups. Let X be a cyclic p-gonal Riemann surface of genus $g \leq (p-1)^2$, and let $\langle\sigma\rangle$ be a p-gonality automorphism group of X. By the Riemann uniformization theorem, $X = \mathcal{H}/\Gamma$ and $G = \mathrm{Aut}\,(X) = \Lambda/\Gamma$ for some Fuchsian groups Γ and Λ with signatures $(g; -)$ and $(h; m_1, \dots, m_r)$, respectively. Let $|G| = N$, and let n be the number of p-gonality automorphism groups of X. Then, as all of them are conjugate [**4**], $n = [G : \mathrm{N}_G(\langle\sigma\rangle)]$ and so by Theorem 2.2, every period of Λ produces at most N/np fixed points of σ and therefore, in particular,

$$2(g+p-1)/(p-1) \leq sN/np, \tag{1}$$

where s is the number of periods which are multiples of p.

Now, for $h \neq 0$, $\mu(\Lambda) \geq 2\pi s(p-1)/p$, where $\mu(\Lambda)$ is the hyperbolic area of a fundamental domain of the group Λ, and so by (1) and by the Hurwitz Riemann formula $2(g-1) \geq 2n(g+p-1)$, gives

$$n \leq (g-1)/(g+p-1). \tag{2}$$

So let $h = 0$. But then $r \geq 3$.

First, let $r \geq 4$. Clearly $s \geq 1$ and since $(0; 2, \overset{r-s}{\dots}, 2, p, \overset{s}{\dots}, p)$ is the signature of a Fuchsian group with a minimal area in this case, we have

$$\begin{aligned} \mu(\Lambda) &\geq 2\pi(-2 + (r-s)/2 + s(p-1)/p) \\ &\geq 2\pi(-2 + (4-s)/2 + s(p-1)/p) \\ &= \pi s(p-2)/p. \end{aligned}$$

So, by the Hurwitz Riemann formula and by (1), $2(g-1) \geq n(p-2)(g+p-1)/(p-1)$ gives

$$n \leq 2(g-1)(p-1)/(g+p-1)(p-2). \tag{3}$$

Now let $r = 3$. Observe that, since $\mu(\Lambda) > 0$, $1/m_1 + 1/m_2 + 1/m_3 < 1$, and consider first the case $p \geq 5$. If $s = 3$, then $\mu(\Lambda) \geq 2\pi(p-3)/p$

since $(0; p, p, p)$ is the signature of a Fuchsian group with the minimal area in this case. So, by the Hurwitz Riemann formula and by (1), $g-1 \geq n(g+p-1)(p-3)/3(p-1)$, which gives

$$n \leq 3(g-1)(p-1)/(g+p-1)(p-3). \tag{4}$$

If $s=2$, then since $(0; 2, p, p)$ is the signature of a Fuchsian group with the minimal area in this case, $\mu(\Lambda) \geq \pi(p-4)/p$. So, by the Hurwitz Riemann formula and by (1), $4(g-1) \geq n(g+p-1)(p-4)/3(p-1)$ gives

$$n \leq 4(g-1)(p-1)/(g+p-1)(p-4). \tag{5}$$

If $s=1$ and $p \geq 7$, then a Fuchsian group with signature $(0; 2, 3, p)$ has the minimal possible area and so $\mu(\Lambda) \geq \pi(p-6)/3p$, and so the Hurwitz Riemann formula and (1) give

$$n \leq 6(g-1)(p-1)/(g+p-1)(p-6). \tag{6}$$

If $s=1$ and $p=5$, then a Fuchsian group with signature $(0; 2, 4, 5)$ has the minimal possible area. So $\mu(\Lambda) \geq \pi/10$, and thus the Hurwitz Riemann formula and (1) give

$$n \leq 16(g-1)/(g+4). \tag{7}$$

So, finally, let $r=p=3$. If $s=3$, then a Fuchsian group with signature $(0; 3, 3, 6)$ has the minimal possible area. So $\mu(\Lambda) \geq \pi/3$, and hence the Hurwitz Riemann formula and (1) give

$$n \leq 12(g-1)/(g+2). \tag{8}$$

If $s=2$, then a Fuchsian group with signature $(0; 2, 3, 9)$ has the minimal possible area. So $\mu(\Lambda) \geq \pi/9$; therefore, the Hurwitz Riemann formula and (1) give

$$n \leq 24(g-1)/(g+2). \tag{9}$$

Finally, if $s=1$, then a Fuchsian group with signature $(0; 2, 3, 7)$ has the minimal possible area. So $\mu(\Lambda) \geq \pi/2$; thus, the Hurwitz Riemann formula and (1) give

$$n \leq 28(g-1)/(g+2). \qquad (10)$$

So, comparing inequalities (2)–(10) we obtain the proof. □

Now since $(g-1)/(g+p-1)$ is an increasing function of g, we obtain from the above theorem

Corollary 3.2. *A Riemann surface of genus $g \geq 2$ admits at most $6(p-2)/(p-6)$, 12 and 14 cyclic p-gonal ramified coverings for $p \geq 7$, $p = 5$ and $p = 3$, respectively.*

And even more:

Corollary 3.3. *A Riemann surface of genus $g \geq 2$ admits at most 30 cyclic p-gonal ramified coverings.*

Corollary 3.4. *A Riemann surface X of genus $g \geq 2$ having less than $3(g-1)$ automorphisms admits at most one cyclic p-gonal ramified covering.*

Proof. Here $\mu(\Lambda) > 4\pi/3$, by the Hurwitz Riemann formula, and so either $h \neq 0$ or $r \geq 5$. However, in the first case X does not admit cyclic p-gonal covering by (2). For $r \geq 5$, $\mu(\Lambda) \geq \pi(s(p-2)+p)/p$, since $(0; 2, \overset{5-s}{\ldots}, 2, p, \overset{s}{\ldots}, p)$ is the signature of a Fuchsian group with a minimal area in this case. So (1) and the Hurwitz Riemann formula give $n \leq 2s(p-1)(g-1)/(s(p-2)+p)(g+p-1)$ which clearly is smaller than 2. □

Remark 3.5. Our bounds do not seem to be sharp for arithmetically admissible p and g. For example, the signature $(2,3,7)$ which gives the bound in our theorem for $p = 3$ is not admissible as was remarked in [**3**] and so a cyclic trigonal Riemann surface has actually strictly less than 14 p-gonality automorphism groups. Our method gives a way to find such precise bounds; however, implementing it in practice seems to be a rather difficult technical problem involving finite group theory.

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