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NUMBER-THEORETIC CONDITIONS WHICH YIELD ISOMORPHISMS AND EQUIVALENCES BETWEEN MATRIX RINGS OVER LEAVITT ALGEBRAS

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ABSTRACT. For each integer $n \geq 2$ let L_n denote the Leavitt algebra of order n. We provide number-theoretic descriptions of the relationships between the integers k, k', n, n' for which there are isomorphisms and/or equivalences between the matrix rings $\mathrm{M}_k(L_n)$ and $\mathrm{M}_{k'}(L_{n'})$ possessing various properties. Such properties include: isomorphism (unrestricted), induced isomorphism, graded isomorphism and graded equivalence. These results extend the isomorphism results achieved in [**2**].

Throughout this note K denotes a field. For $n \geq 2$ we denote by $L_K(1,n)$, or simply L_n when appropriate, the *Leavitt algebra of order n with coefficients in K*. $L_K(1,n)$ is the free associative K-algebra with generators $\{x_i, y_i : 1 \leq i \leq n\}$ and relations

$$x_i y_j = \delta_{ij} \text{ for all } 1 \leq i, j \leq n, \quad \text{and} \quad \sum_{i=1}^{n} y_i x_i = 1.$$

(See [**2**] or [**10**] for additional information about L_n.) $R = L_n$ also may be viewed as the K-algebra universal with respect to the property that ${}_RR \cong {}_RR^n$ as left R-modules. Indeed, an important explicit isomorphism $\phi : {}_RR \to {}_RR^n$ is given by

$$\phi(r) = (ry_1, ry_2, \ldots, ry_n), \text{ with inverse } \phi^{-1}((r_1, r_2, \ldots, r_n)) = \sum_{i=1}^{n} r_i x_i$$

for all $r \in R$ and $(r_1, r_2, \ldots, r_n) \in R^n$.

There has been recent sustained interest in Leavitt algebras, for two important reasons. First, connections between the Leavitt algebras and their C*-algebra counterparts, the so-called *Cuntz algebras* $\mathcal{O}_n$,

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have resulted in advances in both fields of study. Second, the Leavitt algebras provide the motivating examples for a more general class of algebras, the Leavitt path algebras, which have received significant attention in both the algebra and C*-algebra communities. (See the remarks prior to Proposition 1.7 below, as well as [**14**] and [**3**], for further details.)

1. Isomorphisms. The free left L_n-module isomorphism $L_n \cong L_n^n$ described above yields directly that for positive integers a, b, if $a \equiv b \pmod{n-1}$ then $L_n^a \cong L_n^b$. (Leavitt shows in [**10**] that in fact the converse is true as well, a fact we shall use later.) By taking endomorphism rings, the free module isomorphism $L_n^a \cong L_n^b$ for $a \equiv b (\mathrm{mod}\, n-1)$ immediately yields an isomorphism of matrix rings $\mathrm{M}_a(L_n) \cong \mathrm{M}_b(L_n)$. In particular, there are isomorphisms between different-sized matrix rings over L_n. With this as a motivating observation, the question then arises:

Question 1. When are two matrix rings over Leavitt algebras isomorphic?

Question 1 becomes quite natural, and perhaps even more compelling, in light of the question posed in [**13**], which can be rephrased as

Question 1′. When are two matrix rings over Cuntz algebras isomorphic?

While the answer to Question 1′ was established in [**13**], the answer provided there did not yield an explicit description of the germane isomorphisms. However, the following pair of results have been recently established in [**2**].

Theorem ([**2**, Theorem 4.14]). *Let k, n be positive integers, and K any field. Then $L_K(1,n) \cong \mathrm{M}_k(L_K(1,n))$ if and only if $\gcd(k, n-1) = 1$. In this case, isomorphisms $L_K(1,n) \mapsto \mathrm{M}_k(L_K(1,n))$ are explicitly described.*

Theorem ([**2**, Theorem 5.1]). *Let k, n be positive integers. Then $\mathcal{O}_n \cong \mathrm{M}_k(\mathcal{O}_n)$ if and only if $\gcd(k, n-1) = 1$. In this case, isomorphisms $\mathcal{O}_n \mapsto \mathrm{M}_k(\mathcal{O}_n)$ are explicitly described.*

Because all of the results presented in this article hold for all fields K, we will use the L_n notation for $L_K(1, n)$ throughout the sequel. For each $m \in \mathbf{N}$ let $\mathbf{Z}_m$ denote the ring $\mathbf{Z}/m\mathbf{Z}$, and let $U(\mathbf{Z}_m)$ denote the group of units of $\mathbf{Z}_m$. Using [**2**, Theorem 4.14] as the key ingredient, it is demonstrated (although not explicitly so stated) in the proof of [**2**, Theorem 5.2] that the following more general result indeed holds.

Theorem 1.1. (Answer to Question 1.) *Let k, k', n, n' be positive integers with $n, n' \geq 2$. Then*

$$\mathrm{M}_k(L_n) \cong \mathrm{M}_{k'}(L_{n'})$$

if and only if

$$n = n' \text{ and } k \equiv k'l (\operatorname{mod} n-1) \text{ for some } l \in U(\mathbf{Z}_{n-1}).$$

For any group G and any G-graded ring S the matrix ring $\mathrm{M}_k(S)$ is naturally G-graded by setting $(\mathrm{M}_k(S))_g = (\mathrm{M}_k(S_g))$. For any rings S and T graded by a group G, if there is a graded isomorphism $\tau : S \to T$ (notationally: $S \cong^{gr} T$), then τ extends componentwise to yield a graded isomorphism $\tau_k : \mathrm{M}_k(S) \to \mathrm{M}_k(T)$, for any $k \in \mathbf{N}$. Clearly the standard isomorphism $\mathrm{M}_{kl}(S) \mapsto \mathrm{M}_k(\mathrm{M}_l(S))$ preserves this grading for any $l \in \mathbf{N}$.

The Leavitt algebra L_n is $\mathbf{Z}$-graded, as follows. It is easily shown that monomials of the form $y_{i_1} y_{i_2} \cdots y_{i_t} \cdot x_{j_1} x_{j_2} \cdots x_{j_u}$ span L_n as a K-space. We define

$$\deg(y_{i_1} y_{i_2} \cdots y_{i_t} \cdot x_{j_1} x_{j_2} \cdots x_{j_u}) = u - t,$$

and extend K-linearly to all of L_n. This grading gives the same $\mathbf{Z}$-grading on L_n as that induced by setting $\deg(X_i) = 1$, $\deg(Y_i) = -1$ in the free associative K-algebra $T = K\langle X_1, \ldots, X_n, Y_1, \ldots, Y_n\rangle$, and then grading the factor ring $L_n = T/I$ in the natural way. (We note that the relations which define L_n are homogeneous in this grading of T.) In particular, any matrix ring over L_n inherits a $\mathbf{Z}$-grading from L_n in this way. Furthermore, it was shown in [**1**] that, in this grading,

$$(L_n)_0 \cong \varinjlim_{u\in\mathbf{N}}(\mathrm{M}_{n^u}(K)).$$

Here the connecting homomorphisms are unital (so that the direct limit is unital); the homomorphism from $\mathrm{M}_{n^u}(K)$ to $\mathrm{M}_{n^{u+1}}(K)$ is given by sending any matrix of the form $(a_{i,j})$ to the matrix $(a_{i,j}I_n)$.

Let S be any ring, and let N, N' be left S-modules. A ring isomorphism $\Theta : \mathrm{End}_S(N) \to \mathrm{End}_S(N')$ is called *induced* in case there exists a left-module isomorphism $\theta : N \to N'$ such that, for every $f \in \mathrm{End}_S(N)$, $\Theta(f) = \theta \circ f \circ \theta^{-1}$. We write $\mathrm{End}_S(N) \cong^{\mathrm{ind}} \mathrm{End}_S(N')$ to indicate the existence of an induced isomorphism $\Theta : \mathrm{End}_S(N) \to \mathrm{End}_S(N')$.

The goal of this section is to give the number-theoretic solution to each of the following four additional questions.

Question 2. When are two matrix rings over Leavitt algebras isomorphic via a graded isomorphism?

Question 3. When are two matrix rings over Leavitt algebras isomorphic via an induced isomorphism?

Question 4. When are two matrix rings over Leavitt algebras isomorphic via a sequence of graded and/or induced isomorphisms?

Question 5. When are two matrix rings over Leavitt algebras isomorphic via both a graded isomorphism and an induced isomorphism?

As we shall see, the answers to Questions 1 through 5 are each different from the other. Because by Theorem 1 we cannot have an isomorphism between $\mathrm{M}_k(L_n)$ and $\mathrm{M}_{k'}(L_{n'})$ with $n \neq n'$, there is no loss of generality in restricting our attention here to isomorphisms between matrix rings of the form $\mathrm{M}_k(L_n)$ and $\mathrm{M}_{k'}(L_n)$.

Definition 1.2. For positive integers k, n we write

$$k = td \text{ where } d \mid n^i \text{ for some positive integer } i, \text{ and g.c.d.}(t, n) = 1.$$

We will refer to such a factorization as the *factorization of k along n.*

Of course, this definition describes nothing more than the factorization of k into prime powers, grouped in such a way that d is the product of those prime powers for primes dividing n, and t is the product of those prime powers for primes coprime to n. To answer Question 2, we first recall

Proposition [**2**, Proposition 6.3]. *The algebras L_n and $\mathrm{M}_k(L_n)$ are isomorphic as $\mathbf{Z}$-graded algebras if and only if there exists $\alpha \in \mathbf{N}$ such that $k \mid n^\alpha$.*

We extend this result to answer Question 2 as follows.

Proposition 1.3. (Answer to Question 2). *Let k, k', n be positive integers, $n \geq 2$. Let $k = td$, respectively $k' = t'd'$, be the factorization of k, respectively k', along n. Then*

$$\mathrm{M}_k(L_n) \cong^{gr} \mathrm{M}_{k'}(L_n)$$

if and only if

$$t = t'.$$

Proof. If $t = t'$ then twice using [**2**, Proposition 6.3] together with an observation made previously we have

$$\begin{aligned} \mathrm{M}_k(L_n) = \mathrm{M}_{td}(L_n) &\cong^{gr} \mathrm{M}_t(\mathrm{M}_d(L_n)) \cong^{gr} \mathrm{M}_t(L_n) \\ &\cong^{gr} \mathrm{M}_t(\mathrm{M}_{d'}(L_n)) \cong^{gr} \mathrm{M}_{td'}(L_n) = \mathrm{M}_{k'}(L_n). \end{aligned}$$

Conversely, suppose $\mathrm{M}_k(L_n) \cong^{gr} \mathrm{M}_{k'}(L_n)$. Then again using [**2**, Proposition 6.3] we get $\mathrm{M}_t(L_n) \cong^{gr} \mathrm{M}_{t'}(L_n)$. But such a graded isomorphism then restricts to an isomorphism of the respective zero-components, so that we get $(\mathrm{M}_t(L_n))_0 \cong (\mathrm{M}_{t'}(L_n))_0$, which by a previous remark yields

$$\varinjlim_{u \in \mathbf{N}} (\mathrm{M}_{t \cdot n^u}(K)) \cong \varinjlim_{u \in \mathbf{N}} (\mathrm{M}_{t' \cdot n^u}(K)).$$

That $t = t'$ now follows from [**18**, 2.1(b)] (utilizing the notion of *generalized integers*, also known as *Steinitz integers* or *supernatural integers*). (See also [**7**, Theorem 15.26] and [**18**, Section 2] for more information about isomorphisms between these direct limit rings.) □

We note here that the existence of *some* graded isomorphism between $\mathrm{M}_k(L_n)$ and $\mathrm{M}_{k'}(L_n)$ does not imply that *all* isomorphisms between $\mathrm{M}_k(L_n)$ and $\mathrm{M}_{k'}(L_n)$ are necessarily graded. For instance, we have $L_6 = \mathrm{M}_1(L_6) \cong^{gr} \mathrm{M}_4(L_6)$ (since $t = t' = 1$); but an example of a non-graded isomorphism between L_6 and $\mathrm{M}_4(L_6)$ is presented in [**2**, Section 4].

To answer Question 3, we invoke the aforementioned property of L_n verified in [**10**].

Proposition 1.4. (Answer to Question 3). *Let k, k', n be positive integers, $n \geq 2$. Then*

$$\mathrm{M}_k(L_n) \cong^{\mathrm{ind}} \mathrm{M}_{k'}(L_n)$$

if and only if

$$k \equiv k' (\mathrm{mod}\, n-1).$$

Proof. An induced isomorphism between $\mathrm{M}_k(L_n) = \mathrm{End}_{L_n}(L_n^k)$ and $\mathrm{M}_{k'}(L_n) = \mathrm{End}_{L_n}(L_n^{k'})$ implies the existence of an isomorphism between the free left L_n-modules L_n^k and $L_n^{k'}$, which by [**10**, Theorem 1] occurs precisely when $k \equiv k' (\mathrm{mod}\, n-1)$. □

In order to answer Question 4, we develop some group theory.

Definition 1.5. Let $m \in \mathbf{N}$. We define $G(\mathbf{Z}_m)$ by setting

$$G(\mathbf{Z}_m) = \{a \in \mathbf{Z}_m \mid \exists b \in \mathbf{N} \text{ for which } a \equiv b (\mathrm{mod}\, m) \text{ and } b \text{ divides } (m+1)^j \text{ for some } j \in \mathbf{N}\}.$$

Noting that the condition "b divides $(m+1)^j$ for some $j \in \mathbf{N}$" implies that g.c.d.$(b, m) = 1$, the following is straightforward to prove.

Lemma 1.6. *For every* $m \in \mathbf{N}$, $G(\mathbf{Z}_m)$ *is a subgroup of* $U(\mathbf{Z}_m)$.

The congruence condition given in the definition of $G(\mathbf{Z}_m)$ is non-trivial. For instance, if $a = 7$ and $m = 9$, then clearly a does not divide any power of $m + 1 = 10$, but $7 \equiv 16 (\mathrm{mod}\, m)$ and 16 divides $(m+1)^4$. Thus $7 \in G(\mathbf{Z}_9)$. Indeed, $G(\mathbf{Z}_9) = U(\mathbf{Z}_9)$. In contrast, we observe for instance that $G(\mathbf{Z}_4) = \{1\}$, since any divisor of $(4+1)^j$ is a power of 5, and thus is congruent to $1 (\mathrm{mod}\, 4)$. Thus $G(\mathbf{Z}_4) \neq U(\mathbf{Z}_4) = \{1, 3\}$. The question "for which integers m do we have $G(\mathbf{Z}_m) = U(\mathbf{Z}_m)$?" is one of interest in the number theory community, and has connections to the Artin conjecture for primitive roots (see, e.g., [**9**]).

The existence of explicit isomorphisms between L_n and $\mathrm{M}_k(L_n)$ when $k \mid n^j$ for some $j \in \mathbf{N}$ was observed in [**1**, Proposition 2.1], although it was not until the appearance of [**2**] that it was realized these yield the graded isomorphisms between matrix rings over Leavitt algebras. In turn, these isomorphisms arose as the counterparts of some analogous isomorphisms between matrix rings over the previously-mentioned Cuntz algebras $\mathcal{O}_n$ established in the mid-1970's in [**12**, Proposition 2.5]. These graded isomorphisms, together with the induced isomorphisms, and compositions of such, provide a large collection of isomorphisms between matrix rings over Leavitt algebras. Indeed, for many (but not all) values of n, the existence of an isomorphism between $\mathrm{M}_k(L_n)$ and $\mathrm{M}_{k'}(L_n)$ implies the existence of an isomorphism made up of compositions of graded and induced isomorphisms. On the other hand, for other values of n, it is possible to have $\mathrm{M}_k(L_n) \cong \mathrm{M}_{k'}(L_n)$ without the existence of such a composite isomorphism. We show now that the subgroup $G(\mathbf{Z}_{n-1})$ of $U(\mathbf{Z}_{n-1})$ determines exactly when this happens.

Theorem 1.7. (Answer to Question 4). *Let* k, k', n *be positive integers,* $n \geq 2$. *Let* $k = td$, *respectively* $k' = t'd'$, *be the factorization of* k, *respectively* k', *along* n. *Then*

there exists an isomorphism between $M_k(L_n)$ *and* $M_{k'}(L_n)$ *which is the composition of graded and/or induced isomorphisms*

if and only if

$$t \equiv t'g (\mathrm{mod}\, n-1) \text{ for some } g \in G(\mathbf{Z}_{n-1}).$$

Proof. Suppose $t \equiv t'g(\mathrm{mod}\, n-1)$ for some $g \in G(\mathbf{Z}_{n-1})$. Then by definition of $G(\mathbf{Z}_{n-1})$ there exists $g' \in \mathbf{N}$ so that $g \equiv g'(\mathrm{mod}\, n-1)$ and $g' \mid n^j$ for some $j \in \mathbf{N}$. In particular $t \equiv t'g'(\mathrm{mod}\, n-1)$. Now using Propositions 2 and 3 appropriately, we get a sequence of isomorphisms

$$\begin{aligned} \mathrm{M}_k(L_n) &\cong^{gr} \mathrm{M}_t(\mathrm{M}_d(L_n)) \cong^{gr} \mathrm{M}_t(L_n) \cong^{ind} \mathrm{M}_{t'g'}(L_n) \\ &\cong^{gr} \mathrm{M}_{t'}(\mathrm{M}_{g'}(L_n)) \\ &\cong^{gr} \mathrm{M}_{t'}(L_n) \cong^{gr} \mathrm{M}_{t'}(\mathrm{M}_{d'}(L_n)) \cong^{gr} \mathrm{M}_{k'}(L_n), \end{aligned}$$

as desired.

Conversely, suppose such a sequence of isomorphisms exists. That is, there exists a sequence of positive integers $k = k_0, k_1, \dots, k_r = k'$ for which $\mathrm{M}_{k_i}(L_n) \cong \mathrm{M}_{k_{i+1}}(L_n)$ for all $0 \leq i \leq r-1$, via an isomorphism which is either graded or induced. We argue by induction on r. For $r = 1$ we have two cases. If $\mathrm{M}_{k_0}(L_n) \cong^{gr} \mathrm{M}_{k_1}(L_n)$, then $t_0 = t_1$ by Proposition 2, since obviously $t_0 \equiv t_1 \cdot 1(\mathrm{mod}\, n-1)$ and $1 \in G(\mathbf{Z}_{n-1})$, so the result holds. Now suppose $\mathrm{M}_{k_0}(L_n) \cong^{ind} \mathrm{M}_{k_1}(L_n)$. Then $k_0 \equiv k_1(\mathrm{mod}\, n-1)$, so that $t_0 d_0 \equiv t_1 d_1(\mathrm{mod}\, n-1)$. But $d_0, d_1 \in G(\mathbf{Z}_{n-1})$ by definition; and since each is invertible in $\mathbf{Z}_{n-1}$ this gives $t_0 \equiv t_1 d_1 d_0^{-1}(\mathrm{mod}\, n-1)$, so by defining $g_1 = d_1 d_0^{-1} \in G(\mathbf{Z}_{n-1})$ the result again holds.

Now assume the result holds for $i < r$, and show the result follows for $i+1$. By induction we assume that $t_0 \equiv t_i g_i$ for some $g_i \in G(\mathbf{Z}_{n-1})$. As above, we consider two cases. If $\mathrm{M}_{k_i}(L_n) \cong^{gr} \mathrm{M}_{k_{i+1}}(L_n)$ then $t_i = t_{i+1}$ by Proposition 2, so that $t_0 \equiv t_i g_i \equiv t_{i+1} g_i(\mathrm{mod}\, n-1)$, so the result holds. In the other case, suppose $\mathrm{M}_{k_i}(L_n) \cong^{ind} \mathrm{M}_{k_{i+1}}(L_n)$. Then $k_i \equiv k_{i+1}(\mathrm{mod}\, n-1)$, so that $t_i d_i \equiv t_{i+1} d_{i+1}(\mathrm{mod}\, n-1)$. But, arguing as before, this gives $t_i \equiv t_{i+1} d_{i+1} d_i^{-1}(\mathrm{mod}\, n-1)$, so that

$$t_0 \equiv t_i g_i \equiv t_{i+1} d_{i+1} d_i^{-1} g_i(\mathrm{mod}\, n-1).$$

Defining $g_{i+1} = d_{i+1} d_i^{-1} g_i \in G(\mathbf{Z}_{n-1})$ completes the argument. □

As a consequence of Theorem 1.7, we conclude that the isomorphisms constructed in [**2**] in order to answer Question 1 cannot in general be made up of compositions of graded and induced isomorphisms. For instance, Theorem 1.1 yields that $L_5 \cong \mathrm{M}_3(L_5)$, since g.c.d. $(3, 5-1) = 1$. However, Theorem 1.7 shows that these two algebras cannot be isomorphic via a sequence of graded and/or induced isomorphisms, since $t = 1$, $t' = 3$, and $G(\mathbf{Z}_4) = \{1\}$. (See [**2**, Section 2] for further details.)

With the answers to Questions 2 and 3 in hand, it is of course easy to answer Question 5. We record that answer below, and then provide some additional information about a special case. In order to analyze the special case, we now explicitly describe isomorphisms between specific matrix rings over Leavitt algebras. Let S_n denote the set $\{1, 2, \ldots, n\}$, and for $j \in \mathbf{N}$ let S_n^j denote the direct product of j copies of S_n. For $I = (i_1, i_2, \ldots, i_j) \in S_n^j$, let x_I denote $x_{i_1} x_{i_2} \cdots x_{i_j} \in L_n$; analogously, let y_I denote $y_{i_1} y_{i_2} \cdots y_{i_j} \in L_n$. We order S_n^j lexicographically.

There is an involution $*$ on L_n defined by setting $x_i^* = y_i$, $y_i^* = x_i$, and extending appropriately. The involution $*$ then lifts to an involution on any matrix ring over L_n, where we set $(r_{i,j})^* = (r_{i,j}^*)^t$ for each $(r_{i,j}) \in \mathrm{M}_k(L_n)$.

Lemma 1.8. (1) (Special case of Answer to Question 2). *Suppose $q \mid n$. We define a graded isomorphism $\Psi_q : L_n \to \mathrm{M}_q(L_n)$, as follows. For each $1 \leq i \leq n$ write $i = (n/q)v + r$ with $1 \leq r \leq (n/q)$. (Note then that necessarily $0 \leq v \leq q - 1$.) We define $\Psi_q(x_i) = X_i$, where X_i is the $q \times q$ matrix whose $(v+1)^{st}$ column is*

$$\begin{pmatrix} x_{(r-1)q+1} \\ x_{(r-1)q+2} \\ \vdots \\ x_{rq} \end{pmatrix}$$

and whose remaining entries are zero. We then define $\Psi_q(y_i) = X_i^*$.

In particular, if $q = n$, then $\Psi_n(x_i)$ is the matrix whose ith column is

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and whose remaining entries are zero.

(2) (Special case of Answer to Question 3). *Let R denote L_n, and let $j \in \mathbf{N}$. Let $\phi : {}_RR \to {}_RR^n$ denote the free module isomorphism given above, and let $\phi^{n^j} : {}_RR \to {}_RR^{n^j}$ denote the isomorphism generated by j iterations of the isomorphism ϕ in each component. Then the isomorphism $\Phi^{n^j} : R \to \mathrm{M}_{n^j}(R)$ which is induced from ϕ^{n^j} is explicitly given by:*

$$\Phi^{n^j}(r) = (r_{I,J}), \text{ where } r_{I,J} = x_I r y_J \text{ for all } I, J \in S_n^j.$$

In particular,

$$\Phi^n(r) = (r_{i,j}), \text{ where } r_{i,j} = x_i r y_j \text{ for all } 1 \leq i, j \leq n.$$

(3) $\Psi_n = \Phi^n$.

Proof. (1) The proof is given (with slightly different notation, and in the context of matrix rings over Cuntz algebras) in [**12**, Proposition 2.5]. It is not hard to verify directly that this map yields a surjective algebra homomorphism. The injectivity follows from the fact established in [**11**] that L_n is a simple ring for all $n \geq 2$.

(2) It is tedious to verify the case of interest ($j = 1$) by using the explicit description of ϕ given in the introduction. The general case follows by a straightforward induction argument.

(3) By (2), for each $1 \leq k \leq n$ we have $\Phi^n(x_k) = (x_i x_k y_j)$, which in turn equals $(x_i \delta_{k,j})$ by the definition of multiplication in L_n, which yields precisely $\Psi_n(x_k)$. A similar observation holds for each y_k, which establishes the result. □

For clarity, we provide an example of an isomorphism which arises in Lemma 1.8 (1).

Example 1.9. Let $n = 6, q = 3$. Then $\Psi_3 : L_6 \to \mathrm{M}_3(L_6)$ is the isomorphism defined by setting $\Psi_3(x_i) = X_i$ for $1 \leq i \leq 6$, where

$$X_1 = \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} x_4 & 0 & 0 \\ x_5 & 0 & 0 \\ x_6 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 0 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0 & x_4 & 0 \\ 0 & x_5 & 0 \\ 0 & x_6 & 0 \end{pmatrix} \quad X_5 = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ 0 & 0 & x_3 \end{pmatrix} \quad X_6 = \begin{pmatrix} 0 & 0 & x_4 \\ 0 & 0 & x_5 \\ 0 & 0 & x_6 \end{pmatrix}$$

By defining $\Psi_3(y_i) = (\Psi_3(x_i))^*$ for $1 \le i \le 6$ we then get $\Psi_3 : L_6 \to \mathrm{M}_3(L_6)$ by extending additively and multiplicatively.

Proposition 1.10. (1) (Answer to Question 5). *Let k, k', n be positive integers, $n \geq 2$. Let $k = td$, respectively $k' = t'd'$, be the factorization of k, respectively k', along n. Then*

there exists both a graded isomorphism as well as an induced isomorphism between $M_k(L_n)$ and $M_{k'}(L_n)$

if and only if

$$t = t' \text{ and } k \equiv k' (\operatorname{mod} n-1).$$

(2) (Special case of Answer to Question 5). *Moreover, in the particular case where $k' = kn^j$ for some $j \in \mathbf{N}$, then there exists an isomorphism between $\mathrm{M}_k(L_n)$ and $\mathrm{M}_{k'}(L_n)$ which itself is both graded and induced.*

Proof. Statement (1) follows directly from Propositions 1.3 and 1.4.

For (2), we argue as follows. Let R denote L_n, and let $\phi : {}_RR \to {}_RR^n$ be the explicit isomorphism described above. Using the description given in Lemma 1.8 (2) it is clear that the induced isomorphism $\Phi^{n^j} : R \to \mathrm{M}_{n^j}(R)$ is in fact graded as well. Thus, the result holds for $k = 1$. But it is then easy to show that for any $k \in \mathbf{N}$ the corresponding isomorphism $\Phi^{kn^j} : \mathrm{M}_k(R) \to \mathrm{M}_{kn^j}(R)$ given by setting $\Phi^{kn^j}((r_{i,\ell})) = (\Phi^{n^j}(r_{i,\ell}))$ is both graded and induced, which gives the result. □

Clearly if $k' = kn^j$ for some $j \in \mathbf{N}$, then the pair $\{k, k'\}$ satisfy the conditions of Proposition 1.10 (1). However, there are in general many other pairs of integers which satisfy the conditions of Proposition 1.10 (1), but for which no such j exists. Indeed, for any positive integer n we define $T_n \subseteq \mathbf{N}$ by setting

$$T_n = \{a \in \mathbf{N} \mid a \equiv 1 (\operatorname{mod} n-1) \text{ and } a \mid n^j \text{ for some } j \in \mathbf{N}\}.$$

Then for any $a \in T_n$ and any $k \in \mathbf{N}$ it is easy to show that the pair $\{k, ka\}$ satisfies these conditions. In general we may have elements of T_n which are not powers of n; for instance, $16 \in T_6$. We do not know whether Proposition 1.10 (2) can be extended to all pairs $\{k, k'\}$ for which $k' = ka$ for some $a \in T_n$.

2. Equivalences. In this second section we investigate Morita equivalences between matrix rings over Leavitt algebras. It is shown in [**4**] that $K_0(L_n) \cong \mathbf{Z}_{n-1}$, where $K_0(R)$ denotes the Grothendieck group of the ring R. Since K_0 is a Morita invariant of a ring, and any ring R is Morita equivalent to the matrix ring $\mathrm{M}_k(R)$ for each $k \in \mathbf{N}$, we get immediately

Proposition 2.1. *Let k, k', n, n' be positive integers with $n, n' \geq 2$. Then*

$$\mathrm{M}_k(L_n) \text{ is Morita equivalent to } \mathrm{M}_{k'}(L_{n'})$$

if and only if

$$n = n'.$$

So on the surface the study of Morita equivalences between matrix rings over Leavitt algebras seems uninteresting. However, as with isomorphisms, one can ask about the *graded* relationship between such matrix rings. For rings R and S graded by a group G, R and S are *graded equivalent* in case the categories $R-gr$ and $S-gr$ are equivalent. Additionally, R and S are *graded Morita equivalent* in case there is a progenerator P of $R-$Mod for which P happens to also be graded, and for which $S \cong^{gr} \mathrm{End}_R(P)$. (See [**15, 16, 17**] for additional information on graded equivalences and graded Morita equivalences, especially in the context where $G = \mathbf{Z}$.)

As done in Section 1, we assume that the grading on any matrix ring is the one arising from the grading on the individual components. A ring R graded by a group G is called *strongly graded* in case $R_g R_{g^{-1}} = R_e$ for all $g \in G$. (Here $R_g R_{g^{-1}}$ denotes sums of elements of the form $r_g s_{g^{-1}}$ with $r, s \in R$, $r_g \in R_g$, $s_{g^{-1}} \in R_{g^{-1}}$, and e denotes the identity element of G.) If R is strongly graded then it is easy to show that $\mathrm{M}_k(R)$ is strongly graded as well.

Lemma 2.2. *For each $n \geq 2$, L_n is strongly* **Z***-graded.*

Proof. Let R denote L_n. By definition of the grading on R, R_0 consists of sums of monomials of the form

$$y_{i_1}y_{i_2}\cdots y_{i_j}\cdot x_{i_1}x_{i_2}\cdots x_{i_{j'}} \text{ for which } j=j'.$$

Pick an arbitrary monomial $r \in R_0$ of this form.

If $z = j$, then clearly $r \in R_z R_{-z}$.

If $z < j$, then $u = j - z > 0$. Now $1 = x_1 y_1$ gives $1 = x_1^u y_1^u$, and we write

$$\begin{aligned} r &= y_{i_1}y_{i_2}\cdots y_{i_j}\cdot 1\cdot x_{i_1}x_{i_2}\cdots x_{i_j} = y_{i_1}y_{i_2}\cdots y_{i_j}\cdot x_1^u y_1^u\cdot x_{i_1}x_{i_2}\cdots x_{i_j} \\ &= y_{i_1}y_{i_2}\cdots y_{i_j}x_1^u\cdot y_1^u x_{i_1}x_{i_2}\cdots x_{i_j} \in R_{j-u}R_{u-j} = R_z R_{-z}. \end{aligned}$$

If $z > j$, then $v = z - j > 0$. Now $1 = \sum_{k=1}^n y_k x_k$, so that

$$\begin{aligned} r &= y_{i_1}y_{i_2}\cdots y_{i_j}\cdot 1\cdot x_{i_1}x_{i_2}\cdots x_{i_j} \\ &= y_{i_1}y_{i_2}\cdots y_{i_j}\cdot\left(\sum_{k=1}^n y_k x_k\right)\cdot x_{i_1}x_{i_2}\cdots x_{i_j} \\ &= \sum_{k=1}^n (y_{i_1}y_{i_2}\cdots y_{i_j}y_k\cdot x_k x_{i_1}x_{i_2}\cdots x_{i_j}) \in R_{j+1}R_{-(j+1)}. \end{aligned}$$

So the result holds for $z = j + 1$. An induction argument using the same substitution $1 = \sum_{k=1}^n y_k x_k$ as appropriate yields that $r \in R_{j+v}R_{-(j+v)}$ for all $v \geq 1$, and we are done. □

Proposition 2.3. *Let k, k', n, n' be positive integers with $n, n' \geq 2$. Then*

$$\mathrm{M}_k(L_n) \textit{ is graded equivalent to } \mathrm{M}_{k'}(L_{n'})$$

if and only if

n and n' have the same set of prime factors.

Proof. By Lemma 2.2 the two rings are strongly **Z**-graded, so that [**6**, Corollary 2.13] applies. In particular, $\mathrm{M}_k(L_n)$ is graded equivalent to

$\mathrm{M}_{k'}(L_{n'})$ if and only if there is an equivalence between the full module categories $(\mathrm{M}_k(L_n))_0 - \mathrm{Mod}$ and $(\mathrm{M}_{k'}(L_{n'}))_0 - \mathrm{Mod}$. But as noted in Section 1, we have an explicit description of each of these zero-components, to wit, we get a Morita equivalence

$$\mathrm{M}_k(\varinjlim_{u\in\mathbf{N}}(\mathrm{M}_{n^u}(K))) \sim \mathrm{M}_{k'}(\varinjlim_{u\in\mathbf{N}}(\mathrm{M}_{(n')^u}(K))).$$

Since matrix rings preserve Morita equivalence, this in turn happens if and only if there is a Morita equivalence

$$\varinjlim_{u\in\mathbf{N}}(\mathrm{M}_{n^u}(K)) \sim \varinjlim_{u\in\mathbf{N}}(\mathrm{M}_{(n')^u}(K)).$$

These direct limit algebras are specific examples of *ultramatricial* algebras; as such, by [**7**, Corollary 15.27] their Morita equivalence classes are exactly determined by their K_0 groups (together with the partial order on these). In our situation these K_0 groups turn out to be the additive groups of the rings $\mathbf{Z}[1/n]$, respectively $\mathbf{Z}[1/n']$. But two such groups are isomorphic precisely when n and n' have the same set of prime factors. □

The condition relating n and n' in the previous proposition in fact arises in a number of contexts (see [**18**] for a substantial list of such places). In particular, this condition arises in C*-algebras, in the context of classifying UHF algebras (see [**8**, Theorem 1.12]).

We conclude this article by observing the following.

Proposition 2.4. *Let k, k', n, n' be positive integers with $n, n' \geq 2$. Then*

$$\mathrm{M}_k(L_n) \textit{ is graded Morita equivalent to } \mathrm{M}_{k'}(L_{n'})$$

if and only if

$$n = n'.$$

Proof. Because the rings $\mathrm{M}_k(L_n)$ and $\mathrm{M}_{k'}(L_{n'})$ are assumed to be Morita equivalent we get $n = n'$ as above. Conversely, we have that L_n is graded Morita equivalent to any ring of the form $\mathrm{M}_k(L_n)$ (with the standard grading), since there is a graded isomorphism $\mathrm{End}_{L_n}(L_n^k) \cong \mathrm{M}_k(L_n)$. The result is now clear. □

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