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A REGULARIZED TRACE FORMULA FOR DIFFERENTIAL EQUATIONS WITH TRACE CLASS OPERATOR COEFFICIENTS

INCI ALBAYRAK, KEVSER KOKLU AND AZAD BAYRAMOV

ABSTRACT. We will obtain a formula for the regularized trace of the differential equation of the second order with trace class operator coefficient.

1. Introduction. Let H be a separable Hilbert space. In the Hilbert space $H_1 = L_2([0,\pi], H)$, we consider the self-adjoint operator L generated by the differential expression

$$\ell(y) = -y''(x) + Q(x)y(x)$$

and the boundary conditions

$$y(0) = 0, \quad y^{\pi}(p) = 0. \tag{1.1}$$

Suppose that the operator function $Q(x)$ in the expression $\ell(y)$ satisfies the following conditions:

(1) For all $x \in [0,\pi]$ $\mathcal{Q}(x) : H \to H$ is a self-adjoint nuclear operator. Moreover, $Q(x)$ has continuous derivative of second order with respect to the norm in space $\sigma_a(H)$ in the interval $[0,\pi]$ and for $x \in [0,\pi]$, $\mathcal{Q}^{(i)}(x) : H \to H$ are selfadjoint operators $(i = 1, 2)$.

(2) $\|\mathcal{Q}\|_{H_1} < 1$.

(3) There is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of the space H such that

$$\sum_{n=1}^{\infty} \|\mathcal{Q}(x)\varphi_n\|_{H_1} < \infty.$$

Here $\sigma_1(H)$ denotes the space of the nuclear operators from H to H, as in Gorbachuk et al. [**7**]. Let L_o be the operator generated by the differential expression $\ell_o(y) = -y''(x)$ and the boundary conditions (1.1).

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The spectrum of this operator L_o is the set $\{(m-(1/2))^2\}_{m=1}^{\infty}$, and every point of this set is an eigenvalue of L_o with infinite multiplicity. The orthonormal eigenfunctions corresponding to the eigenvalue have the form

$$\Psi_{mn}^{o}=\sqrt{\frac{2}{\pi}}\sin\left(m-\frac{1}{2}\right)x\cdot\varphi_n,\quad n=1,2,\ldots. \tag{1.2}$$

In this paper, we investigate the spectrum and regularized trace of the operator L. Gelfand and Levitan [**6**] first obtained a trace formula for a selfadjoint Sturm-Liouville differential equation. After this study several mathematicians were interested in developing trace formulas for different differential operators. The current situation of this subject and studies related to it have been given in the comprehensive survey paper[**11**].

Note that the trace formulas are used in the inverse problems of spectral analysis of differential equations (see, for example [**11**]) and have applications in the approximate calculation of eigenvalues of the related operator [**3, 4**].

The trace formulas of the abstract self-adjoint operators with continuous spectrum were first analyzed by Krein [**9**]. In this work, he also proved the formula mathematically, which was obtained earlier [**10**] through physical theories in quantum statistics and crystal theory. The trace formulas related to the Sturm-Liouville problem with bounded selfadjoint operator given an infinite interval and having a continuous spectrum were considered in [**1, 2**].

Note that Faddev's study of the regularized trace formula for the Sturm-Liouville equation with the matrix coefficient in [**5**] has been predecessor for [**1, 2**].

2. Investigation of the spectrum. Let R_λ^o and R_λ be the resolvents of the operators L_o and L, respectively.

Lemma 2.1. *If $\mathcal{Q}(x)$ satisfies the condition* (3) *and $\lambda\in\rho(L_o)$, then $\mathcal{Q}R_\lambda^o : H_1\to H_1$ is a nuclear operator: $\mathcal{Q}R_\lambda^o\in\sigma_1(H_1)$.*

Proof. The system (1.2) of eigenfunctions of the operator L_o is an orthonormal basis of the space H_1. Then, as shown in Gorbachuk et al. [**8**], it is sufficient to show that the series

$$
\begin{aligned}
(2.1)\quad & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\|\mathcal{Q}R_{\lambda}^{0}\Psi_{mn}^{o}\right\|_{H_1} \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(m-\frac{1}{2}\right)^2-\lambda\right|^{-1}\|\mathcal{Q}\Psi_{mn}^{o}\|_{H_1} \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(m-\frac{1}{2}\right)^2-\lambda\right|^{-1} \\
&\qquad\qquad \times\left[\int_0^{\pi}\frac{2}{\pi}\sin^2\left(m-\frac{1}{2}\right)x\|\mathcal{Q}(x)\varphi_n\|_H^2\,dx\right]^{1/2} \\
&\le \sqrt{\frac{2}{\pi}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(m-\frac{1}{2}\right)^2-\lambda\right|\|\mathcal{Q}(x)\varphi_n\|_{H_1} \\
&\le C_\lambda\sum_{m=1}^{\infty}m^{-2}\sum_{n=1}^{\infty}\|\mathcal{Q}(x)\varphi_n\|_{H_1}
\end{aligned}
$$

is a convergent series in order to prove that $QR_{\lambda}^{0}\in\sigma_1(H_1)$. Here C_λ is a positive constant related only to λ. By virtue of condition (3) from (2.1) we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|\mathcal{Q}R_{\lambda}^{o}\Psi_{mn}^{o}\|_{H_1}<\infty.$$

The lemma is proved. □

Theorem 2.1. *If* $Q(x)$ *satisfies conditions* (1)–(3), *then the spectrum of operator* L *is a subset of the union of intervals*

$$\Omega_m=\left[\left(m-\frac{1}{2}\right)^2-\|\mathcal{Q}\|_{H_1},\ \left(m-\frac{1}{2}\right)^2+\|Q\|_{H_1}\right],\quad m=1,2,\dots,$$

which are pairwise disjoint and

(a) *Every point different from* $(m-(1/2))^2$, *belonging to the interval* Ω_m *of the spectrum of operator* L, *is a discrete eigenvalue, whose multiplicity is finite.*

(b) $(m-(1/2))^2$ *may be an eigenvalue of the operator* L, *whose multiplicity is finite or infinite.*

(c) *The equality* $\lim_{n\to\infty}\lambda_{mn} = (m-(1/2))^2$ *holds. Here* $\{\lambda_{mn}\}_{n=1}^{\infty}$ *are the eigenvalues, belonging to the interval* Ω_m *of the operator* L*, and each eigenvalue has been repeated according to multiplicity.*

Proof. The resolvent R_λ of operator L satisfies the equation

$$R_\lambda^o - R_\lambda \mathcal{Q} R_\lambda^o = R_\lambda. \tag{2.2}$$

If $\lambda \in R\setminus \cup_{m=1}^{\infty}\Omega_m$, then

$$\left|\lambda - \left(m-\frac{1}{2}\right)^2\right| > \|\mathcal{Q}\|_{H_1}, \quad m=1,2,\dots . \tag{2.3}$$

For the self-adjoint operator $R_\lambda^o = (L_o - \lambda I)^{-1}$,

$$\|R_\lambda^o\|_{H_1} = \max_m \left|\lambda - \left(m-\frac{1}{2}\right)^2\right|^{-1}$$

holds. From here and (2.3), we obtain

$$\|R_\lambda^o\|_{H_1} < \|\mathcal{Q}\|_{H_1}^{-1}.$$

Hence,

$$\|\mathcal{Q}R_\lambda^o\|_{H_1} \le \|\mathcal{Q}\|_{H_1}\|R_\lambda^o\|_{H_1} < 1.$$

Thus, $A(B) = R_\lambda^o - B\mathcal{Q}R_\lambda^o$ is a contraction operator from $\mathcal{L}(H_1)$ to $\mathcal{L}(H_1)$. Here $\mathcal{L}(H_1)$ is the linear bounded operator space from H_1 to H_1. According to this, $A(R_\lambda) = R_\lambda$, that is, equation (2.2) has a single solution $R_\lambda \in \mathcal{L}(H_1)$. Thus, every point $\lambda \notin \cup_{m=1}^{\infty}\Omega_m$ is the regular point of the self-adjoint operator L. So, the spectrum of operator L is $\sigma(L)\subset \cup_{m=1}^{\infty}\Omega_m$. From formula (2.2) and Lemma 2.1, for every $\lambda \in \rho(L)\cap\rho(L_0)$, $R_\lambda - R_\lambda^o$ belongs to $\sigma_1(H_1)$, that is, $R_\lambda - R_\lambda^o$ is the nuclear operator. In this case, as was proved in Kato [**8**, page 244], the continuous parts of the spectra of operators L_o and L coincide. According to this, and since the spectrum of operator L_o is continuous, the continuous part of the spectra operator L is the set $\{(m-(1/2))^2\}_{m=1}^{\infty}$. This also means that assertions (a), (b) and (c) of Theorem 2.1 are satisfied.

3. A formula for the regularized trace. Let $\{\Psi_{mn}\}_{m,n=1}^{\infty}$ be the orthonormal eigenfunctions corresponding to the eigenvalues $\{\lambda_{mn}\}_{m,n=1}^{\infty}$ of the operator L and

$$\Gamma_p = \left\{\lambda, \left|\lambda - \left(p - \frac{1}{2}\right)^2\right| = 1\right\},$$

$$B_{mn}^0 = (\cdot, \Psi_{mn}^o)_{H_1}\Psi_{mn}^o, \qquad B_{mn} = (\cdot, \Psi_{mn})_{H_1}\Psi_{mn},$$

$$L_{om}^{(r)} = \sum_{n=1}^{\infty}\left(m - \frac{1}{2}\right)^{2r} B_{mn}^o, \qquad L_m^{(r)} = \sum_{\substack{n=1 \\ \lambda_{mn}\neq 0}}^{\infty} \lambda_{mn}^r B_{mn}, \quad r = -1, 1.$$

Theorem 3.1. *If the operator function* $Q(x)$ *satisfies conditions* (2) *and* (3), *then the series*

$$\sum_{n=1}^{\infty}\left(\lambda_{pn} - \left(p - \frac{1}{2}\right)^2\right), \quad p = 1, 2, \dots,$$

are absolutely convergent series.

Proof. The difference $R_\lambda - R_\lambda^o$ satisfies the following formula

$$R_\lambda - R_\lambda^o = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}}{\lambda_{mn} - \lambda} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}^o}{(m - (1/2))^2 - \lambda}, \tag{3.1}$$

since, except for $(p - (1/2))^2$ and the $\{\lambda_{pn}\}_{n=1}^{\infty}$ eigenvalues of the operators L_0 and L, all their eigenvalues are outside of the circle Γ_p. From the last formula, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\Gamma_p}\lambda(R_\lambda - R_\lambda^0 \\ &= \sum_{n=1}^{\infty}\left(B_{pn}^0\frac{1}{2\pi i}\int_{\Gamma_p}\frac{\lambda d\lambda}{\lambda - (p - (1/2))^2} - B_{pn}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{\lambda d\lambda}{\lambda - \lambda_{pn}}\right) \\ &= \sum_{n=1}^{\infty}\left(\left(p - \frac{1}{2}\right)^2 B_{pn}^0 - \lambda_{pn}B_{pn}\right) = L_{op}^{(1)} - L_p^{(1)}. \end{aligned} \tag{3.2}$$

Since the operator function $R_\lambda - R_\lambda^0$ is analytic with respect to the norm in the $\sigma_1(H_1)$ space in the region $\rho(L)$, from (3.2) we obtain

$$L_p^{(1)} - L_{op}^{(1)} \in \sigma_1(H_1), \quad p = 1, 2, \dots . \tag{3.3}$$

Similarly, it can be shown that

$$L_p^{(-1)} - L_{op}^{(-1)} \in \sigma_1(H_1), \quad p = 1, 2, \dots . \tag{3.4}$$

Since the operator L may only have negative eigenvalues of finite number, in order to prove the theorem, it is necessary to show that

$$\sum_{\substack{n \\ \lambda_{pn} > 0}} \left| \lambda_{pn} - \left(m - \frac{1}{2} \right)^2 \right| < \infty, \quad p = 1, 2, \dots .$$

For this reason, we shall accept in the following that $\lambda_{pn} > 0$, $p, n = 1, 2, \dots$. Since the spectrum of operator $L_{op}^{(r)}$ is the set $\{0, (p - (1/2))^{2r}\}$, we have

$$\left(p - \frac{1}{2} \right)^{2r} \ge (L_{op}^{(r)} \Psi_{pn}, \Psi_{pn})_{H_1}, \qquad \lambda_{pn}^r = (L_p^{(r)} \Psi_{pn}, \Psi_{pn})_{H_1}$$

$$\begin{aligned} \sum_{\substack{n \\ \lambda_{pn}^r > (p-(1/2))^{2r}}} & \left(\lambda_{pn}^r - \left(p - \frac{1}{2} \right)^{2r} \right) \\ & \le \sum_{\substack{n \\ \lambda_{pn}^r > (p-(1/2))^{2r}}} ((L_p^{(r)} - L_{op}^{(r)}) \Psi_{pn}, \Psi_{pn})_{H_1} \\ & \le \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \left((L_p^{(r)} - L_{op}^{(r)}) \Psi_{mn}, \Psi_{mn} \right)_{H_1} \right| \\ & \le \| L^{(r)})p - L_{op}^{(r)} \|_{\sigma_1(H_1)}. \end{aligned}$$

Using formulas (3.3) and (3.4), from the above inequalities, we find

$$\sum_{\substack{n \\ \lambda_{pn} > (p-(1/2))^2}} \left(\lambda_{pn} - \left(p - \frac{1}{2} \right)^2 \right) < \infty.$$

$$\sum_{\substack{n \\ \lambda_{pn} > (p-(1/2))^2}} \left(\left(p - \frac{1}{2}\right)^2 - \lambda_{pn}\right)$$
$$\leq \text{const} \sum_{\substack{n \\ \lambda_{pn} > (p-(1/2))^2}} \left(\left(p - \frac{1}{2}\right)^2 - \lambda_{pn}\right)\left(p - \frac{1}{2}\right)^{-2} \lambda_{pn}^{-1}$$
$$= \text{const} \sum_{\substack{n \\ \lambda_{pn} > (p-(1/2))^2}} \left(\lambda_p^{-1} - \left(p - \frac{1}{2}\right)^{-2}\right) < \infty.$$

From the last relations, we obtain

$$\sum_{n=1}^{\infty} \left|\lambda_{pn} - \left(p - \frac{1}{2}\right)^2\right| < \infty, \quad p = 1, 2, \dots .$$

The theorem is proved. □

Since the operator function $R_\lambda - R_\lambda^0$ belongs to $\sigma_1(H_1)$ for every $\lambda \in \rho(\lambda)$ from formula (3.1) and Theorem 3.1, we have

$$\operatorname{tr}(R_\lambda - R_\lambda^{\cdot}) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{(m - (1/2))^2 - \lambda}\right).$$

Multiplying both sides of this equality by $\lambda/(2\pi)$ and integrating over the circle $|\lambda| = b_p = (p - (1/2))^2 + p$, we have

$$(3.5) \quad \frac{1}{2\pi i} \int_{|\lambda| = b_p} \lambda \operatorname{tr}(R_\lambda - R_\lambda^o)\, d\lambda$$
$$= \frac{1}{2\pi i} \int_{|\lambda| = b_p} \lambda \sum_{m=1}^{p} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{(m - (1/2))^2 - \lambda}\right) d\lambda$$
$$+ \frac{1}{2\pi i} \int_{|\lambda| = bp} \lambda \sum_{m=p+1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{(m - (1/2))^2 - \lambda}\right) d\lambda.$$

For $m \leq p$ and $p \geq 1$, by condition (2),

$$\left(m - \frac{1}{2}\right)^2 - \|\mathcal{Q}\|_{H_1} \leq \lambda_{mn} \leq \left(m - \frac{1}{2}\right)^2 + \|\mathcal{Q}\|_{H_1} < \left(m - \frac{1}{2}\right)^2 + p = b_p.$$

Hence,

$$|\lambda_{mn}| < b_p, \quad m \le p, \quad p \ge 1; \quad n = 1, 2, \dots . \tag{3.6}$$

Moreover, for $m > p$,

$$\lambda_{mn} \ge \left(m - \frac{1}{2}\right)^2 - \|Q\|_{H_1} \ge \left(p + 1 - \frac{1}{2}\right)^2 - \|Q\|_{H_1} > \left(p - \frac{1}{2}\right)^2 + p = b_p. \tag{3.7}$$

Using (3.5), (3.6) and (3.7),

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \mathrm{tr}\,(R_\lambda - R_\lambda^o)\, d\lambda = \sum_{m=1}^{p}\sum_{n=1}^{\infty}\left(\left(m - \frac{1}{2}\right)^2 - \lambda_{mn}\right). \tag{3.8}$$

On the other hand, from the formula $R_\lambda = R_\lambda^o - R_\lambda \mathcal{Q} R_\lambda^o$, the equality

$$R_\lambda - R_\lambda^o = \sum_{j=1}^{N}(-1)^j R_\lambda^o(\mathcal{Q}R_\lambda^o)^j + (-1)^{N+1} R_\lambda(\mathcal{Q}R_\lambda^o)^{N+1} \tag{3.9}$$

is obtained for every natural number N. From (3.8) and (3.9), we have

$$\begin{aligned}\sum_{m=1}^{p}\sum_{n=1}^{\infty}&\left(\left(m - \frac{1}{2}\right)^2 - \lambda_{mn}\right)\\ &= \sum_{j=1}^{N}\frac{(-1)^j}{2\pi i}\int_{|\lambda|=b_p}\lambda \mathrm{tr}\,\left[R_\lambda^o(\mathcal{Q}R_\lambda^o)^j\right]\, d\lambda\\ &\quad + \frac{(-1)^{N+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda \mathrm{tr}\,\left[R_\lambda(\mathcal{Q}R_\lambda^o)^{N+1}\right]\, d\lambda.\end{aligned} \tag{3.10}$$

Let

$$M_p^j = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda \mathrm{tr}\,\left[R_\lambda^o(\mathcal{Q}R_\lambda^o)^j\right]\, d\lambda, \tag{3.11}$$

$$M_{pN} = \frac{(-1)^{N}}{2\pi i}\int_{|\lambda|=b_p}\lambda \mathrm{tr}\,\left[R_\lambda(\mathcal{Q}R_\lambda^o)^{N+1}\right]\, d\lambda. \tag{3.12}$$

Then from (3.10), (3.11) and (3.12) we have

$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left(\lambda_m - \left(m - \frac{1}{2}\right)^2\right) = \sum_{j=1}^{N} M_p^j + M_{pN}. \tag{3.13}$$

Now we shall compute the right side of (3.13). Since the operator function $\mathcal{Q}R_\lambda^0$ in the domain $\mathbf{C}\setminus\{(m-(1/2))^2\}_{m=1}^\infty$ is analytic with respect to the norm in $\sigma_1(H_1)$, one can show that for M_p^j the following formula is true

$$M_p^j = \frac{(-1)^j}{2\pi i j}\int_{|\lambda|=b_p} \operatorname{tr}(\mathcal{Q}R_\lambda^o)^j\,d\lambda. \tag{3.14}$$

From (1.2) and (3.14), we have

$$\begin{aligned}
M_p^1 &= -\frac{1}{2\pi i}\int_{|\lambda|=b_p} \operatorname{tr}(\mathcal{Q}R_\lambda^o)\,d\lambda\\
&= -\frac{1}{2\pi i}\int_{|\lambda|=b_p}\sum_{m=1}^\infty\sum_{n=1}^\infty (\mathcal{Q}R_\lambda^o\Psi_{mn}^o,\Psi_{mn}^o)_{H_1}\,d\lambda\\
&= \sum_{m=1}^\infty\sum_{n=1}^\infty (\mathcal{Q}\Psi_{mn}^o,\Psi_{mn}^o)_{H_1}\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{\lambda-(m-(1/2))^2}\\
&= \sum_{m=1}^p\sum_{n=1}^\infty (\mathcal{Q}\Psi_{mn}^o,\Psi_{mn}^o)_{H_1}\\
&= \sum_{m=1}^p\sum_{n=1}^\infty \frac{2}{\pi}\int_0^\pi (\mathcal{Q}(x)\varphi_n,\varphi_n)\sin^2\left(m-\frac{1}{2}\right)x\,dx\\
&= \frac{1}{\pi}\sum_{m=1}^p\sum_{n=1}^\infty\int_0^\pi (\mathcal{Q}(x)\varphi_n,\varphi_n)(1-\cos(2m-1)x)\,dx\\
&= \frac{p}{\pi}\int_0^\pi \operatorname{tr}Q(x)\,dx\\
&\quad -\frac{1}{\pi}\sum_{m=1}^p\int_0^\pi \operatorname{tr}\mathcal{Q}(x)\cos(2m-1)x\,dx.
\end{aligned}\tag{3.15}$$

Theorem 3.2. *If the operator function $Q(x)$ satisfies conditions* (1)–(3), *then*

$$\lim_{p\to\infty} = M_p^j = 0,\quad j\ge 2 \tag{3.16}$$

$$\lim_{p\to\infty} = M_{pN} = 0,\quad N\ge 4. \tag{3.17}$$

Proof. For $j = 2$ from (3.13), we write

(3.18)
$$\begin{aligned} M_p^2 &= \frac{1}{4\pi i}\int_{|\lambda|=b_p} \operatorname{tr}(\mathcal{Q}R_\lambda^o)^2\, d\lambda \\ &= \frac{1}{4\pi i}\int_{|\lambda|=b_p} \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}((\mathcal{Q}R_\lambda^o)^2\Psi_{mn}^0, \Psi_{mn}^0)_{H_1}\right] d\lambda. \end{aligned}$$

Moreover, we know that

$$\mathcal{Q}R_\lambda^o\Psi_{mn}^o = \frac{\mathcal{Q}\Psi_{mn}^o}{(m-(1/2))^2-\lambda}$$

and
(3.19)
$$\begin{aligned} (\mathcal{Q}R_\lambda^o)^2\Psi_{mn}^o &= \left(\left(m-\frac{1}{2}\right)^2-\lambda\right)^{-1}\mathcal{Q}R_\lambda^o\mathcal{Q}\Psi_{mn}^o \\ &= \left(\left(m-\frac{1}{2}\right)^2-\lambda\right)^{-1}\mathcal{Q}R_\lambda^0\left\{\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}(\mathcal{Q}\Psi_{mn}^o, \Psi_{rq})_{H_1}\Psi_{rq}^o\right\} \\ &= \left(\left(m-\frac{1}{2}\right)^2-\lambda\right)^{-1}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\left(\left(r-\frac{1}{2}\right)^2-\lambda\right)^{-1} \\ &\qquad\times(\mathcal{Q}\Psi_{mn}^o, \psi_{rq}^o)_{H_1}\mathcal{Q}\Psi_{rq}^o. \end{aligned}$$

From (3.18) and (3.19), we have
(3.20)
$$M_p^2 = \frac{1}{4\pi i}\int_{|\lambda|=bp}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{(\mathcal{Q}\Psi_{mn}^o,\Psi_{rq}^o)_{H_1}(\mathcal{Q}\Psi_{rq}^o,\Psi_{mn}^o)_{H_1}}{(\lambda-(m-(1/2))^2)(\lambda-(r-(1/2))^2)}\right]d\lambda.$$

It is easy to verify that, for $m, r \le p$,

(3.21)
$$\int_{|\lambda|=bp}\frac{d\lambda}{(\lambda-(m-(1/2))^2)(\lambda-(r-(1/2))^2)} = 0.$$

This formula is true also for $m, r > p$. Then, from (3.20) and (3.21), we have

$$M_p^2 = \frac{1}{2\pi i}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|(\mathcal{Q}\Psi_{mn}^{o},\Psi_{rq}^{o})_{H_1}\right|^2$$
$$\times\int_{|\lambda|=b_p}\left(\lambda-\left(m-\frac{1}{2}\right)^2\right)^{-1}\left(\lambda-\left(r-\frac{1}{2}\right)^2\right)^{-1}d\lambda$$
$$=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left(\left(m-\frac{1}{2}\right)^2-\left(r-\frac{1}{2}\right)^2\right)^{-1}$$
$$\times\left|(\mathcal{Q}\Psi_{mn}^{o},\Psi_{rq}^{o})_{H_1}\right|^2,$$

which implies that

$$(3.22)\qquad \begin{aligned}|M_p^2| &\le \sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left(\left(r-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2\right)^{-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\\ &\qquad\times\left|(\mathcal{Q}\Psi_{rq}^{o},\Psi_{mn}^{o})_{H_1}\right|^2\\ &=\sum_{r=p+1}^{\infty}\left(\left(r-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2\right)^{-1}\sum_{q=1}^{\infty}\|\mathcal{Q}\Psi_{rq}^{o}\|_{H_1}^2.\end{aligned}$$

Using (1.2) and condition (3) we estimate expression $\sum_{q=1}^{\infty}\|\mathcal{Q}\Psi_{rq}^{o}\|_{H_1}^2$.

$$(3.23)\qquad \begin{aligned}\sum_{q=1}^{\infty}\|\mathcal{Q}\Psi_{rq}^{o}\|_{H_1}^2 &= \sum_{q=1}^{\infty}\int_0^{\pi}\left\|\mathcal{Q}(x)\sqrt{\frac{2}{\pi}}\sin\left(r-\frac{1}{2}\right)x\varphi_q\right\|_H^2 dx\\ &\le\sum_{q=1}^{\infty}\int_0^{\pi}\|\mathcal{Q}(x)\varphi_q\|_H^2\,dx\\ &=\sum_{q=1}^{\infty}\|\mathcal{Q}(x)\varphi_q\|_{H_1}^2 < C,\end{aligned}$$

where C is a positive constant. From (3.22) and (3.23), we have

$$|M_p^2|\le C\sum_{r=p+1}^{\infty}\left(\left(r-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2\right)^{-1}.$$

It can be shown that the inequality

$$\sum_{r=p+1}^{\infty}\left(\left(r-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2\right)^{-1}<2\left(p-\frac{1}{2}\right)^{-1/2} \tag{3.24}$$

is true. From the last two inequalities, we have

$$\lim_{p\to\infty} M_p^2 = 0.$$

In a similar form it can be proved that

$$\lim_{p\to\infty} M_p^3 = 0.$$

We shall now prove formula (3.16) for $j \geq 4$. For this we estimate the expression $\|\mathcal{Q}R_\lambda^o\|_{\sigma_1(H_1)}$ on the circle $|\lambda| = b_p$. As shown in [**10**],

$$\|\mathcal{Q}R_\lambda^o\|_{\sigma_1(H_1)} \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\|\mathcal{Q}R_\lambda^o\Psi_{mn}^o\|_{H_1}.$$

By (2.1) and condition (3), we get

$$\begin{aligned}
\|\mathcal{Q}R_\lambda^o\|_{\sigma_1(H_1)} &\leq C\sum_{m=1}^{\infty}\left|\left(m-\frac{1}{2}\right)^2-\lambda\right|^{-1} \\
&\leq C\Bigg(\sum_{m=1}^{p}\left(|\lambda|-\left(m-\frac{1}{2}\right)^2\right)^{-1} \\
&\qquad + \sum_{m=p+1}^{\infty}\left(\left(m-\frac{1}{2}\right)^2-|\lambda|\right)^{-1}\Bigg) \\
&= C\Bigg(\sum_{m=1}^{p}\left(\left(p-\frac{1}{2}\right)^2+p-\left(m-\frac{1}{2}\right)^2\right)^{-1} \\
&\qquad + \sum_{m=p+1}^{\infty}\left(\left(m-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2-p\right)^{-1}\Bigg) \\
&< C\Bigg(1+\sum_{m=p+1}^{\infty}\Bigg(\frac{1}{2}\left(\left(m-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2\right)
\end{aligned}$$

$$+\frac{1}{2}\left(\left(\left(m-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2\right)-p\right)^{-1}\right)$$

$$< C\left(1+\sum_{m=p+1}^{\infty}\left(\frac{1}{2}\left(\left(m-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2\right)\right.\right.$$

$$\left.\left.+\frac{1}{2}\left(\left(p+1-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2-p\right)^{-1}\right)\right.$$

$$< C\left(1+\sum_{m=p+1}^{\infty}2\left(\left(m-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2\right)^{-1}\right).$$

From here and (3.24), we have

$$\|\mathcal{Q}R_\lambda^o\|_{\sigma_1(H)} < C_1, \qquad |\lambda| = b_p = \left(p-\frac{1}{2}\right)^2+p, \quad C_1>0. \tag{3.25}$$

Now we estimate $\|R_\lambda^o\|_{H_1}$ on the circle $|\lambda| = b_p$. For $m \le p$,

$$\left|\left(m-\frac{1}{2}\right)^2-\lambda\right| \ge p,$$

and, for $m \ge p+1$,

$$\left|\left(m-\frac{1}{2}\right)^2-\lambda\right| \ge \left(m-\frac{1}{2}\right)^2-|\lambda| \ge \left(p+1-\frac{1}{2}\right)^2-\left(p-\frac{1}{2}\right)^2-p \ge p.$$

Hence,

$$\left|\left(m-\frac{1}{2}\right)^2-\lambda\right|^{-1}, \qquad |\lambda| = b_p = \left(p-\frac{1}{2}\right)^2+p. \tag{3.26}$$

On the other hand,

$$\|R_\lambda^o\|_{H_1} = \max_m\left\{\left|\left(m-\frac{1}{2}\right)^2-\lambda\right|^{-1}\right\}.$$

From here and (3.26), we have

$$\|R_\lambda^o\|_{H_1} < p^{-1}. \tag{3.27}$$

Using Theorem 2.1 and condition (3) it can be shown that, on the circle $|\lambda| = b_p$, for sufficiently large p,

$$\|R_\lambda\|_{H_1} < C_2 p^{-1}, \qquad C_2 > 0, \tag{3.28}$$

is true. From (3.14), (3.25) and (3.27), and since $Q(x)$ satisfies condition (2), we have

$$\begin{aligned}
|M_p^j| &= \frac{1}{2\pi j}\left|\int_{|\lambda|=b_p} \operatorname{tr}(\mathcal{Q}R_\lambda^o)^j \, d\lambda\right| \\
&\le \frac{1}{2\pi j}\int_{|\lambda|=b_p} \|(\mathcal{Q}R_\lambda^o)^j\|_{\sigma_1(H_1)}|d\lambda| \\
&\le \frac{1}{2\pi j}\int_{|\lambda|=b_p} \|\mathcal{Q}R_\lambda^o\|_{\sigma_1(H_1)}\|(\mathcal{Q}R_\lambda^o)^{j-1}\|_{H_1}|d\lambda| \\
&\le C_1\int_{|\lambda|=b_p} \|\mathcal{Q}\|_{H_1}^{j-1}\|R_\lambda^o\|_{H_1}^{j-1}|d\lambda| \\
&< C_1 2^{1-j}\int_{|\lambda|=b_p} p^{1-j}|d\lambda| < C_3 p^{3-j}, \quad C_3 > 0.
\end{aligned}$$

From here, we get

$$\lim_{p\to\infty} M_p^j = 0, \quad j \ge 4,$$

and so formula (3.16) is proved.

We now prove formula (3.17). From (3.12), (3.25), (3.27) and (3.28), we have

$$\begin{aligned}
|M_{pN}| &= \frac{1}{2\pi}\left|\int_{|\lambda|=b_p} \lambda \operatorname{tr}\left[R_\lambda(\mathcal{Q}R_\lambda^o)^{N+1}\right] d\lambda\right| \\
&\le \int_{|\lambda|=b_p} |\lambda| \left|\operatorname{tr}\left[R_\lambda(\mathcal{Q}R_\lambda^o)^{N+1}\right]\right| |d\lambda| \\
&\le b_p\int_{|\lambda|=b_p} \left\|R_\lambda(\mathcal{Q}R_\lambda^o)^{N+1}\right\|_{\sigma_1(H_1)} |d\lambda| \\
&\le b_p\int_{|\lambda|=b_p} \|R_\lambda\|_{H_1} \left\|(\mathcal{Q}R_\lambda^o)^{N+1}\right\|_{\sigma_1(H_1)} |d\lambda| \\
&\le C_2 b_p p^{-1}\int_{|\lambda|} = b_p\|QR_\lambda\|_{H_1}^N\|\mathcal{Q}R_\lambda^o\|_{\sigma_1(H_1)} |d\lambda| \\
&\le C_2 b_p p^{-1}\|\mathcal{Q}\|_{H_1}^N p^{-N} C_1 2\pi b_p \le C_4 p^{3-N}, \quad C_4 > 0.
\end{aligned}$$

From here, we get

$$\lim_{p\to\infty} M_{PN} = 0, \quad N \geq 4.$$

The main result of this article is given by the following theorem.

Theorem 3.3. *If the operator function $Q(x)$ satisfies conditions* (1)–(3) *then*

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}-\left(m-\frac{1}{2}\right)^2\right)-\frac{1}{\pi}\int_0^{\pi}\operatorname{tr}\mathcal{Q}(x)\,dx\right]=\frac{1}{4}[\operatorname{tr}\mathcal{Q}(\pi)-\operatorname{tr}\mathcal{Q}(0)].$$

The series on the left side of this equality is called the regularized trace of operator L.

Proof. From (3.13), (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned}
&\lim_{p\to\infty}\sum_{m=1}^{p}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}-\left(m-\frac{1}{2}\right)^2\right)-\frac{p}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx\right]\\
&=-\frac{1}{\pi}\lim_{p\to\infty}\int_0^{\pi}\operatorname{tr}\mathcal{Q}(x)\cos(2m-1)x\,dx\\
&=-\frac{1}{2\pi}\sum_{m=1}^{\infty}\left[\int_0^{\pi}\operatorname{tr}\mathcal{Q}(x)\cos mx\,dx-(-1)^m\int_0^{\pi}\operatorname{tr}\mathcal{Q}(x)\cos mx\,dx\right]\\
&=-\frac{1}{4}\sum_{m=1}^{\infty}\left\{\left[\int_0^{\pi}\operatorname{tr}\mathcal{Q}(x)\sqrt{\frac{2}{\pi}}\cos mx\,dx\right]\sqrt{\frac{2}{\pi}}\cos m.0\right.\\
&\quad\left.-\left[\int_0^{\pi}\operatorname{tr}\mathcal{Q}(x)\sqrt{\frac{2}{\pi}}\cos mx\,dx\right]\sqrt{\frac{2}{\pi}}\cos m\pi\right\}\\
&=-\frac{1}{4}[\operatorname{tr}\mathcal{Q}(0)-\operatorname{tr}\mathcal{Q}(\pi)].
\end{aligned}$$

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Yildiz Technical University, Faculty of Chemistry and Metallurgy, Department of Mathematical Engineering, Davutpasa Campus, 34210, Istanbul, Turkey
Email address: ibayrak@yildiz.edu.tr

Yildiz Technical University, Faculty of Chemistry and Metallurgy, Department of Mathematical Engineering, Davutpasa Campus, 34210, Istanbul, Turkey
Email address: ozkoklu@yildiz.edu.tr

Yildiz Technical University, Faculty of Art and Science, Department of Statistics, Davutpasa 34210, Istanbul, Turkey
Email address: azadbay@gmail.com