

ON THE SPACE OF ORIENTED GEODESICS OF HYPERBOLIC 3-SPACE

NIKOS GEORGIIOU AND BRENDAN GUILFOYLE

ABSTRACT. We construct a Kähler structure $(\mathbf{J}, \Omega, \mathbf{G})$ on the space $\mathbf{L}(\mathbf{H}^3)$ of oriented geodesics of hyperbolic 3-space \mathbf{H}^3 and investigate its properties. We prove that $(\mathbf{L}(\mathbf{H}^3), \mathbf{J})$ is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 - \overline{\Delta}$, where $\overline{\Delta}$ is the reflected diagonal, and that the Kähler metric \mathbf{G} is of neutral signature, conformally flat and scalar flat. We establish that the identity component of the isometry group of the metric \mathbf{G} on $\mathbf{L}(\mathbf{H}^3)$ is isomorphic to the identity component of the hyperbolic isometry group. Finally, we show that the geodesics of \mathbf{G} correspond to ruled minimal surfaces in \mathbf{H}^3 , which are totally geodesic if and only if the geodesics are null.

1. Introduction. The space $\mathbf{L}(\mathbf{M}^3)$ of oriented geodesics of a 3-manifold \mathbf{M}^3 of constant curvature is a 4-dimensional manifold which carries a natural complex structure \mathbf{J} . In the case where \mathbf{M}^3 is an Euclidean 3-space \mathbf{E}^3 , this complex structure can be traced back to Weierstrass [13] and Whittaker [14], with its modern reemergence occurring in Hitchin's study of monopoles on \mathbf{E}^3 [5].

More recently, this structure has been supplemented by a compatible symplectic structure, so that $\mathbf{L}(\mathbf{M}^3)$ inherits a natural Kähler structure. This has been investigated when $\mathbf{M}^3 = \mathbf{E}^3$ and $\mathbf{M}^3 = \mathbf{E}_1^3$ [2, 3, 4], and the purpose of this paper is to study the hyperbolic 3-space case $\mathbf{M}^3 = \mathbf{H}^3$.

From a topological point of view, $\mathbf{L}(\mathbf{M}^3)$ is homeomorphic to $\mathbf{S}^2 \times \mathbf{S}^2 - \Delta$, where Δ is the diagonal. However, from holomorphic point of view we show that:

Theorem 1. *The complex surface $(\mathbf{L}(\mathbf{H}^3), \mathbf{J})$ is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 - \overline{\Delta}$, where $\overline{\Delta}$ is the reflected diagonal (see Definition 2).*

2010 AMS *Mathematics subject classification.* Primary 51M09, Secondary 51M30.

Keywords and phrases. Kaehler structure, hyperbolic 3-space, isometry group.
Received by the editors on November 6, 2007, and in revised form on March 3, 2008.

DOI:10.1216/RMJ-2010-40-4-1183 Copyright ©2010 Rocky Mountain Mathematics Consortium

The \mathbf{P}^1 in Theorem 1 refers to the boundary of the Poincaré ball model of \mathbf{H}^3 , considered as the past and future infinities of the oriented geodesics, from which $\mathbf{L}(\mathbf{H}^3)$ inherits its complex structure.

We then turn to the Kähler metric \mathbf{G} and prove:

Theorem 2. *The Kähler metric \mathbf{G} is of neutral signature, conformally flat and scalar flat.*

We also show that, despite the $(++--)$ signature, this metric on $\mathbf{L}(\mathbf{H}^3)$ faithfully reflects the hyperbolic metric g on \mathbf{H}^3 , in the following sense:

Theorem 3. *The identity component of the isometry group of the metric \mathbf{G} on $\mathbf{L}(\mathbf{H}^3)$ is isomorphic to the identity component of the hyperbolic isometry group.*

A curve in $\mathbf{L}(\mathbf{H}^3)$ is a 1-parameter family of oriented geodesics in \mathbf{H}^3 : a ruled surface. Our final result characterizes the ruled surfaces that arise as geodesics in $\mathbf{L}(\mathbf{H}^3)$:

Theorem 4. *The geodesics of the Kähler metric \mathbf{G} are generated by the 1-parameter subgroups of the isometry group of \mathbf{G} .*

A ruled surface generated by a geodesic of \mathbf{G} is a minimal surface in \mathbf{H}^3 , and the geodesic is null if and only if the ruled surface is totally geodesic.

In the next section we describe the space of oriented geodesics of hyperbolic 3-space from a topological and a differentiable point of view, using the ball and upper half-space models of \mathbf{H}^3 . In Section 2 we define and investigate the Kähler structure on $\mathbf{L}(\mathbf{H}^3)$ and prove Theorems 1 and 2. The proof of Theorem 3, which is contained in Section 3, consists of a number of steps, formulated as propositions. We first find the Killing vectors of \mathbf{G} . We then compute the action induced on $\mathbf{L}(\mathbf{H}^3)$ by isometries of \mathbf{H}^3 and prove that the infinitesimal generators of this action coincide precisely with the Killing vectors of \mathbf{G} . Finally, we

study the geodesics of the neutral Kähler metric and the ruled surfaces they generate in \mathbf{H}^3 in Section 4.

The uniqueness of this Kähler structure has recently been established by Salvai [11] (cf. [10] for the \mathbf{E}^n case). Indeed, a number of our results overlap with those of Salvai, who utilizes techniques of Lie groups to obtain his results. Our approach is particularly geared to the study of surfaces in $\mathbf{L}(\mathbf{H}^3)$ [1] and, given recent interest in flat and CMC surfaces in \mathbf{H}^3 [7, 9, 12], we hope that this line of inquiry will prove fruitful.

1. The space of oriented geodesics of \mathbf{H}^3 .

Definition 1. Let $\mathbf{L}(\mathbf{H}^3)$ be the space of oriented geodesics in \mathbf{H}^3 .

The topology of $\mathbf{L}(\mathbf{H}^3)$ is most easily seen using the Poincaré ball model. This has underlying space

$$B^3 = \{(y^1, y^2, y^3) \in \mathbf{R}^3 \mid (y^1)^2 + (y^2)^2 + (y^3)^2 < 1\},$$

for standard coordinates (y^1, y^2, y^3) on \mathbf{R}^3 , with hyperbolic metric

$$ds^2 = \frac{4[(dy^1)^2 + (dy^2)^2 + (dy^3)^2]}{[1 - (y^1)^2 - (y^2)^2 - (y^3)^2]^2}.$$

Definition 2. Let $\iota : S^2 \rightarrow S^2$ be the antipodal map and define the *reflected diagonal* by

$$\bar{\Delta} = \{(\mu_1, \mu_2) \in S^2 \times S^2 \mid \mu_1 = \iota(\mu_2)\}.$$

We can now identify the space of oriented geodesics of \mathbf{H}^3 :

Proposition 1. *The space $\mathbf{L}(\mathbf{H}^3)$ of oriented geodesics of hyperbolic 3-space is homeomorphic to $S^2 \times S^2 - \bar{\Delta}$.*

Proof. Consider the unit ball model of \mathbf{H}^3 . In this model the geodesics are either diameters, or circles which are asymptotically orthogonal to

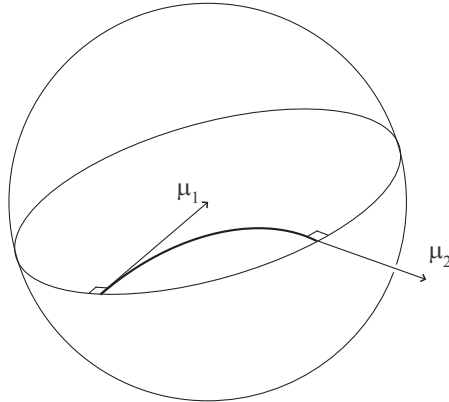


FIGURE 1. Hyperbolic geodesics in the ball model.

the boundary 2-sphere. An oriented geodesic can thus be uniquely identified by its beginning and end point on the boundary. Moreover, any ordered pair of points on the boundary 2-sphere define a unique oriented geodesic, as long as the points are distinct. Thus, the space of oriented geodesics is homeomorphic to $S^2 \times S^2 - \{\text{diag}\}$.

In fact, for geometric reasons which will become clear below, we will identify an oriented geodesic by the direction of its tangent vector at past and future infinity, see Figure 1. Since these directions are inward and outward pointing (respectively), we see that the oriented geodesics can also be identified with $S^2 \times S^2$ minus antipodal directions, as claimed. \square

For computational purposes we use the upper half-space model of \mathbf{H}^3 . Thus, the underlying space is

$$\mathbf{R}_+^3 = \{(x^0, x^1, x^2) \in \mathbf{R}^3 \mid x^0 > 0\},$$

for standard coordinates (x^0, x^1, x^2) on \mathbf{R}^3 . In these coordinates the hyperbolic metric g has expression:

$$d\tilde{s}^2 = \frac{(dx^0)^2 + (dx^1)^2 + (dx^2)^2}{(x^0)^2}.$$

This is related to the ball model by the mapping $P: \mathbf{R}_+^3 \rightarrow B^3 : (x^0, x^1, x^2) \mapsto (y^1, y^2, y^3)$ defined by

$$(1.1) \quad \begin{aligned} y^1 &= \frac{2x^1}{(x^0 + 1)^2 + (x^1)^2 + (x^2)^2}, \\ y^2 &= \frac{2x^2}{(x^0 + 1)^2 + (x^1)^2 + (x^2)^2}, \\ y^3 &= \frac{(x^0)^2 + (x^1)^2 + (x^2)^2 - 1}{(x^0 + 1)^2 + (x^1)^2 + (x^2)^2}. \end{aligned}$$

The map P is a diffeomorphism. In fact, it is an isometry: $P^*(ds^2) = d\tilde{s}^2$ [8].

We now describe the geodesics of \mathbf{H}^3 in this model.

Proposition 2. *The geodesics of \mathbf{H}^3 that are not parallel to the x^0 -axis are*

$$\begin{aligned} x^0 &= \frac{1}{\sqrt{c_1^2 + c_2^2} \cosh(r + r_0)}, \\ x^1 &= \frac{c_1}{c_1^2 + c_2^2} \tanh(r + r_0) + c_3, \\ x^2 &= \frac{c_2}{c_1^2 + c_2^2} \tanh(r + r_0) + c_4, \end{aligned}$$

where r is the arclength of the geodesic, $c_1, c_2, c_3, c_4, r_0 \in \mathbf{R}$ and c_1, c_2 are not both 0. Geodesics parallel to the x^0 -axis are given by

$$x^0(r) = e^{\pm r+r_0} \quad x^1(r) = c_3 \quad x^2(r) = c_4,$$

for $c_3, c_4, r_0 \in \mathbf{R}$.

Proof. Let (x^0, x^1, x^2) be local coordinates on the space \mathbf{H}^3 with metric g as defined above. The only nonvanishing Christoffel symbols of the metric are

$$\Gamma_{00}^0 = \Gamma_{01}^1 = \Gamma_{02}^2 = -\frac{1}{x^0}, \quad \Gamma_{11}^0 = \Gamma_{22}^0 = \frac{1}{x^0}.$$

The geodesic equations (using the summation convention here and throughout)

$$\frac{d^2x^k}{dr^2} + \Gamma_{ij}^k \frac{dx^i}{dr} \frac{dx^j}{dr} = 0, \quad k = 0, 1, 2,$$

then turn out to be:

$$\begin{aligned} \frac{d^2 x^0}{dr^2} + \frac{1}{x^0} \left[\left(\frac{dx^1}{dr} \right)^2 + \left(\frac{dx^2}{dr} \right)^2 - \left(\frac{dx^0}{dr} \right)^2 \right] &= 0, \\ \frac{d^2 x^1}{dr^2} - \frac{2}{x^0} \frac{dx^1}{dr} \frac{dx^0}{dr} &= 0, \quad \frac{d^2 x^2}{dr^2} - \frac{2}{x^0} \frac{dx^2}{dr} \frac{dx^0}{dr} = 0, \end{aligned}$$

where r is an affine parameter along the geodesic.

These can be integrated to yield the first integrals:

$$I_1 = \frac{(\dot{x}^0)^2 + (\dot{x}^1)^2 + (\dot{x}^2)^2}{(x^0)^2}, \quad I_2 = \frac{2\dot{x}^2}{(x^0)^2}, \quad I_3 = \frac{2\dot{x}^1}{(x^0)^2},$$

where the dot denotes differentiation with respect to r . Thus, I_1, I_2, I_3 are constant along any geodesic.

By parameterizing the geodesic by arc-length we can set $I_1 = 1$. Let $I_2 = 2c_1$ and $I_3 = 2c_2$, so that

$$(1.2) \quad (\dot{x}^0)^2 + (\dot{x}^1)^2 + (\dot{x}^2)^2 = (x^0)^2, \quad \dot{x}^1 = c_1(x^0)^2, \quad \dot{x}^2 = c_2(x^0)^2.$$

Combining these equations, we then get

$$c_1^2(x^0)^4 + c_2^2(x^0)^4 + (\dot{x}^0)^2 = (x^0)^2,$$

or, rearranging:

$$(\dot{x}^0)^2 = (x^0)^2 - (c_1^2 + c_2^2)(x^0)^4.$$

Integrating, we have that:

$$(1.3) \quad \int \frac{dx^0}{x^0 \sqrt{1 - K(x^0)^2}} = r + r_0 \quad K = c_1^2 + c_2^2.$$

For $c_1 \neq 0$ or $c_2 \neq 0$ we find that

$$\begin{aligned} \int \frac{dx^0}{x^0 \sqrt{1 - K(x^0)^2}} &= \frac{1}{2} \log \left| \frac{1 - \sqrt{1 - K(x^0)^2}}{1 + \sqrt{1 - K(x^0)^2}} \right| \\ &= \frac{1}{2} \log \left(\frac{1 - \sqrt{1 - K(x^0)^2}}{1 + \sqrt{1 - K(x^0)^2}} \right). \end{aligned}$$

So we have

$$1 - \sqrt{1 - K(x^0)^2} = e^{2(r+r_0)}(1 + \sqrt{1 - K(x^0)^2}),$$

and hence

$$x^0 = \frac{1}{\sqrt{c_1^2 + c_2^2} \cosh(r + r_0)},$$

as claimed.

Now, from the second of equation (1.2),

$$\dot{x}^1 = c_1(x^0)^2,$$

we obtain that

$$\dot{x}^1 = \frac{c_1}{(c_1^2 + c_2^2) \cosh^2(r + r_0)}.$$

Similarly we obtain

$$\dot{x}^2 = \frac{c_2}{(c_1^2 + c_2^2) \cosh^2(r + r_0)},$$

and integrating we finally get

$$x^1(r) = \frac{c_1}{c_1^2 + c_2^2} \tanh(r + r_0) + c_3, \quad x^2(r) = \frac{c_2}{c_1^2 + c_2^2} \tanh(r + r_0) + c_4,$$

as claimed.

The case $c_1 = c_2 = 0$ follows easily by integration of (1.3). □

We see that the geodesics in \mathbf{H}^3 , where c_1, c_2 are not both 0, are semi-circles in 3-space \mathbf{R}^3 with center $(0, c_3, c_4)$ and radius $(c_1^2 + c_2^2)^{-1/2}$. If we let

$$\xi = c_1 + ic_2, \quad \eta = c_3 + ic_4,$$

then the geodesics can be labeled as shown in Figure 2.

In fact, initially we will define a Kähler structure only on an open subset of $\mathbf{L}(\mathbf{H}^3)$. We define this subset as follows.

Definition 3. Let $\mathbf{U} \subset \mathbf{L}(\mathbf{H}^3)$ be the set of oriented geodesics in the upper-half space model that are not parallel to the x^0 -axis.

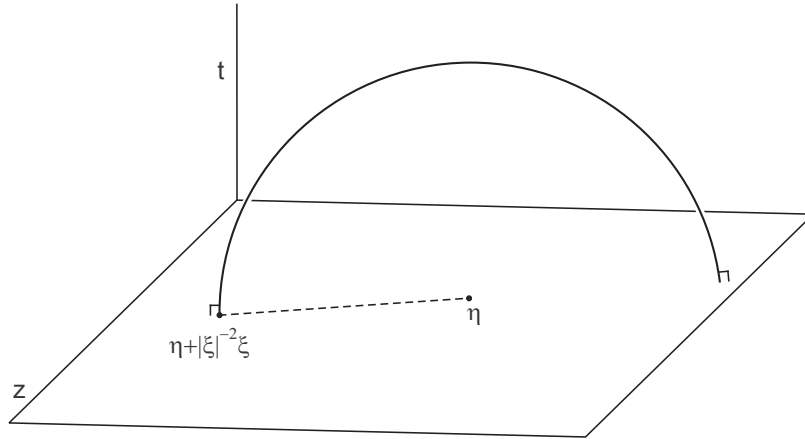


FIGURE 2. Hyperbolic geodesics in the upper half-space model.

Introducing complex coordinates $t = x^0, z = x^1 + ix^2$, so that

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} \right), \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial x^0},$$

the metric tensor g becomes

$$ds^2 = \frac{dzd\bar{z} + dt^2}{t^2}.$$

From Proposition 2 the points of \mathbf{U} are the oriented geodesics in \mathbf{H}^3 that are given in terms of the constants ξ and η by

$$(1.4) \quad t = \frac{1}{|\xi| \cosh(r + r_0)}, \quad z = \eta + \frac{\tanh(r + r_0)}{\bar{\xi}},$$

for $\xi \in \mathbf{C} - \{0\}, \eta \in \mathbf{C}$ and $r_0 \in \mathbf{R}$.

Thus, (ξ, η) are local coordinates on $\mathbf{U} \subset \mathbf{L}(\mathbf{H}^3)$ and by setting $r_0 = 0$ we fix the parameterization on these geodesics. In the next section we define a Kähler structure on \mathbf{U} and extend it to all of $\mathbf{L}(\mathbf{H}^3)$. First,

we must explicitly identify the tangent space \mathbf{TU} with the orthogonal Jacobi fields along the associated geodesics in \mathbf{H}^3 [6].

Definition 4. Let $\Phi : \mathbf{U} \times \mathbf{R} \rightarrow \mathbf{H}^3$ be the map $\Phi(\xi, \eta, r) = (t(\xi, \eta, r), z(\xi, \eta, r))$ given by equations (1.4) with $r_0 = 0$.

For later use we note that:

Proposition 3. The derivative $D\Phi : T_{(\xi, \eta, r)}(\mathbf{U} \times \mathbf{R}) \rightarrow T_{\Phi(\xi, \eta, r)}(\mathbf{H}^3)$ is given by

$$\begin{aligned}
 D\Phi_{(\xi, \eta, r)} \left(\frac{\partial}{\partial \xi} \right) &= -\frac{\tanh r}{\xi^2} \frac{\partial}{\partial \bar{z}} - \frac{1}{2\xi|\xi| \cosh r} \frac{\partial}{\partial t}, \\
 D\Phi_{(\xi, \eta, r)} \left(\frac{\partial}{\partial \eta} \right) &= \frac{\partial}{\partial z}, \\
 D\Phi_{(\xi, \eta, r)} \left(\frac{\partial}{\partial r} \right) &= \frac{1}{\bar{\xi} \cosh^2 r} \frac{\partial}{\partial z} + \frac{1}{\xi \cosh^2 r} \frac{\partial}{\partial \bar{z}} - \frac{\sinh r}{|\xi| \cosh^2 r} \frac{\partial}{\partial t}.
 \end{aligned}$$

Proof. This is found by differentiation of Φ . \square

Definition 5. A null frame at a point p in \mathbf{H}^3 is a trio of vectors $e_{(0)}, e_{(+)}, e_{(-)} \in \mathbf{C} \otimes T_p \mathbf{H}^3$ such that:

$$\begin{aligned}
 e_{(0)} &= \overline{e_{(0)}}, & e_{(+)} &= \overline{e_{(-)}}, \\
 g(e_{(0)}, e_{(0)}) &= g(e_{(+)}, e_{(-)}) = 1, & g(e_{(0)}, e_{(+)}) &= 0,
 \end{aligned}$$

where the hyperbolic inner product g is extended bilinearly over \mathbf{C} .

Given an oriented geodesic in \mathbf{H}^3 , an adapted null frame is a null frame along the geodesic such that $e_{(0)}$ is the tangent to the geodesic.

Proposition 4. An adapted null frame to the oriented geodesic $(\xi, \eta) \in \mathbf{U}$ is given by

$$\begin{aligned}
 e_{(0)} &= \frac{1}{\bar{\xi} \cosh^2 r} \frac{\partial}{\partial z} + \frac{1}{\xi \cosh^2 r} \frac{\partial}{\partial \bar{z}} - \frac{\sinh r}{|\xi| \cosh^2 r} \frac{\partial}{\partial t}, \\
 e_{(+)} &= \frac{1}{\sqrt{2} \cosh^2 r} \left[-\frac{e^{-r}}{\bar{\xi}} \frac{\partial}{\partial z} + \frac{e^r}{\xi} \frac{\partial}{\partial \bar{z}} \right] + \frac{1}{\sqrt{2}|\xi| \cosh^2 r} \frac{\partial}{\partial t}.
 \end{aligned}$$

Proof. We note that $e_{(0)} = D\Phi_{(\xi,\eta,r)}(\partial/\partial r)$ and a straightforward computation gives

$$g(e_{(\alpha)}, e_{(\beta)}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \alpha, \beta = 0, +, -.$$

Thus, we have an adapted null frame as claimed. \square

Note. An orthonormal frame along the geodesic (ξ, η) is given by

$$\begin{aligned} e_{(0)} &= -\frac{1}{\sqrt{c_1^2 + c_2^2}} \frac{\sinh r}{\cosh^2 r} \frac{\partial}{\partial x^0} \\ &\quad + \frac{c_1}{c_1^2 + c_2^2} \frac{1}{\cosh^2 r} \frac{\partial}{\partial x^1} + \frac{c_2}{c_1^2 + c_2^2} \frac{1}{\cosh^2 r} \frac{\partial}{\partial x^2}, \\ e_{(1)} &= \frac{1}{\sqrt{c_1^2 + c_2^2}} \frac{1}{\cosh^2 r} \frac{\partial}{\partial x^0} + \frac{c_1}{c_1^2 + c_2^2} \frac{\sinh r}{\cosh^2 r} \frac{\partial}{\partial x^1} + \frac{c_2}{c_1^2 + c_2^2} \frac{\sinh r}{\cosh^2 r} \frac{\partial}{\partial x^2}, \\ e_{(2)} &= -\frac{c_2}{c_1^2 + c_2^2} \frac{1}{\cosh r} \frac{\partial}{\partial x^1} + \frac{c_1}{c_1^2 + c_2^2} \frac{1}{\cosh r} \frac{\partial}{\partial x^2}. \end{aligned}$$

In particular,

$$e_{(+)} = \frac{1}{\sqrt{2}} \{e_{(1)} + ie_{(2)}\}.$$

Proposition 5. *The inverse of the mapping of Proposition 4 is*

$$\begin{aligned} \frac{\partial}{\partial t} &= -|\xi| \sinh r e_{(0)} + \frac{|\xi|}{\sqrt{2}} e_{(+)} + \frac{|\xi|}{\sqrt{2}} e_{(-)}, \\ \frac{\partial}{\partial z} &= \frac{\bar{\xi}}{2} e_{(0)} - \frac{\bar{\xi} e^{-r}}{2\sqrt{2}} e_{(+)} + \frac{\bar{\xi} e^r}{2\sqrt{2}} e_{(-)}. \end{aligned}$$

Proof. From Proposition 4 we have the linear system

$$\begin{bmatrix} e_{(0)} \\ e_{(+)} \\ e_{(-)} \end{bmatrix} = A \begin{bmatrix} \partial/\partial z \\ \partial/\partial \bar{z} \\ \partial/\partial t \end{bmatrix},$$

where

$$A = \frac{1}{\sqrt{2}\xi\bar{\xi} \cosh^2 r} \begin{bmatrix} \xi\sqrt{2} & \bar{\xi}\sqrt{2} & -\sqrt{2}|\xi| \sinh r \\ -\xi e^{-r} & \bar{\xi} e^r & |\xi| \\ \xi e^r & -\bar{\xi} e^{-r} & |\xi| \end{bmatrix}.$$

So the inverse of A is

$$A^{-1} = \frac{1}{2\sqrt{2}} \begin{bmatrix} \bar{\xi}\sqrt{2} & -\bar{\xi} e^{-r} & \bar{\xi} e^r \\ \xi\sqrt{2} & \xi e^r & -\xi e^{-r} \\ -2\sqrt{2}|\xi| \sinh r & 2|\xi| & 2|\xi| \end{bmatrix}.$$

The result follows. \square

Definition 6. Let $\gamma : \mathbf{R} \rightarrow \mathbf{H}^3$ be an oriented geodesic. A *Jacobi field* along a geodesic describes an infinitesimal variation of the geodesic. In \mathbf{H}^3 this is given by a vector field X along γ satisfying the equation

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X - X = 0.$$

The solutions of this equation form a 6-dimensional vector space, which we denote by $\mathcal{J}(\gamma)$. Let $\mathcal{J}^\perp(\gamma)$ be the 4-dimensional vector space of Jacobi fields that are orthogonal to γ .

Definition 7. For an oriented geodesic γ in \mathbf{H}^3 , let $\text{Pr}_\gamma : T_\gamma \mathbf{H}^3 \rightarrow T_\gamma \mathbf{H}^3$ be the projection onto the plane orthogonal to the geodesic.

Let $h : T_\gamma \mathbf{U} \rightarrow T_\gamma \mathbf{H}^3$ be defined by $h = \text{Pr}_\gamma \circ D\Phi$.

Proposition 6. *The map h has local coordinate description on \mathbf{U}*

$$\begin{aligned} h_{(\xi,\eta,r)} \left(\frac{\partial}{\partial \xi} \right) &= -\frac{e^r}{2\sqrt{2}\xi} e_{(+)} - \frac{e^{-r}}{2\sqrt{2}\xi} e_{(-)}, \\ h_{(\xi,\eta,r)} \left(\frac{\partial}{\partial \eta} \right) &= -\frac{\bar{\xi} e^{-r}}{2\sqrt{2}} e_{(+)} + \frac{\bar{\xi} e^r}{2\sqrt{2}} e_{(-)}. \end{aligned}$$

Proof. From Propositions 3 and 5 we get

$$\begin{aligned} D\Phi_{(\xi,\eta,r)} \left(\frac{\partial}{\partial \xi} \right) &= -\frac{e^r}{2\sqrt{2}\xi} e_{(+)} - \frac{e^{-r}}{2\sqrt{2}\xi} e_{(-)}, \\ D\Phi_{(\xi,\eta,r)} \left(\frac{\partial}{\partial \eta} \right) &= \frac{\bar{\xi}}{2} e_{(0)} - \frac{\bar{\xi} e^{-r}}{2\sqrt{2}} e_{(+)} + \frac{\bar{\xi} e^r}{2\sqrt{2}} e_{(-)}, \end{aligned}$$

and therefore the projection of $D\Phi_{(\xi,\eta,r)}(\partial/\partial\xi)$ and $D\Phi_{(\xi,\eta,r)}(\partial/\partial\eta)$ gives the expressions of the proposition. \square

The next proposition shows that the tangent space $T_{(\xi,\eta)}\mathbf{U}$ can be identified with the orthogonal Jacobi fields along the geodesic (ξ, η) .

Proposition 7. *The map h is a vector space isomorphism between $T_\gamma\mathbf{U}$ and $\mathcal{J}^\perp(\gamma)$.*

Proof. First we show that $e_{(+)}$ is a parallel vector field along the geodesic, that is,

$$\nabla_{e_{(0)}}e_{(+)} = 0.$$

To do this we compute the nonvanishing covariant derivatives:

$$\begin{aligned} \nabla_{\partial/\partial t}\frac{\partial}{\partial t} &= -\frac{1}{2}|\xi|\cosh r\frac{\partial}{\partial t}, & \nabla_{\partial/\partial z}\frac{\partial}{\partial \bar{z}} &= \frac{1}{2}|\xi|\cosh r\frac{\partial}{\partial t}, \\ \nabla_{\partial/\partial t}\frac{\partial}{\partial z} &= -\frac{1}{2}|\xi|\cosh r\frac{\partial}{\partial z}, & \nabla_{\partial/\partial t}\frac{\partial}{\partial \bar{z}} &= -\frac{1}{2}|\xi|\cosh r\frac{\partial}{\partial \bar{z}}. \end{aligned}$$

Thus,

$$\begin{aligned} \nabla_{e_{(0)}}\frac{\partial}{\partial z} &= \tanh r\frac{\partial}{\partial z} + \frac{|\xi|}{2\xi\cosh r}\frac{\partial}{\partial t}, \\ \nabla_{e_{(0)}}\frac{\partial}{\partial \bar{z}} &= \tanh r\frac{\partial}{\partial \bar{z}} + \frac{|\xi|}{2\bar{\xi}\cosh r}\frac{\partial}{\partial t}, \\ \nabla_{e_{(0)}}\frac{\partial}{\partial t} &= -\frac{|\xi|}{\xi\cosh r}\frac{\partial}{\partial z} - \frac{|\xi|}{\xi\cosh r}\frac{\partial}{\partial \bar{z}} + \tanh r\frac{\partial}{\partial t}, \end{aligned}$$

and finally we find that

$$\nabla_{e_{(0)}}e_{(+)} = 0.$$

Now, since the frames are parallel, for $\alpha = -, +$ and any function $f(r)$,

$$\begin{aligned} \nabla_{e_{(0)}}(fe_{(\alpha)}) &= \frac{df}{dr}e_{(\alpha)} + f\nabla_{e_{(0)}}e_{(\alpha)} = \frac{df}{dr}e_{(\alpha)}, \\ \nabla_{e_{(0)}}^2(fe_{(\alpha)}) &= \nabla_{e_{(0)}}\left(\frac{df}{dr}e_{(\alpha)}\right) = \frac{d^2f}{dr^2}e_{(\alpha)} + \frac{df}{dr}\nabla_{e_{(0)}}e_{(\alpha)} \\ &= \frac{d^2f}{dr^2}e_{(\alpha)}. \end{aligned}$$

Now we are ready to prove our claim. Since the frame is adapted, we have that $\dot{\gamma} = e_{(0)}$ and

$$\begin{aligned} \nabla_{e_{(0)}}^2 h\left(\frac{\partial}{\partial \xi}\right) &= \nabla_{e_{(0)}}^2 \left(-\frac{e^r}{2\sqrt{2}\xi} e_{(+)} - \frac{e^{-r}}{2\sqrt{2}\xi} e_{(-)} \right), \\ &= \frac{d^2}{dr^2} \left(-\frac{e^r}{2\sqrt{2}\xi} \right) e_{(+)} + \frac{d^2}{dr^2} \left(-\frac{e^{-r}}{2\sqrt{2}\xi} \right) e_{(-)}, \\ &= -\frac{e^r}{2\sqrt{2}\xi} e_{(+)} - \frac{e^{-r}}{2\sqrt{2}\xi} e_{(-)}, \\ &= h\left(\frac{\partial}{\partial \xi}\right). \end{aligned}$$

Thus $h(\partial/\partial \xi)$ is a Jacobi field along the geodesic. Similarly $h(\partial/\partial \eta)$ can be shown to be a Jacobi field along the geodesic. Moreover, these vector fields span the space of orthogonal Jacobi fields along the geodesic. \square

2. The Kähler structure on $\mathbf{L}(\mathbf{H}^3)$. A Kähler structure on a 4-manifold \mathbf{M} is a triple $(\mathbf{J}, \Omega, \mathbf{G})$, where \mathbf{J} is a complex structure, Ω is a symplectic 2-form and \mathbf{G} is an inner product. These are required to satisfy the conditions:

$$\Omega(\mathbf{J}\cdot, \mathbf{J}\cdot) = \Omega(\cdot, \cdot), \quad \mathbf{G}(\cdot, \cdot) = \Omega(\mathbf{J}\cdot, \cdot).$$

We now construct a Kähler structure on $\mathbf{L}(\mathbf{H}^3)$. We first define the structure on the open subset $\mathbf{U} \subset \mathbf{L}(\mathbf{H}^3)$ and then show that it extends to the whole space.

Definition 8. Given an oriented geodesic γ in \mathbf{H}^3 , let $\mathcal{R}_\gamma : T_\gamma \mathbf{H}^3 \rightarrow T_\gamma \mathbf{H}^3$ be rotation through 90° about the tangent vector to the geodesic.

Because we are working in a space of constant curvature this rotation preserves Jacobi fields:

Proposition 8 [5]. *The map \mathcal{R}_γ takes \mathcal{J}^\perp to \mathcal{J}^\perp .*

We now define our complex structure:

Definition 9. Let $\mathbf{J} : T_\gamma \mathbf{U} \rightarrow T_\gamma \mathbf{U}$ be defined to be $\mathbf{J} = h^{-1} \circ \mathcal{R}_\gamma \circ h$.

It is clear that $\mathbf{J}^2 = -\text{Id}$, so that we have an almost complex structure. In order to be a complex structure \mathbf{J} must also satisfy a certain integrability condition. This is equivalent to the existence of holomorphic coordinates, which we demonstrate below.

Proposition 9. *The following two vectors form an eigenbasis for \mathbf{J} at (ξ, η) :*

$$\bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}} + \frac{\partial}{\partial \eta}, \quad \xi^2 \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \bar{\eta}}.$$

Proof. Note that $\mathcal{R}_\gamma(e_{(+)}) = ie_{(+)}$, and thus we have

$$\mathbf{J} \left(\bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}} + \frac{\partial}{\partial \eta} \right) = h^{-1} \circ \mathcal{R}_\gamma \left(-\frac{\bar{\xi}e^{-r}}{\sqrt{2}} e_{(+)} \right) = i \left(\bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}} + \frac{\partial}{\partial \eta} \right).$$

Similarly, we have

$$\mathbf{J} \left(\xi^2 \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \bar{\eta}} \right) = i \left(\xi^2 \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \bar{\eta}} \right),$$

as claimed. \square

We introduce holomorphic coordinates on \mathbf{U} as follows:

Definition 10. Define $(\mu_1, \mu_2) \in \mathbf{C}^2$ by

$$\mu_1 = -\eta + \frac{1}{\bar{\xi}}, \quad \mu_2 = \frac{1}{\bar{\eta} + (1/\xi)},$$

the inverse relation being

$$\eta = -\frac{1}{2} \left(\mu_1 - \frac{1}{\bar{\mu}_2} \right), \quad \xi = \frac{2}{\bar{\mu}_1 + (1/\mu_2)}.$$

We are now in a position to prove one of our main results:

Theorem 1. *The almost complex structure \mathbf{J} extends to the whole of $\mathbf{L}(\mathbf{H}^3)$ and is integrable. Moreover, the complex surface $(\mathbf{L}(\mathbf{H}^3), \mathbf{J})$ is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 - \bar{\Delta}$.*

Proof. We first note that (μ_1, μ_2) are holomorphic coordinates on \mathbf{U} .
Indeed,

$$\begin{aligned} \frac{\partial}{\partial \mu_1} &= -\frac{1}{2} \left[\frac{2\bar{\mu}_2}{(\mu_1\bar{\mu}_2 + 1)} \right]^2 \frac{\partial}{\partial \bar{\xi}} - \frac{1}{2} \frac{\partial}{\partial \eta} \\ &= -\frac{1}{2} \left(\bar{\xi}^2 \frac{\partial}{\partial \bar{\xi}} + \frac{\partial}{\partial \eta} \right), \\ \frac{\partial}{\partial \mu_2} &= \frac{1}{2\mu_2^2} \left[\left(\frac{2\mu_2}{\bar{\mu}_1\mu_2 + 1} \right)^2 \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \bar{\eta}} \right] \\ &= \frac{1}{2} \left(\bar{\eta} + \frac{1}{\xi} \right)^2 \left(\xi^2 \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \bar{\eta}} \right), \end{aligned}$$

and hence we find that

$$\mathbf{J} \left(\frac{\partial}{\partial \mu_1} \right) = i \frac{\partial}{\partial \mu_1}, \quad \mathbf{J} \left(\frac{\partial}{\partial \mu_2} \right) = i \frac{\partial}{\partial \mu_2},$$

so we have holomorphic coordinates for \mathbf{J} and therefore the almost complex structure is integrable.

These coordinates come from the past and future boundaries of the oriented geodesic in the ball model. To see this explicitly, introduce coordinates (w, ρ) on the 3-ball by $w = y^1 + iy^2$ and $\rho = y^3$. The isometry P given by equations (1.1) becomes $P(z, t) = (w, \rho)$, with

$$w = \frac{2z}{z\bar{z} + (t+1)^2}, \quad \rho = 1 - \frac{2(t+1)}{z\bar{z} + (t+1)^2}.$$

Mapping a geodesic $\gamma \in \mathbf{U}$ to the Poincaré ball we get

$$\begin{aligned} w &= \frac{2[(\eta + (1/\bar{\xi}))e^{2r} + \eta - (1/\bar{\xi})]}{(\eta\bar{\eta} + (1/\xi\bar{\xi}) + (\eta/\xi) + (\bar{\eta}/\bar{\xi}) + 1)e^{2r} + (4/|\xi|)e^r + (\eta\bar{\eta} + (1/\xi\bar{\xi}) - (\eta/\xi) - (\bar{\eta}/\bar{\xi}) + 1)}, \\ \rho &= 1 - \frac{2(e^{2r} + (2/|\xi|)e^r + 1)}{(\eta\bar{\eta} + (1/\xi\bar{\xi}) + (\eta/\xi) + (\bar{\eta}/\bar{\xi}) + 1)e^{2r} + (4/|\xi|)e^r + (\eta\bar{\eta} + (1/\xi\bar{\xi}) - (\eta/\xi) - (\bar{\eta}/\bar{\xi}) + 1)}. \end{aligned}$$

Define

$$w_{\pm} = \lim_{r \rightarrow \pm\infty} w(r), \quad \rho_{\pm} = \lim_{r \rightarrow \pm\infty} \rho(r),$$

and find

$$\begin{aligned} w_+ &= \frac{2(\eta + (1/\bar{\xi}))}{\eta\bar{\eta} + (1/\xi\bar{\xi}) + (\eta/\xi) + (\bar{\eta}/\bar{\xi}) + 1} = \frac{2\mu_2}{\mu_2\bar{\mu}_2 + 1}, \\ \rho_+ &= 1 - \frac{2}{\eta\bar{\eta} + (1/\xi\bar{\xi}) + (\eta/\xi) + (\bar{\eta}/\bar{\xi}) + 1} = \frac{1 - \mu_2\bar{\mu}_2}{1 + \mu_2\bar{\mu}_2}, \\ w_- &= \frac{2(\eta - (1/\bar{\xi}))}{\eta\bar{\eta} + (1/\xi\bar{\xi}) - (\eta/\xi) - (\bar{\eta}/\bar{\xi}) + 1} = -\frac{2\mu_1}{\mu_1\bar{\mu}_1 + 1}, \\ \rho_- &= 1 - \frac{2}{\eta\bar{\eta} + (1/\xi\bar{\xi}) - (\eta/\xi) - (\bar{\eta}/\bar{\xi}) + 1} = \frac{\mu_1\bar{\mu}_1 - 1}{\mu_1\bar{\mu}_1 + 1}. \end{aligned}$$

We see that $w_{\pm}\bar{w}_{\pm} + \rho_{\pm}^2 = 1$ and so $(w_{\pm}, \rho_{\pm}) \in \partial B^3 = S^2$, as expected. In fact, μ_2 is the holomorphic coordinate obtained on S^2 by stereographic projection from the south pole, while μ_1 is the anti-holomorphic coordinate (the composition of stereographic projection and the antipodal map ι).

It is now clear that the definition of \mathbf{J} as rotation of Jacobi fields about the oriented geodesic extends to all of $\mathbf{L}(\mathbf{H}^3)$ and the theorem follows. \square

We now define the symplectic structure on \mathbf{U} .

Definition 11. Let $\mathbf{X}, \mathbf{Y} \in T_{\gamma}\mathbf{U}$. Define a 2-form on \mathbf{U} by

$$\Omega(\mathbf{X}, \mathbf{Y}) = g(h(\mathbf{X}), \nabla_{\dot{\gamma}}(h(\mathbf{Y}))) - g(h(\mathbf{Y}), \nabla_{\dot{\gamma}}(h(\mathbf{X}))).$$

Proposition 10. *The 2-form Ω is given in local coordinates (ξ, η) by*

$$\Omega = -\frac{1}{2}\{d\xi \wedge d\bar{\eta} + d\bar{\xi} \wedge d\eta\},$$

and so is closed and nondegenerate, that is, it is a symplectic structure.

Proof. We have found that

$$\begin{aligned} h\left(\frac{\partial}{\partial\xi}\right) &= -\frac{e^r}{2\sqrt{2}\xi}e_{(+)} - \frac{e^{-r}}{2\sqrt{2}\xi}e_{(-)}, \\ \nabla_{\dot{\gamma}}h\left(\frac{\partial}{\partial\xi}\right) &= -\frac{e^r}{2\sqrt{2}\xi}e_{(+)} + \frac{e^{-r}}{2\sqrt{2}\xi}e_{(-)}, \\ h\left(\frac{\partial}{\partial\eta}\right) &= -\frac{\bar{\xi}e^{-r}}{2\sqrt{2}}e_{(+)} + \frac{\bar{\xi}e^r}{2\sqrt{2}}e_{(-)}, \\ \nabla_{\dot{\gamma}}h\left(\frac{\partial}{\partial\eta}\right) &= \frac{\bar{\xi}e^{-r}}{2\sqrt{2}}e_{(+)} + \frac{\bar{\xi}e^r}{2\sqrt{2}}e_{(-)}. \end{aligned}$$

Thus, the only nonvanishing components of Ω can be computed to be

$$\Omega\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\bar{\eta}}\right) = \Omega\left(\frac{\partial}{\partial\bar{\xi}}, \frac{\partial}{\partial\eta}\right) = -\frac{1}{2}. \quad \square$$

The complex structure and symplectic form are compatible in the following sense:

Proposition 11. $\Omega(\mathbf{J}\cdot, \mathbf{J}\cdot) = \Omega(\cdot, \cdot)$.

Proof. Let $\mathbf{X}, \mathbf{Y} \in T_{\gamma}\mathbf{U}$, and we compute

$$\begin{aligned} \Omega(\mathbf{J}(\mathbf{X}), \mathbf{J}(\mathbf{Y})) &= g(h(\mathbf{J}(\mathbf{X})), \nabla_{\dot{\gamma}}(h(\mathbf{J}(\mathbf{Y}))) \\ &\quad - g(h(\mathbf{J}(\mathbf{Y})), \nabla_{\dot{\gamma}}(h(\mathbf{J}(\mathbf{X}))), \\ &= g(\mathcal{R}_{\gamma}(h(\mathbf{X})), \nabla_{\dot{\gamma}}(\mathcal{R}_{\gamma}(h(\mathbf{Y}))) \\ &\quad - g(\mathcal{R}_{\gamma}(h(\mathbf{Y})), \nabla_{\dot{\gamma}}(\mathcal{R}_{\gamma}(h(\mathbf{X}))). \end{aligned}$$

It is clear that Proposition 11 holds if $\nabla_{\dot{\gamma}}$ and \mathcal{R}_{γ} commute on orthogonal vector fields along the geodesic. To see this, let $X = f e_{(+)} + \bar{f} e_{(-)}$ be an orthogonal vector field along the geodesic γ . Then

$$\begin{aligned} \nabla_{\dot{\gamma}}(\mathcal{R}_{\gamma}(X)) &= \nabla_{\dot{\gamma}}i(f e_{(+)} - \bar{f} e_{(-)}) \\ &= i(\dot{f} e_{(+)} - \dot{\bar{f}} e_{(-)}), \\ &= \mathcal{R}_{\gamma}(\dot{f} e_{(+)} + \dot{\bar{f}} e_{(-)}) \\ &= \mathcal{R}_{\gamma}(\nabla_{\dot{\gamma}}(X)), \end{aligned}$$

as claimed. \square

Definition 12. The Kähler metric \mathbf{G} on \mathbf{U} is defined by

$$\mathbf{G}(\cdot, \cdot) = \Omega(\mathbf{J}\cdot, \cdot).$$

This has local coordinate expression:

Proposition 12. *The Kähler metric has the following expression in local coordinates (ξ, η) :*

$$\mathbf{G} = -\frac{i}{4} \left(\frac{1}{\xi^2} d\xi^2 - \frac{1}{\bar{\xi}^2} d\bar{\xi}^2 + \xi^2 d\eta^2 - \xi^2 d\bar{\eta}^2 \right).$$

Proof. First we express the symplectic form Ω in terms of μ_1, μ_2 . By Definition 10 we have

$$\begin{aligned} d\eta &= -\frac{1}{2} \left(d\mu_1 + \frac{1}{\bar{\mu}_2} d\bar{\mu}_2 \right), \\ d\xi &= -\frac{2}{(1 + \bar{\mu}_1\mu_2)^2} (\mu_2^2 d\bar{\mu}_1 - d\mu_2). \end{aligned}$$

By Proposition 10 we have $\Omega = -\mathbf{Re}[d\xi \wedge d\bar{\eta}]$, and since

$$\begin{aligned} d\xi \wedge d\bar{\eta} &= \frac{1}{(1 + \bar{\mu}_1\mu_2)^2} (\mu_2^2 d\bar{\mu}_1 - d\mu_2) \wedge \left(d\bar{\mu}_1 + \frac{1}{\mu_2^2} d\mu_2 \right), \\ &= \frac{2}{(1 + \bar{\mu}_1\mu_2)^2} d\bar{\mu}_1 \wedge d\mu_2, \end{aligned}$$

we get

$$(2.1) \quad \Omega = - \left[\frac{1}{(1 + \mu_1\bar{\mu}_2)^2} d\mu_1 \wedge d\bar{\mu}_2 + \frac{1}{(1 + \bar{\mu}_1\mu_2)^2} d\bar{\mu}_1 \wedge d\mu_2 \right].$$

Now we find \mathbf{G} in these coordinates

$$\begin{aligned} \mathbf{G} &= -i \left[\frac{1}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1 \otimes d\bar{\mu}_2 - \frac{1}{(1 + \bar{\mu}_1 \mu_2)^2} d\bar{\mu}_1 \otimes d\mu_2 \right], \\ &= -i \frac{(\eta \bar{\xi} + 1)^2}{4} \frac{1}{2} \left(d\eta + \frac{1}{\bar{\xi}^2} d\bar{\xi} \right) \\ &\quad \otimes \frac{2}{(\eta \bar{\xi} + 1)^2} (\bar{\xi}^2 d\eta - d\bar{\xi}), + i \frac{(\bar{\eta} \xi + 1)^2}{4} \frac{1}{2} \left(d\bar{\eta} + \frac{1}{\xi^2} d\xi \right) \\ &\quad \otimes \frac{2}{(\bar{\eta} \xi + 1)^2} (\xi^2 d\bar{\eta} - d\xi), \\ &= -i \left[\frac{1}{4} \left(d\eta + \frac{1}{\bar{\xi}^2} d\bar{\xi} \right) \otimes (\bar{\xi}^2 d\eta - d\bar{\xi}) \right. \\ &\quad \left. - \frac{1}{4} \left(d\bar{\eta} + \frac{1}{\xi^2} d\xi \right) \otimes (\xi^2 d\bar{\eta} - d\xi) \right], \\ &= -\frac{i}{4} \left(\frac{1}{\bar{\xi}^2} d\xi^2 - \frac{1}{\bar{\xi}^2} d\bar{\xi}^2 + \bar{\xi}^2 d\eta^2 - \xi^2 d\bar{\eta}^2 \right), \end{aligned}$$

as claimed. \square

We are now in a position to prove our second result:

Theorem 2. *The Kähler structure is defined on the whole of the space $\mathbf{L}(\mathbf{H}^3)$. The metric \mathbf{G} is of neutral signature, is conformally flat and scalar flat.*

Proof. It is clear from the expression of \mathbf{G} and Ω in holomorphic coordinates (equations (2.1) and (2.2)) that these are well defined except where $\mu_1 = -\mu_2^{-1}$. But this is just the reflected diagonal, and so the Kähler structure is well defined on the whole of the space $\mathbf{L}(\mathbf{H}^3)$.

The signature of the metric is $(+ + - -)$, and the curvature can be computed directly from the coordinate expressions given above. The only nonvanishing components of the Riemann curvature tensor are

$$R_{\bar{\mu}_1 \mu_2 \mu_2}^{\mu_2} = -R_{\bar{\mu}_1 \mu_2 \bar{\mu}_1}^{\bar{\mu}_1} = \frac{2}{(1 + \bar{\mu}_1 \mu_2)^2}.$$

The results are then as stated. \square

As in the case of oriented lines of \mathbf{E}^3 , this metric has the following mechanical interpretation: the length of a vector $\mathbf{X} \in T_\gamma \mathbf{L}(\mathbf{H}^3)$ is the angular momentum of the Jacobi field $h(\mathbf{X})$ about the geodesic γ in \mathbf{H}^3 .

3. The isometry group of the neutral Kähler metric. We now find the isometry group of the space of oriented geodesics endowed with the above Kähler metric.

Theorem 3. *The identity component of the isometry group of the metric \mathbf{G} on $\mathbf{L}(\mathbf{H}^3)$ is isomorphic to the identity component of the hyperbolic isometry group.*

We prove this by way of a number of propositions: first we find the Killing vectors of \mathbf{G} and of the hyperbolic metric g . Then we integrate those of g up to explicit expressions for the isometry group of \mathbf{H}^3 and find how this group acts on $\mathbf{L}(\mathbf{H}^3)$. Finally we show that the infinitesimal generators of this action are precisely the Killing vectors of \mathbf{G} .

We start then with:

Proposition 13. *The Killing vectors of \mathbf{G} form a 6-parameter Lie algebra given by*

$$\mathbf{K} = \mathbf{Re} \left[(c_1 + c_2\mu_1 + c_3\mu_1^2) \frac{\partial}{\partial \mu_1} + (\bar{c}_3 - \bar{c}_2\mu_2 + \bar{c}_1\mu_2^2) \frac{\partial}{\partial \mu_2} \right],$$

where $c_1, c_2, c_3 \in \mathbf{C}$.

Proof. Let

$$\mathbf{K} = \mathbf{Re} \left(\mathbf{K}^{\mu_1} \frac{\partial}{\partial \mu_1} + \mathbf{K}^{\mu_2} \frac{\partial}{\partial \mu_2} \right),$$

be a vector field on $\mathbf{L}(\mathbf{H}^3)$ with $\mathbf{K}^{\mu_i} = \mathbf{K}^{\mu_i}(\mu_1, \bar{\mu}_1, \mu_2, \bar{\mu}_2)$ for $i = 1, 2$.

We will solve the Killing equations for \mathbf{K} :

$$(3.1) \quad \mathbf{K}^i \partial_i \mathbf{G}_{jk} + \mathbf{G}_{ki} \partial_j \mathbf{K}^i + \mathbf{G}_{ji} \partial_k \mathbf{K}^i = 0,$$

where ∂ represents partial differentiation.

In what follows we denote the Killing equation with specific values of the indices j and k simply by (j, k) . Thus, for example, $(\mu_1, \bar{\mu}_1)$ will

mean the Killing equation (3.1) with $j = \mu_1, k = \bar{\mu}_1$. Similarly for the derivative of these equations.

To start then, the $(\bar{\mu}_1, \bar{\mu}_1)$ and $(\bar{\mu}_2, \bar{\mu}_2)$ equations are

$$\partial_{\bar{\mu}_1} \mathbf{K}^{\mu_2} = 0, \quad \partial_{\bar{\mu}_2} \mathbf{K}^{\mu_1} = 0,$$

and so

$$\mathbf{K}^{\mu_1} = \mathbf{K}^{\mu_1}(\mu_1, \bar{\mu}_1, \mu_2), \quad \mathbf{K}^{\mu_2} = \mathbf{K}^{\mu_2}(\mu_1, \mu_2, \bar{\mu}_2).$$

Now, differentiating the (μ_1, μ_2) equation with respect to μ_2 gives

$$(\mu_2 \bar{\mu}_1 + 1)^2 \partial_{\mu_2}^2 \bar{\mathbf{K}}^{\mu_2} + 2(\mu_2 \bar{\mu}_1 + 1) \partial_{\mu_2} \bar{\mathbf{K}}^{\mu_2} = 0.$$

This can be rewritten

$$\partial_{\mu_2} \left[(\mu_2 \bar{\mu}_1 + 1)^2 \partial_{\mu_2} \bar{\mathbf{K}}^{\mu_2} \right] = 0,$$

which integrates to

$$\mathbf{K}^{\mu_2} = \alpha_2 + \frac{\alpha_1}{\mu_1 \bar{\mu}_2 + 1}, \text{ for } \alpha_i = \alpha_i(\mu_1, \bar{\mu}_1, \mu_2) \in \mathbf{C}, \quad i = 1, 2.$$

Since $\partial_{\bar{\mu}_1} \mathbf{K}^{\mu_2} = 0$ we get

$$(\mu_1 \bar{\mu}_2 + 1) \partial_{\bar{\mu}_1} \alpha_2 + \partial_{\bar{\mu}_1} \alpha_1 = 0.$$

Differentiating this with respect to $\bar{\mu}_2$ we see that $\partial_{\bar{\mu}_1} \alpha_2 = 0$, and hence $\partial_{\bar{\mu}_1} \alpha_1 = 0$. Thus, α_1 and α_2 are holomorphic, that is, $\alpha_1 = \alpha_1(\mu_1, \mu_2)$ and $\alpha_2 = \alpha_2(\mu_1, \mu_2)$.

We now differentiate the equation (μ_1, μ_2) with respect to μ_1 yielding

$$(\mu_1 \bar{\mu}_2 + 1)^2 \partial_{\mu_1}^2 \bar{\mathbf{K}}^{\mu_1} + 2(\mu_1 \bar{\mu}_2 + 1) \partial_{\mu_1} \bar{\mathbf{K}}^{\mu_1} = 0,$$

which, by a similar argument, has solution

$$\mathbf{K}^{\mu_1} = \alpha_3 + \frac{\alpha_4}{\bar{\mu}_1 \mu_2 + 1}, \text{ for } \alpha_3(\mu_1, \mu_2), \alpha_4(\mu_1, \mu_2) \in \mathbf{C}.$$

The equation $(\bar{\mu}_1, \bar{\mu}_2)$ now gives $\alpha_1 = \alpha_4 = 0$, while $\partial_{\mu_1} \partial_{\mu_2}^2 (\mu_1, \bar{\mu}_1)$ says that

$$\mu_1 \partial_{\mu_1}^2 \alpha_2 + 2 \partial_{\mu_1} \alpha_2 = 0,$$

with solution

$$\alpha_2 = \alpha_5 + \frac{\alpha_6}{\mu_1}, \text{ for } \alpha_5(\mu_2), \alpha_6(\mu_2) \in \mathbf{C}.$$

Now, $\partial_{\mu_1}^2 \partial_{\mu_2}(\mu_1, \bar{\mu}_1)$ reads

$$-(\mu_1 \bar{\mu}_2 + 1)^2 \partial_{\mu_2} \alpha_6 + 2\bar{\alpha}_6 \mu_1^2 \mu_2 = 0,$$

which implies that $\alpha_6 = 0$. On the other hand, the derived equation $\partial_{\mu_1}^2 \partial_{\mu_2}(\mu_2, \bar{\mu}_2)$ implies that

$$\mu_2 \partial_{\mu_2}^2 \alpha_3 + 2\partial_{\mu_2} \alpha_3 = 0,$$

and so

$$\alpha_3 = \alpha_7 + \frac{\alpha_8}{\mu_2},$$

where $\alpha_7 = \alpha_7(\mu_1), \alpha_8 = \alpha_8(\mu_1) \in \mathbf{C}$.

Substituting the previous results in equation $(\mu_2, \bar{\mu}_2)$ we find that

$$(\bar{\mu}_1 \mu_2 + 1)^2 \partial_{\mu_1} \alpha_8 - 2\bar{\alpha}_8 \mu_1 \mu_2^2 = 0,$$

from which we conclude that $\alpha_8 = 0$.

The derived equation $\partial_{\bar{\mu}_2} \partial_{\mu_1}^2(\mu_1, \bar{\mu}_2)$ implies that $\partial_{\mu_1}^3 \alpha_7 = 0$, and so

$$\alpha_7 = c_1 + c_2 \mu_1 + c_3 \mu_1^2, \text{ for } c_1, c_2, c_3 \in \mathbf{C}.$$

Similarly, $\partial_{\bar{\mu}_1} \partial_{\mu_2}^2(\bar{\mu}_1, \mu_2)$ implies that $\partial_{\mu_2}^3 \alpha_5 = 0$, and we therefore have

$$\alpha_5 = c_4 + c_5 \mu_2 + c_6 \mu_2^2, \text{ for } c_4, c_5, c_6 \in \mathbf{C}.$$

Finally, putting all of the preceding together we obtain

$$\mathbf{K}^{\mu_1} = c_1 + c_2 \mu_1 + c_3 \mu_1^2, \quad \mathbf{K}^{\mu_2} = \bar{c}_3 - \bar{c}_2 \mu_2 + \bar{c}_1 \mu_2^2,$$

as claimed. \square

By a similar method we compute the isometry group of the upper half-space model of \mathbf{H}^3 and its Lie algebra.

Proposition 14. *The Killing vectors of the hyperbolic metric g form a 6-parameter Lie algebra given by*

$$K = K^0 \frac{\partial}{\partial x^0} + K^1 \frac{\partial}{\partial x^1} + K^2 \frac{\partial}{\partial x^2},$$

where

$$\begin{aligned} K^0 &= (a_4 + a_5x^1 + a_6x^2)x^0, \\ K^1 &= a_{11} - \frac{1}{2}a_5((x^0)^2 - (x^1)^2 + (x^2)^2) \\ &\quad - a_{10}x^2 + a_4x^1 + a_6x^1x^2, \\ K^2 &= a_9 - \frac{1}{2}a_6((x^0)^2 + (x^1)^2 - (x^2)^2) \\ &\quad + a_{10}x^1 + a_4x^2 + a_5x^1x^2, \end{aligned}$$

for $a_4, a_5, a_6, a_9, a_{10}, a_{11} \in \mathbf{R}$.

Proof. Let $K = K^0(\partial/\partial x^0) + K^1(\partial/\partial x^1) + K^2(\partial/\partial x^2)$ be a Killing vector of the hyperbolic metric g . Then it satisfies

$$(3.2) \quad K^i \partial_i g_{jk} + g_{ki} \partial_j K^i + g_{ji} \partial_k K^i = 0,$$

where $g_{ij} = (x^0)^{-2} \delta_{ij}$.

As before, we denote the Killing equation with specific values of the indices j and k simply by (j, k) . Thus, for example, (x^1, x^2) will mean the Killing equation (3.2) with $j = x^1, k = x^2$. Throughout, all functions will be real-valued.

To start then, the (x^0, x^0) equation is

$$x^0 \partial_{x^0} K^0 - K^0 = 0,$$

with solution $K^0 = a_1 x^0$ for $a_1 = a_1(x^1, x^2)$. Substituting this into the equations (x^1, x^0) and (x^2, x^0) , we obtain

$$x^0 \partial_{x^1} a_1 + \partial_{x^0} K^1 = 0, \quad x^0 \partial_{x^2} a_1 + \partial_{x^0} K^2 = 0,$$

which we integrate to $K^1 = a_2 - (1/2)(x^0)^2 \partial_{x^1} a_1$ and $K^2 = a_3 - (1/2)(x^0)^2 \partial_{x^2} a_1$ for $a_2 = a_2(x^1, x^2)$ and $a_3 = a_3(x^1, x^2)$.

Now the derived equations $\partial_{x^0}^2(x^2, x^2)$, $\partial_{x^0}^2(x^1, x^1)$ and $\partial_{x^0}^2(x^2, x^1)$ yield

$$\partial_{x^2}^2 a_1 = 0, \quad \partial_{x^1}^2 a_1 = 0, \quad \partial_{x^2} \partial_{x^1} a_1 = 0.$$

The first of these implies $a_1 = a + bx^2$ for $a = a(x^1)$ and $b = b(x^1)$, while the last equation means that $\partial_{x^1}b = 0$ and so b is constant. The middle equation says that $\partial_{x^1}(\partial_{x^1}a_1) = \partial_{x^1}^2a = 0$, and hence $a = c + dx^1$, where c, d are constants. Therefore, we have $a_1 = a_4 + a_5x^1 + a_6x^2$, where a_4, a_5, a_6 are constants.

Substituting the above results to the equations (x^1, x^1) and (x^2, x^2) , we obtain

$$\partial_{x^1}a_2 - a_4 - a_5x^1 - a_6x^2 = 0, \quad \partial_{x^2}a_3 - a_4 - a_5x^1 - a_6x^2 = 0,$$

with solutions

$$a_2 = a_7 + a_4x^1 + \frac{1}{2}a_5(x^1)^2 + a_6x^1x^2,$$

and

$$a_3 = a_8 + a_4x^2 + \frac{1}{2}a_6(x^2)^2 + a_5x^1x^2,$$

where $a_7 = a_7(x^2)$ and $a_8 = a_8(x^1)$.

Now the equation $\partial_{x^1}(x^2, x^1)$ gives $\partial_{x^1}^2a_8 + a_6 = 0$, and therefore

$$a_8 = a_9 + a_{10}x^1 - \frac{1}{2}a_6(x^1)^2,$$

where a_9 and a_{10} are constants.

From $\partial_{x^2}(x^2, x^1)$, it implies that $\partial_{x^2}^2a_7 + a_5 = 0$, and therefore

$$a_7 = a_{11} + a_{12}x^2 - \frac{1}{2}a_5(x^2)^2,$$

where a_{11} and a_{12} are constants. The equation (x^2, x^1) gives $a_{10} = -a_{12}$.

Finally, assembling expressions for K^i we get:

$$\begin{aligned} K^0 &= (a_4 + a_5x^1 + a_6x^2)x^0, \\ K^1 &= a_{11} - \frac{1}{2}a_5((x^0)^2 - (x^1)^2 + (x^2)^2) \\ &\quad - a_{10}x^2 + a_4x^1 + a_6x^1x^2, \\ K^2 &= a_9 - \frac{1}{2}a_6((x^0)^2 + (x^1)^2 - (x^2)^2) \\ &\quad + a_{10}x^1 + a_4x^2 + a_5x^1x^2, \end{aligned}$$

as claimed. \square

Corollary 1. *The Killing vectors of the hyperbolic metric g are*

$$K = \mathbf{R}e \left[t(\gamma + 2\alpha\bar{z}) \frac{\partial}{\partial t} + 2(\beta + \gamma z - \alpha t^2 + \bar{\alpha}z^2) \frac{\partial}{\partial z} \right],$$

for $\alpha, \beta, \gamma \in \mathbf{C}$.

Proof. If we reintroduce variables (t, z) in the upper half-space model of \mathbf{H}^3 by

$$t = x^0 \in \mathbf{R}_+, \quad z = x^1 + ix^2 \in \mathbf{C},$$

and set

$$\alpha = \frac{1}{2}(a_5 + ia_6), \quad \beta = a_{11} + ia_9, \quad \gamma = a_4 + ia_{10},$$

$$\alpha, \beta, \gamma \in \mathbf{C},$$

the result follows from Proposition 14. \square

We now integrate the Killing vectors of \mathbf{H}^3 to get the group action:

Proposition 15. *The isometry group of \mathbf{H}^3 is 6-dimensional, parameterized by $\alpha, \beta, \gamma \in \mathbf{C}$, and the associated 1-parameter group of local isometries which map $((t_0, z_0), s) \rightarrow (t(s), z(s))$, for α and γ not both zero, is*

$$t(s) = \frac{t_0(|z_0 - \tau|^2 + t_0^2)}{|\bar{z}_0 - \bar{\tau} - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(|z_0 - \tau|^2 + t_0^2)|^2 + t_0^2} e^{(\gamma_1 + \bar{\gamma}_1)s/2},$$

$$z(s) = \frac{z_0 - \tau - \frac{\alpha}{\bar{\gamma}_1}(e^{\bar{\gamma}_1 s} - 1)(|z_0 - \tau|^2 + t_0^2)}{|\bar{z}_0 - \bar{\tau} - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(|z_0 - \tau|^2 + t_0^2)|^2 + t_0^2}$$

$$\times (|z_0 - \tau|^2 + t_0^2)e^{\gamma_1 s} + \tau,$$

where τ is a solution of $\bar{\alpha}\tau^2 + \gamma\tau + \beta = 0$ and $\gamma_1 = \gamma + 2\bar{\alpha}\tau$.

For $\alpha = \gamma = 0$, the isometries are

$$t(s) = t_0, \quad z(s) = \beta s + z_0.$$

Proof. Consider the integral curve $\mathbf{R} \rightarrow \mathbf{H}^3: s \mapsto t(s)(\partial/\partial t) + z(s)(\partial/\partial z)$, of the vector field K . To find the integral curves explicitly of K , we have to solve the system of differential equations

$$(3.3) \quad t' = t \left(\frac{1}{2}(\gamma + \bar{\gamma}) + \alpha \bar{z} + \bar{\alpha} z \right), \quad z' = \beta + \gamma z - \alpha t^2 + \bar{\alpha} z^2,$$

where a prime denotes differentiation with respect to the variable s .

For $\alpha = \gamma = 0$ the result follows immediately, so we now consider the case where α and γ are not both zero.

First let us assume that $\beta = 0$. The system (3.3) can be written as

$$(3.4) \quad V' = VAV + BV + VB^T,$$

where

$$V = \begin{bmatrix} z & t \\ -t & \bar{z} \end{bmatrix}, \quad A = \begin{bmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{bmatrix}, \quad B = \frac{1}{2} \begin{bmatrix} \gamma & 0 \\ 0 & \bar{\gamma} \end{bmatrix}.$$

System (3.4) is a matrix Riccati equation and has general solution [15]

$$V = Q \left[V_0^{-1} - \int_0^s Q^T A Q ds \right]^{-1} Q^T,$$

where $V_0 = V(0)$, and Q is a 2×2 matrix satisfying the equation

$$Q' = BQ, \quad Q(0) = I,$$

I being the 2×2 identity matrix.

This has the solution

$$Q = \begin{bmatrix} e^{(\gamma/2)s} & 0 \\ 0 & e^{(\bar{\gamma}/2)s} \end{bmatrix}, \quad \text{and so } Q^T A Q = \begin{bmatrix} \bar{\alpha} e^{\gamma s} & 0 \\ 0 & \alpha e^{\bar{\gamma} s} \end{bmatrix}.$$

Thus,

$$\int_0^s Q^T A Q ds = \begin{bmatrix} (\bar{\alpha}/\gamma)(e^{\gamma s} - 1) & 0 \\ 0 & (\alpha/\bar{\gamma})(e^{\bar{\gamma} s} - 1) \end{bmatrix}.$$

If the initial value of V^{-1} is

$$V_0^{-1} = \begin{bmatrix} v_1 & v_2 \\ -v_2 & \bar{v}_1 \end{bmatrix},$$

for $v_2 \in \mathbf{R}$ and $v_1 \in \mathbf{C}$, then

$$C = V_0^{-1} - \int_0^s Q^T A Q ds = \begin{bmatrix} v_1 - (\bar{\alpha}/\gamma)(e^{\gamma s} - 1) & v_2 \\ -v_2 & \bar{v}_1 - (\alpha/\bar{\gamma})(e^{\bar{\gamma} s} - 1) \end{bmatrix},$$

and the determinant of C is

$$\det C = \left| v_1 - \frac{\bar{\alpha}}{\gamma}(e^{\gamma s} - 1) \right|^2 + v_2^2.$$

So, we obtain the inverse of C :

$$\left[V_0^{-1} - \int_0^s Q^T A Q ds \right]^{-1} = \frac{1}{\det C} \begin{bmatrix} \bar{v}_1 - (\alpha/\bar{\gamma})(e^{\bar{\gamma}s} - 1) & -v_2 \\ v_2 & v_1 - (\bar{\alpha}/\gamma)(e^{\gamma s} - 1) \end{bmatrix}.$$

Finally, we find the solution

$$V = Q C^{-1} Q^T = \frac{1}{\det C} \begin{bmatrix} e^{\gamma s}(\bar{v}_1 - (\alpha/\bar{\gamma})(e^{\bar{\gamma}s} - 1)) & -v_2 e^{(\gamma + \bar{\gamma})s/2} \\ v_2 e^{(\gamma + \bar{\gamma})s/2} & e^{\bar{\gamma}s}(v_1 - (\bar{\alpha}/\gamma)(e^{\gamma s} - 1)) \end{bmatrix}.$$

For $\beta = 0$ then, the integral curves are

$$t(s) = -\frac{v_2}{|v_1 - (\bar{\alpha}/\gamma)(e^{\gamma s} - 1)|^2 + v_2^2} e^{(\gamma + \bar{\gamma})s/2},$$

$$z(s) = \frac{\bar{v}_1 - (\alpha/\bar{\gamma})(e^{\bar{\gamma}s} - 1)}{|v_1 - (\bar{\alpha}/\gamma)(e^{\gamma s} - 1)|^2 + v_2^2} e^{\gamma s}.$$

To solve the general case with $\beta \neq 0$, choose a complex number τ satisfying $\beta = -\gamma\tau - \bar{\alpha}\tau^2$. Then a shift $z \rightarrow z + \tau$ and completing the squares on the z term of the righthand side of the second equation of (3.3) reduces the equations to system (3.4). Thus, the general solution turns out to be

$$t(s) = -\frac{v_2}{|v_1 - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)|^2 + v_2^2} e^{(\gamma_1 + \bar{\gamma}_1)s/2},$$

$$z(s) = \frac{\bar{v}_1 - \frac{\alpha}{\bar{\gamma}_1}(e^{\bar{\gamma}_1 s} - 1)}{|v_1 - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)|^2 + v_2^2} e^{\gamma_1 s} + \tau,$$

where $\gamma_1 = \gamma + 2\bar{\alpha}\tau$.

Setting

$$v_1 = \frac{\bar{z}_0 - \bar{\tau}}{|z_0 - \tau|^2 + t_0^2}, \quad v_2 = -\frac{t_0}{|z_0 - \tau|^2 + t_0^2},$$

we obtain that $z(0) = z_0$ and $t(0) = t_0$, and the result follows. \square

Since the above transformations $((t_0, z_0), s) \mapsto (t(s), z(s))$ are isometries of \mathbf{H}^3 , they map oriented geodesics to oriented geodesics. The following explicitly describes this as a map from $\mathbf{L}(\mathbf{H}^3)$ to itself.

Proposition 16. *The above action maps oriented geodesics according to: $(\xi, \eta) \mapsto (\xi', \eta')$ where, for α and γ not both zero:*

$$\begin{aligned} \xi' &= \xi e^{-\bar{\gamma}_1 s} \left[\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} + \frac{1}{\xi} - \bar{\tau} \right) - 1 \right] \\ &\quad \times \left[\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} - \frac{1}{\xi} - \bar{\tau} \right) - 1 \right], \\ \eta' &= \frac{\eta - \tau - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi}) - \tau)(\eta - (1/\bar{\xi}) - \tau)}{[(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi}) - \tau) - 1][(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta - (1/\bar{\xi}) - \tau) - 1]} e^{\gamma_1 s} + \tau, \end{aligned}$$

and for $\alpha = \gamma = 0$,

$$\xi' = \xi, \quad \eta' = \eta + \beta s.$$

Proof. It suffices to work on $\mathbf{U} \subset \mathbf{L}(\mathbf{H}^3)$ and consider the action on the oriented geodesics

$$z_0 = \eta + \frac{\tanh r}{\bar{\xi}}, \quad t_0 = \frac{1}{\sqrt{\xi \bar{\xi}} \cosh r}.$$

We will find the oriented geodesic that is obtained by mapping this oriented geodesic by the 1-parameter group of actions in the last proposition.

The case $\alpha = \gamma = 0$ follows trivially, so we omit the proof and consider the case where α and γ are not both zero.

Denote

$$\begin{aligned} \Lambda &= |z_0 - \tau|^2 + t_0^2 \\ &= \left(\eta - \tau + \frac{\tanh r}{\bar{\xi}} \right) \left(\bar{\eta} - \bar{\tau} + \frac{\tanh r}{\xi} \right), \\ &= |\eta - \tau|^2 + \frac{1}{|\xi|^2} + \left(\frac{\eta - \tau}{\xi} + \frac{\bar{\eta} - \bar{\tau}}{\bar{\xi}} \right) \tanh r. \end{aligned}$$

Then we obtain $v_1\bar{v}_1 + v_2^2 = \Lambda^{-1}$. Set $\lambda = \alpha\gamma_1^{-1}(e^{\gamma_1 s} - 1)$, and

$$\begin{aligned} |v_1 - \lambda|^2 + v_2^2 &= \frac{1 - \lambda(z_0 - \tau) - \bar{\lambda}(\bar{z}_0 - \bar{\tau}) + |\lambda|^2\Lambda}{\Lambda}, \\ &= \frac{1 - \lambda(\eta - \tau) - \bar{\lambda}(\bar{\eta} - \bar{\tau}) + |\lambda|^2 \left(|\eta - \tau|^2 + \frac{1}{|\xi|^2} \right)}{\Lambda}, \\ &\quad + \frac{((|\lambda|^2(\bar{\eta} - \bar{\tau}) - \lambda)/\bar{\xi} + (|\lambda|^2(\eta - \tau) - \bar{\lambda})/\xi) \tanh r}{\Lambda}. \end{aligned}$$

We now shift η to $\eta + \tau$; this will simplify the calculations and can be undone at the end by shifting η to $\eta - \tau$. Thus, the above expression becomes

$$\begin{aligned} |v_1 - \lambda|^2 + v_2^2 &= \frac{1 - \lambda\eta - \bar{\lambda}\bar{\eta} + |\lambda|^2 (|\eta|^2 + (1/|\xi|^2))}{\Lambda} \\ &\quad + \frac{((|\lambda|^2\bar{\eta} - \lambda)/\bar{\xi} + (|\lambda|^2\eta - \bar{\lambda})/\xi) \tanh r}{\Lambda}. \end{aligned}$$

Then,

$$t = -\frac{v_2}{|v_1 - \lambda|^2 + v_2^2} e^{(\gamma_1 + \bar{\gamma}_1)s/2},$$

and therefore

$$t = \frac{|\xi|^{-1} e^{(\gamma_1 + \bar{\gamma}_1)s/2}}{\left[(1 - \lambda\eta - \bar{\lambda}\bar{\eta} + |\lambda|^2|\eta|^2 + (|\lambda|^2/|\xi|^2)) \cosh r + (|\lambda|^2\bar{\eta} - \lambda)/\bar{\xi} + (|\lambda|^2\eta - \bar{\lambda})/\xi \sinh r \right]}$$

Introducing values A_1 and A_2 by

$$\begin{aligned} A_1 &= \left| \frac{\bar{\alpha}}{\gamma_1} (e^{\gamma_1 s} - 1) \left(\eta + \frac{1}{\xi} \right) - 1 \right|, \\ A_2 &= \left| \frac{\bar{\alpha}}{\gamma_1} (e^{\gamma_1 s} - 1) \left(\eta - \frac{1}{\xi} \right) - 1 \right|. \end{aligned}$$

Then we may rewrite t as follows

$$t = \frac{1}{\sqrt{(\xi e^{-\bar{\gamma}_1 s})(\bar{\xi} e^{-\gamma_1 s}) A_1^2 A_2^2 \cosh(r + \log(A_1/A_2))}}.$$

Hence, we have the following map

$$\begin{aligned}\xi' &= \xi e^{-\bar{\gamma}_1 s} \left[\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} + \frac{1}{\xi} \right) - 1 \right] \\ &\quad \times \left[\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} - \frac{1}{\xi} \right) - 1 \right], \\ r' &= r + \log \left(\frac{A_1}{A_2} \right).\end{aligned}$$

It remains now to find how the η transforms. To find this we have to find the z .

$$\begin{aligned}z &= \frac{\bar{v}_1 - (\alpha/\bar{\gamma}_1)(e^{\bar{\gamma}_1 s} - 1)}{|v_1 - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)|^2 + v_2^2} e^{\gamma_1 s}, \\ &= \frac{z_0 - (\alpha/\bar{\gamma}_1)(e^{\bar{\gamma}_1 s} - 1)\Lambda}{1 - \lambda\eta - \bar{\lambda}\bar{\eta} + |\lambda|^2|\eta|^2 + (|\lambda|^2/|\xi|^2) + ((|\lambda|^2\bar{\eta} - \lambda)/\bar{\xi} + (|\lambda|^2\eta - \bar{\lambda})/\xi) \tanh r} e^{\gamma_1 s}, \\ &= \frac{(\eta + (\tanh r/\bar{\xi}) - (\alpha/\bar{\gamma}_1)(e^{\bar{\gamma}_1 s} - 1)) [|\eta|^2 + (1/|\xi|^2) + ((\eta/\xi) + (\bar{\eta}/\bar{\xi})) \tanh r]}{(1 - \lambda\eta - \bar{\lambda}\bar{\eta} + |\lambda|^2|\eta|^2 + (|\lambda|^2/|\xi|^2)) \cosh r + ((|\lambda|^2\bar{\eta} - \lambda)/\bar{\xi} + (|\lambda|^2\eta - \bar{\lambda})/\xi) \sinh r} e^{\gamma_1 s \cosh r}, \\ &= \frac{\eta - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi}))(\eta - (1/\bar{\xi}))}{[(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi})) - 1][(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta - (1/\bar{\xi})) - 1]} e^{\gamma_1 s} \\ &\quad + \frac{\tanh(r + \log(A_1/A_2))}{\xi e^{-\gamma_1 s} ((\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi})) - 1)((\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta - (1/\bar{\xi})) - 1)}.\end{aligned}$$

Therefore, we have the mapping

$$\begin{aligned}\eta' &= \frac{\eta - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi}))(\eta - (1/\bar{\xi}))}{[(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi})) - 1][(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta - (1/\bar{\xi})) - 1]} e^{\gamma_1 s}, \\ \xi' &= \xi e^{-\bar{\gamma}_1 s} \left(\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} + \frac{1}{\xi} \right) - 1 \right) \left(\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} - \frac{1}{\xi} \right) - 1 \right), \\ r' &= r + \log \left(\frac{A_1}{A_2} \right).\end{aligned}$$

Finally, we shift η back to $\eta - \tau$ to yield the stated result:

$$\begin{aligned}\eta' &= \frac{\eta - \tau - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi}) - \tau)(\eta - (1/\bar{\xi}) - \tau)}{[(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi}) - \tau) - 1][(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta - (1/\bar{\xi}) - \tau) - 1]} e^{\gamma_1 s} + \tau, \\ \xi' &= \xi e^{-\bar{\gamma}_1 s} \left[\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} + \frac{1}{\xi} - \bar{\tau} \right) - 1 \right] \\ &\quad \times \left[\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} - \frac{1}{\xi} - \bar{\tau} \right) - 1 \right]. \quad \square\end{aligned}$$

Proof of Theorem 3. We have obtained a map

$$F: \mathbf{L}(\mathbf{H}^3) \times \mathbf{R} \longrightarrow \mathbf{L}(\mathbf{H}^3): ((\xi, \eta), s) \longmapsto (f_1(\xi, \eta, s), f_2(\xi, \eta, s)),$$

where, for α and γ not both zero:

$$\begin{aligned} f_1(\xi, \eta, s) &= \xi e^{-\bar{\gamma}_1 s} \left[\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} + \frac{1}{\xi} - \bar{\tau} \right) - 1 \right] \\ &\quad \times \left[\frac{\alpha}{\bar{\gamma}_1} (e^{\bar{\gamma}_1 s} - 1) \left(\bar{\eta} - \frac{1}{\xi} - \bar{\tau} \right) - 1 \right], \\ f_2(\xi, \eta, s) &= \frac{\eta - \tau - (\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi}) - \tau)(\eta - (1/\bar{\xi}) - \tau)}{[(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta + (1/\bar{\xi}) - \tau) - 1][(\bar{\alpha}/\gamma_1)(e^{\gamma_1 s} - 1)(\eta - (1/\bar{\xi}) - \tau) - 1]} \\ &\quad \times e^{\gamma_1 s} + \tau, \end{aligned}$$

and, for $\alpha = \gamma = 0$,

$$f_1(\xi, \eta, s) = \xi, \quad f_2(\xi, \eta, s) = \eta + \beta s.$$

We will show that for any $s \in \mathbf{R}$ this map is an isometry of \mathbf{G} . In order to do so, we first find the derivative of F at the point $s = 0$

$$F_* \left(\frac{d}{ds} \Big|_0 \right) = \frac{df_1}{ds} \Big|_0 \frac{\partial}{\partial \xi} + \frac{d\bar{f}_1}{ds} \Big|_0 \frac{\partial}{\partial \bar{\xi}} + \frac{df_2}{ds} \Big|_0 \frac{\partial}{\partial \eta} + \frac{d\bar{f}_2}{ds} \Big|_0 \frac{\partial}{\partial \bar{\eta}}.$$

Calculating the derivatives of F in the case $\alpha = \gamma = 0$ and converting to (μ_1, μ_2) coordinates we obtain the vector field

$$\mathbf{K} = F_* \left(\frac{d}{ds} \Big|_0 \right) = \mathbf{Re} \left(-\beta \frac{\partial}{\partial \mu_1} - \bar{\beta} \mu_2^2 \frac{\partial}{\partial \mu_2} \right),$$

which are the Killing vectors of \mathbf{G} found in Proposition 13 with $c_1 = -\beta$, $c_2 = 0$ and $c_3 = 0$.

On the other hand, suppose that α and γ are not both zero. Then

$$\frac{df_1}{ds} \Big|_0 = -\bar{\gamma}_1 \xi - 2\alpha \xi (\bar{\eta} - \bar{\tau}), \quad \frac{df_2}{ds} \Big|_0 = \gamma_1 (\eta - \tau) + \bar{\alpha} \left[(\eta - \tau)^2 + \frac{1}{\bar{\xi}^2} \right].$$

Recalling now the change of coordinates to (μ_1, μ_2) in $\mathbf{L}(\mathbf{H}^3)$, we obtain

$$\begin{aligned} \mathbf{K}^{\mu_1} &= -\frac{1}{4} \left(\mu_1 + \frac{1}{\bar{\mu}_2} \right)^2 \frac{d\bar{f}_1}{ds} \Big|_0 - \frac{df_2}{ds} \Big|_0 \\ &= \gamma_1 \tau - \bar{\alpha} \tau^2 + (\gamma_1 - 2\bar{\alpha} \tau) \mu_1 - \bar{\alpha} \mu_1^2, \\ \mathbf{K}^{\mu_2} &= \frac{\mu_2^2}{4} \left(\bar{\mu}_1 + \frac{1}{\mu_2} \right)^2 \frac{df_1}{ds} \Big|_0 - \mu_2^2 \frac{d\bar{f}_2}{ds} \Big|_0 \\ &= -\alpha - (\bar{\gamma}_1 - 2\alpha \bar{\tau}) \mu_2 + (\bar{\gamma}_1 \bar{\tau} - \alpha \bar{\tau}^2) \mu_2^2. \end{aligned}$$

Now, substituting $\gamma_1 = \gamma + 2\bar{\alpha} \tau$ and $\gamma \tau + \bar{\alpha} \tau^2 = -\beta$ back in the above expressions, we find that

$$\mathbf{K}^{\mu_1} = -\beta + \gamma \mu_1 - \bar{\alpha} \mu_1^2, \quad \mathbf{K}^{\mu_2} = -\alpha - \bar{\gamma} \mu_2 - \bar{\beta} \mu_2^2.$$

These are precisely the Killing vectors of \mathbf{G} found in Proposition 13 with $c_1 = -\beta$, $c_2 = \gamma$ and $c_3 = -\bar{\alpha}$, and hence

$$\text{Iso}_0(\mathbf{L}(\mathbf{H}^3), \mathbf{G}) = \text{Iso}_0(\mathbf{H}^3, g),$$

as claimed. \square

4. The geodesics of \mathbf{G} . A curve in $\mathbf{L}(\mathbf{H}^3)$ is a 1-parameter family of oriented geodesics in \mathbf{H}^3 , which we refer to as a *ruled surface* in \mathbf{H}^3 . The ruled surfaces that come from geodesics of the neutral Kähler metric in $\mathbf{L}(\mathbf{H}^3)$ have a particularly elegant characterization:

Theorem 4. *The geodesics of the Kähler metric \mathbf{G} are generated by the 1-parameter subgroups of the isometry group of \mathbf{G} .*

A ruled surface generated by a geodesic of \mathbf{G} is a minimal surface in \mathbf{H}^3 , and the geodesic is null if and only if the ruled surface is totally geodesic.

Proof. For $I \subset \mathbf{R}$, let $c : I \rightarrow \mathbf{L}(\mathbf{H}^3)$ be a geodesic in $\mathbf{L}(\mathbf{H}^3)$ with affine parameter t . By an isometry we can move the geodesic to lie in $\mathbf{U} \subset \mathbf{L}(\mathbf{H}^3)$ and use coordinates (ξ, η) as earlier. Thus, the geodesic is given by $c(t) = (\xi(t), \eta(t))$ satisfying the geodesic equation

$$\nabla_{\dot{c}} \dot{c} = 0,$$

where \dot{c} is the tangent vector to c :

$$\dot{c} = \dot{\xi}(t) \frac{\partial}{\partial \xi} + \dot{\eta}(t) \frac{\partial}{\partial \eta} + \dot{\bar{\xi}}(t) \frac{\partial}{\partial \bar{\xi}} + \dot{\bar{\eta}}(t) \frac{\partial}{\partial \bar{\eta}},$$

the dot denoting differentiation with respect to t .

This equation, using the metric expression in Proposition 12 turns out to be

$$(4.1) \quad \xi \ddot{\xi} - \dot{\xi}^2 + (\dot{\bar{\eta}})^2 \xi^4 = 0, \quad \bar{\xi} \ddot{\eta} + 2\dot{\bar{\xi}} \dot{\eta} = 0,$$

which we solve as follows. From the second equation we obtain

$$\frac{d}{dt} \left(\frac{\bar{\xi}^2 d\eta}{\xi} \right) = 0,$$

from which we get that

$$\dot{\eta} = \frac{b_3}{\bar{\xi}^2},$$

for some complex constant b_3 .

Substituting this into the first equation of (4.1), we obtain

$$\xi \ddot{\xi} - \dot{\xi}^2 + \bar{b}_3^2 = 0.$$

This has solution

$$\xi(t) = \frac{\bar{b}_3 \sinh(b_2 t + b_1)}{b_2}.$$

We now get that

$$\dot{\eta} = \frac{\bar{b}_2^2}{b_3 \sinh^2(\bar{b}_2 t + \bar{b}_1)},$$

which we integrate to find the geodesics:

$$\xi(t) = \frac{\bar{b}_3 \sinh(b_2 t + b_1)}{b_2}, \quad \eta(t) = b_4 - \frac{\bar{b}_2 \cosh(\bar{b}_2 t + \bar{b}_1)}{b_3 \sinh(\bar{b}_2 t + \bar{b}_1)},$$

for $b_1, b_2, c_3, c_4 \in \mathbf{C}$. We note that the length of the tangent vector to the geodesic is the constant $i(\bar{b}_2^2 - b_2^2)/4$.

The geodesic in terms of (μ_1, μ_2) coordinates is given by

$$\begin{aligned}\mu_1 &= -b_4 + \frac{\bar{b}_2[1 + \cosh(\bar{b}_2 t + \bar{b}_1)]}{b_3 \sinh(\bar{b}_2 t + \bar{b}_1)}, \\ \mu_2 &= \frac{\bar{b}_3 \sinh(b_2 t + b_1)}{\bar{b}_3 \bar{b}_4 \sinh(b_2 t + b_1) + b_2[1 - \cosh(b_2 t + b_1)]}.\end{aligned}$$

The tangent vector of the geodesic is

$$\dot{c}(t) = \mathbf{Re} \left(\dot{\mu}_1 \frac{\partial}{\partial \mu_1} + \dot{\mu}_2 \frac{\partial}{\partial \mu_2} \right),$$

where

$$\begin{aligned}\dot{\mu}_1 &= -\frac{\bar{b}_2^2[1 + \cosh(\bar{b}_2 t + \bar{b}_1)]}{b_3 \sinh^2(\bar{b}_2 t + \bar{b}_1)}, \\ \dot{\mu}_2 &= \frac{b_2^2 \bar{b}_3 [\cosh(b_2 t + b_1) - 1]}{(\bar{b}_3 \bar{b}_4 \sinh(b_2 t + b_1) + b_2[1 - \cosh(b_2 t + b_1)])^2}.\end{aligned}$$

Thus, for all t ,

$$\dot{\mu}_1(t) = c_1 + c_2 \mu_1(t) + c_3 \mu_1^2(t), \quad \dot{\mu}_2(t) = \bar{c}_3 - \bar{c}_2 \mu_2(t) + \bar{c}_1 \mu_2^2(t),$$

where

$$c_1 = -\frac{b_3^2 b_4^2 - \bar{b}_2^2}{2b_3}, \quad c_2 = -b_3 b_4, \quad c_3 = -\frac{b_3}{2}.$$

We conclude from Proposition 13 that the tangent vector of the geodesic is the restriction of a Killing vector of \mathbf{G} .

We now compute the second fundamental form of the ruled surface in \mathbf{H}^3 generated by the geodesics of \mathbf{G} . In order to simplify the calculations we first utilize an isometry of \mathbf{U} , cf. Proposition 16 with

$$\alpha = 0, \quad s = 1, \quad e^{\gamma_1} = b_3, \quad \text{and} \quad \tau = \frac{b_3 b_4}{b_3 - 1}.$$

This simplifies the geodesic to

$$\xi(t) = \frac{\sinh(b_2 t + b_1)}{b_2}, \quad \eta(t) = -\frac{\bar{b}_2 \cosh(\bar{b}_2 t + \bar{b}_1)}{\sinh(\bar{b}_2 t + \bar{b}_1)}.$$

The map $\Phi: \mathbf{U} \times \mathbf{R} \rightarrow \mathbf{H}^3: ((\xi, \eta), r) \mapsto (x^0, x^1, x^2)$, where

$$x^0 = \frac{1}{|\xi| \cosh r}, \quad x^1 = \frac{\eta + \bar{\eta}}{2} + \frac{\tanh r}{2} \left(\frac{1}{\xi} + \frac{1}{\bar{\xi}} \right),$$

$$x^2 = \frac{i(\bar{\eta} - \eta)}{2} + \frac{i \tanh r}{2} \left(\frac{1}{\xi} - \frac{1}{\bar{\xi}} \right),$$

now yields the parametrization of the ruled surface S in \mathbf{H}^3 . In particular, we have a surface given by $(r, t) \mapsto (x^0(r, t), x^1(r, t), x^2(r, t))$ which has induced metric

$$\bar{g}_{ik} = g_{ij} \frac{\partial x^i}{\partial y^l} \frac{\partial x^j}{\partial y^k},$$

and normal vector \bar{N}

$$\begin{aligned} \bar{N} = & \left(\frac{\partial x^1}{\partial y^1} \frac{\partial x^2}{\partial y^2} - \frac{\partial x^1}{\partial y^2} \frac{\partial x^2}{\partial y^1} \right) \frac{\partial}{\partial x^0} \\ & - \left(\frac{\partial x^0}{\partial y^1} \frac{\partial x^2}{\partial y^2} - \frac{\partial x^0}{\partial y^2} \frac{\partial x^2}{\partial y^1} \right) \frac{\partial}{\partial x^1} \\ & + \left(\frac{\partial x^0}{\partial y^1} \frac{\partial x^1}{\partial y^2} - \frac{\partial x^0}{\partial y^2} \frac{\partial x^1}{\partial y^1} \right) \frac{\partial}{\partial x^2}, \end{aligned}$$

where $y^1 = r$ and $y^2 = t$.

The second fundamental form K_{ab} of S is given by

$$K_{ab} = \frac{\partial x^k}{\partial y^a} \left(\frac{\partial N_k}{\partial y^b} - \tilde{\Gamma}_{ik}^i \frac{\partial x^l}{\partial y^b} N_l \right), \quad a, b = 1, 2,$$

where N is the unit normal of S and $\tilde{\Gamma}_{ik}^i$ are the Christoffel symbols of the hyperbolic metric g .

After some lengthy calculations, the components of the second fundamental form of the ruled surface S in \mathbf{H}^3 are found to be:

$$K_{rr} = 0, \quad K_{rt} = \frac{|\sinh(b_2 t + b_1)|}{2M} (b_2^2 - \bar{b}_2^2),$$

$$K_{tt} = \frac{\mathbf{Re} [\bar{b}_2 \sinh(b_2 t + b_1)]}{|\sinh(b_2 t + b_1)| M} (b_2^2 - \bar{b}_2^2),$$

where

$$\begin{aligned} M^2 &= 2|b_2|^2 \cosh r \sinh r \operatorname{Re} [\cosh(b_2 t + b_1)] \\ &\quad - 4|b_2|^2 \sinh^2 r (|\cosh(b_2 t + b_1)|^2 + 1) \\ &\quad - 2|b_2|^2 |\cosh(b_2 t + b_1)|^2 \\ &\quad - |\sinh(b_2 t + b_1)|^2 (b_2^2 + \bar{b}_2^2) - 2|b_2|^2. \end{aligned}$$

Computing the mean curvature of S , we get

$$H = \bar{g}^{ab} K_{ab} = 0,$$

and hence every geodesic in $\mathbf{L}(\mathbf{H}^3)$ is a ruled minimal surface in \mathbf{H}^3 .

In addition, we see that the second fundamental form vanishes when $b_2 \in \mathbf{R}$ or $b_2 \in i\mathbf{R}$. As noted earlier, the length of the tangent vector to the geodesic is $i(\bar{b}_2^2 - b_2^2)/4$, and so the geodesic in $\mathbf{L}(\mathbf{H}^3)$ is null if and only if the ruled surface S in \mathbf{H}^3 is totally geodesic. \square

Note. We can also parameterize the geodesics in \mathbf{H}^3 so that the closest point to the origin in the ball model is at affine parameter $r = 0$. In this case, the map $\Phi : \mathbf{L}(\mathbf{H}^3) \times \mathbf{R} \rightarrow \mathbf{H}^3$ is $((\mu_1, \mu_2), r) \mapsto (y^1, y^2, y^3)$, where

$$\begin{aligned} y^1 + iy^2 &= \frac{\mu_2(1+|\mu_1|^2)e^r - \mu_1(1+|\mu_2|^2)e^{-r}}{(1+|\mu_1|^2)(1+|\mu_2|^2) \cosh r + |1+\mu_1\bar{\mu}_2|\sqrt{(1+|\mu_1|^2)(1+|\mu_2|^2)}} \\ y^3 &= \frac{(1+|\mu_1|^2)(1-|\mu_2|^2)e^r - (1+|\mu_2|^2)(1-|\mu_1|^2)e^{-r}}{2[(1+|\mu_1|^2)(1+|\mu_2|^2) \cosh r + |1+\mu_1\bar{\mu}_2|\sqrt{(1+|\mu_1|^2)(1+|\mu_2|^2)}}]. \end{aligned}$$

REFERENCES

1. N. Georgiou and B. Guilfoyle, *A characterization of Weingarten surfaces in hyperbolic 3-space*, preprint, (2007) [math.DG/0709.2441].
2. B. Guilfoyle and W. Klingenberg, *An indefinite Kähler metric on the space of oriented lines*, J. London Math. Soc. **72** (2005), 497–509.
3. ———, *A neutral Kähler metric on the space of time-like lines in Lorentzian 3-spaces*, preprint, (2006) math.DG/0608782.
4. ———, *A neutral Kähler surface with applications in geometric optics*, in *Recent developments in pseudo-Riemannian geometry*, D.V. Alekseevsky and H. Baum, eds., European Mathematical Society Publishing House, to appear.
5. N.J. Hitchin, *Monopoles and geodesics*, Comm. Math. Phys. **83** (1982), 579–602.

6. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Volume II, Wiley and Sons, New York, 1996.
7. M. Kokubu, et al., *Singularities of flat fronts in hyperbolic space*, Pacific J. Math. **221** (2005), 303–352.
8. J.G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer-Verlag, New York, 1994.
9. P. Roitman, *Flat surfaces in hyperbolic space as normal surfaces to a congruence of geodesics*, Tohoku Math. J. **59** (2007), 21–37.
10. M. Salvai, *On the geometry of the space of oriented lines in Euclidean space*, Manuscr. Math. **118** (2005), 181–189.
11. ———, *On the geometry of the space of oriented lines of hyperbolic space*, Glasgow Math. J. **49** (2007), 357–366.
12. A. Small, *Surfaces of constant mean curvature 1 in \mathbf{H}^3 and algebraic curves on a quadric*, Proc. Amer. Math. Soc. **122** (1994), 1211–1220.
13. K. Weierstrass, *Untersuchungen über die Flächen, deren mittlere Krümmung überall gleich Null ist*, Monats. Akad. Wiss. Berlin (1866), 612–625.
14. E.T. Whittaker, *On the partial differential equations of mathematical physics*, Math. Ann. **57** (1903), 333–355.
15. D. Zwillinger, *Handbook of differential equations*, Academic Press, New York, 1989.

DEPARTMENT OF MATHEMATICS, INSTITUTE OF TECHNOLOGY, TRALEE, CLASH,
TRALEE, CO. KERRY, IRELAND
Email address: nikos.georgiou@research.ittralee.ie

DEPARTMENT OF MATHEMATICS, INSTITUTE OF TECHNOLOGY, TRALEE, CLASH,
TRALEE, CO. KERRY, IRELAND
Email address: brendan.guilfoyle@ittralee.ie