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AN ALGORITHM TO COMPUTE THE KAUFFMAN POLYNOMIAL OF 2-BRIDGE KNOTS

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ABSTRACT. The 2-bridge knots are a family of knots with bridge number 2 [**1, 13**]. In this paper, we compute the Kauffman polynomial of 2-bridge knots using the Kauffman skein theory and linear algebraic techniques. Our calculation can be easily carried out using Mathematica, Maple, Mathcad, etc.

1. Introduction. The 2-bridge knots (or links) are a family of knots with bridge number 2. A 2-bridge knot (link) has at most 2 components. Except for the knot 8_5, the first 25 knots in the Rolfsen knot table are 2-bridge knots. A 2-bridge knot is also called a *rational knot* because it can be obtained as the numerator or denominator closure of a rational tangle. The rich mathematical aspects of 2-bridge knots can be found in many references such as [**3, 4, 5, 7, 13–15**]. The regular diagram D of a 2-bridge knot can be drawn as shown in Figure 1 [**13**].

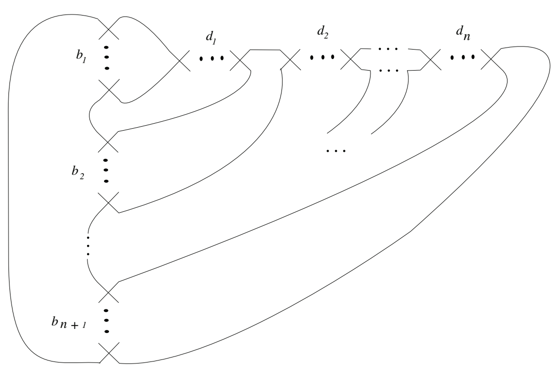

FIGURE 1. Regular diagram D of a 2-bridge knot.

Keywords and phrases. Kauffman skein modules, relative skein modules, Kauffman polynomials, rational knots.
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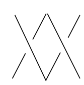
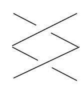
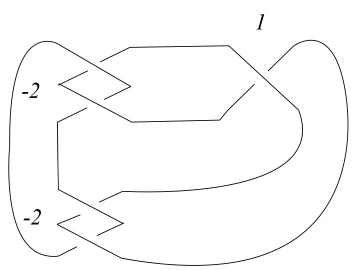
In the diagram, $d_1, d_2, \ldots, d_n$ and $b_1, b_2, \ldots, b_{n+1}$, are nonzero integers whose absolute values indicate the number of crossings. The continued fraction notation for D is $F(D) := [b_1, d_1, b_2, d_2, \ldots, d_n, b_{n+1}]$ [**5**]. We will work with the above diagram to calculate the Kauffman polynomial of 2-bridge knots. Specific information about twists (crossings) is necessary to identify a 2-bridge knot. If d_i (or b_i) is positive, it indicates d_i (or b_i) crossings look this way



(or );

the crossings are changed otherwise.

For example, the Whitehead link is a 2-component 2-bridge link with a regular diagram given by

W= 

The continued fraction notation for the Whitehead link is $F(W) = [-2, 1, -2]$.

Let $\mathbf{Q}(\alpha, s)$ be the field of rational functions in α, s. By a *framed link* we mean an unoriented link equipped with a nonsingular normal vector field up to homotopy. The links described by figures in this paper will be assigned the vertical framing pointing towards the reader.

There are various versions of the *Kauffman polynomial* in the literature [**12**]. Here we adopt the version, sometimes called the *Dubrovnic* polynomial [**8**]. Hence, the Kauffman polynomial of a framed unoriented link L is the unique two-variable rational function $\langle L \rangle$ in α, s defined by the following:

(i) $\langle$  $\rangle - \langle$  $\rangle = (s - s^{-1})(\langle \,)(\, \rangle - \langle$  $\rangle)$, where the four diagrams are exactly the same except near a point where they are shown;

(ii) $\langle$  $\rangle = \alpha \langle \; / \; \rangle$;

(iii) $\langle L \sqcup$  $\rangle = \delta \langle L \rangle$, where $\delta = ((\alpha - \alpha^{-1})/(s - s^{-1}) + 1)$;

(iv) $\langle \varnothing \rangle = 1$, where $\varnothing$ represents the empty link;

(v) $\langle L \rangle$ is unchanged by Reidemeister moves of Type II and III on the diagram of L.

If an oriented link L is represented by a diagram L_1 by forgetting the orientation, the Kauffman polynomial of L is $\langle L \rangle^* = \alpha^{-w(L_1)} \langle L_1 \rangle$, where $w(L_1)$ is the writhe of diagram L_1.

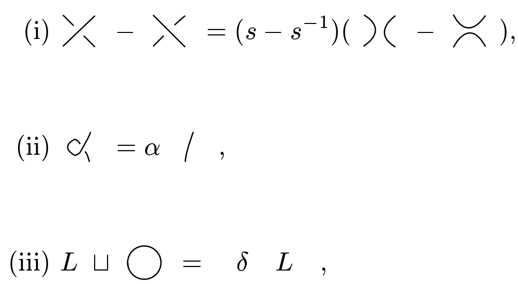
In Section 2, we study the Kauffman skein space of the 3-ball B^3 with possible boundary points. In Section 3, we define linear skein maps on the Kauffman skein space of the 3-ball B^3 with four boundary points and compute the matrices of these linear maps. In Section 4, we present our main theorem of calculating the Kauffman polynomial of a 2-bridge knot by decomposing it as compositions of linear skein maps from Section 3. In Section 5, we calculate the Kauffman polynomial of the Whitehead link as an example.

We acknowledge here that the original idea of calculating link polynomials via linear skein theory is due to Lickorish [**9**], Lickorish and Millet [**10**], while our method of using an orthogonal basis in the skein algebra makes the calculations much easier.

2. The Kauffman skein space of the 3-ball B^3.

2.1. The Kauffman skein space of the 3-ball B^3. The Kauffman skein space [**2, 16**] of the 3-ball B^3, denoted by $K(B^3)$, is the $\mathbf{Q}(\alpha, s)$-

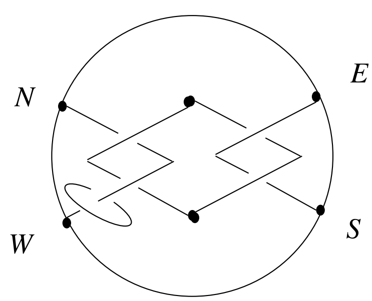
space freely generated by framed isotopic links L in B^3 quotient by the subspace generated by the Kauffman skein relations:



where $\delta = ((\alpha - \alpha^{-1})/(s - s^{-1}) + 1)$.

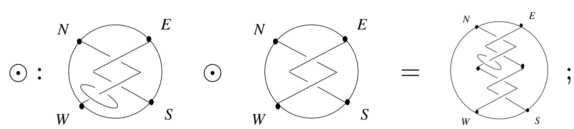
Given any framed link L in B^3, it can be simplified to $\langle L \rangle \varnothing$ by applying the Kauffman skein relation, where $\langle L \rangle$ is the Kauffman polynomial of L. Hence, the Kauffman skein space $K(B^3)$ is generated by the empty link $\varnothing$.

2.2. The Kauffman skein space of B^3 with four boundary points. We place a distinguished set of four coplanar points $\{N, E, S, W\}$ on the sphere S^2, the boundary of the 3-ball B^3. A framed link in $(B^3, NESW)$ is a collection of closed curves and arcs joining the distinguished boundary points N, E, S, W. Two framed links are equivalent if one can be obtained from the other by isotopy. We define the Kauffman skein space $K(B^3, NESW)$ to be the $\mathbf{Q}(\alpha, s)$-space freely generated by framed links L in (B^3, S^2) such that $L \cap S^2 = \partial L = \{N, E, S, W\}$, considered up to an ambient isotopy fixing S^2, quotient by the subspace generated by the Kauffman skein relations. A skein element in $K(B^3, NESW)$ is illustrated below.

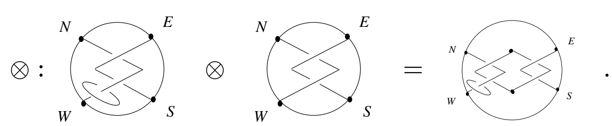


There are two natural multi-linear multiplication operations in $K(B^3, NESW)$:

(1) **Concatenation.** By stacking the first on top of the second through gluing points W, S in the first with N, E in the second,



(2) **Juxtaposition.** By putting two skein elements next to each other through gluing points E, S in the first with N, W in the second,



Note that the skein element $)($ is the identity with respect to the $\odot$ operation, and the skein element $\asymp$ is the identity with respect to the $\otimes$ operation.

The Kauffman skein space $K(B^3, NESW)$ is three-dimensional and has a basis $\{e_1, e_2, e_3\}$ [**2**] given by

$$e_1 = \frac{1}{s+s^{-1}}\left(s^{-1}\ )(\ + \times - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1}) \asymp \right);$$

$$e_2 = \frac{1}{s+s^{-1}}\left(s\ )(\ - \times + (-\delta^{-1}s + \delta^{-1}\alpha^{-1}) \asymp \right);$$

$$e_3 = \delta^{-1} \asymp .$$

In the remaining part of this section, we study properties of these basis elements which are crucial in constructing our calculation techniques.

Proposition 1. *With respect to the* $\odot$ *operation,*

(1) *the basis elements* e_1, e_2, e_3 *are orthogonal, i.e.,*

$$\begin{aligned} e_1 \odot e_2 &= e_2 \odot e_1 = 0, \\ e_1 \odot e_3 &= e_3 \odot e_1 = 0, \\ e_2 \odot e_3 &= e_3 \odot e_2 = 0; \end{aligned}$$

(2) *the basis elements* e_1, e_2 *and* e_3 *are idempotents, i.e.,* $e_1 \odot e_1 = e_1$, $e_2 \odot e_2 = e_2$, $e_3 \odot e_3 = e_3$;

(3) *the basis elements* e_1, e_2, e_3 *add to the identity with respect to the* $\odot$ *operation, i.e.,* $e_1 + e_2 + e_3 =$)(;

(4) *Let* $\sigma =$  . *Then* $\sigma \odot e_1 = e_1 \odot \sigma = se_1$, $\sigma \odot e_2 = e_2 \odot \sigma = -s^{-1}e_2$, $\sigma \odot e_3 = e_3 \odot \sigma = \alpha^{-1}e_3$. *It follows that* $\sigma^{-1} \odot e_1 = e_1 \odot \sigma^{-1} = s^{-1}e_1$, $\sigma^{-1} \odot e_2 = e_2 \odot \sigma^{-1} = -se_2$, $\sigma^{-1} \odot e_3 = e_3 \odot \sigma^{-1} = \alpha e_3$.

Let $\sigma^n_\odot$ represent n copies of σ multiplied through the "$\odot$" multiplication structure, it follows that $\sigma^n_\odot \odot e_1 = e_1 \odot \sigma^n_\odot = s^n e_1$, $\sigma^n_\odot \odot e_2 = e_2 \odot \sigma^n_\odot = (-s^{-1})^n e_2$, $\sigma^n_\odot \odot e_3 = e_3 \odot \sigma^n_\odot = \alpha^{-n} e_3$.

Proof. The proofs follow by linearity of the $\odot$ operation and (repeatedly) applying the Kauffman skein relations and substituting

$$\delta = \left(\frac{\alpha - \alpha^{-1}}{s - s^{-1}} + 1\right).$$

Here we show $e_1 \odot e_2 = 0$ as an example.

$$e_1 \odot e_2 = \frac{1}{(s+s^{-1})^2}\Big(s^{-1}\ )(\ + \times - (\delta^{-1}s^{-1}+\delta^{-1}\alpha^{-1}) \asymp \Big)$$

$$\odot \Big(s\ )(\ - \times + (-\delta^{-1}s+\delta^{-1}\alpha^{-1}) \asymp \Big)$$

$$= \frac{1}{(s+s^{-1})^2}\Big(\ )(\ - s^{-1} \times + s^{-1}(-\delta^{-1}s+\delta^{-1}\alpha^{-1}) \asymp$$

$$+ s \times - \times \odot \times$$

$$+ (-\delta^{-1}s+\delta^{-1}\alpha^{-1})\alpha^{-1} \asymp - (\delta^{-1}s^{-1}+\delta^{-1}\alpha^{-1})s \asymp$$

$$+ (\delta^{-1}s^{-1}+\delta^{-1}\alpha^{-1})\alpha^{-1} \asymp$$

$$- (\delta^{-1}s^{-1}+\delta^{-1}\alpha^{-1})(-\delta^{-1}s+\delta^{-1}\alpha^{-1})\delta \asymp \Big) = 0. \qquad \square$$

If we rotate the basis elements e_1, e_2, e_3 in the plane by 90°, we obtain another basis for $K(B^3, NESW)$. We present the basis elements e_1, e_2, e_3 using subscripts h (vs v) to indicate the basis elements after (vs. before) the rotation:

$$e_{1_v} = e_1 = \frac{1}{s+s^{-1}}(s^{-1}\ )(\ + \times - (\delta^{-1}s^{-1}+\delta^{-1}\alpha^{-1}) \asymp);$$

$e_{2v} = e_2 = \frac{1}{s+s^{-1}}(s$)($-$  $+(-\delta^{-1}s+\delta^{-1}\alpha^{-1})$ );

$e_{3v} = e_3 = \delta^{-1}$  .

$e_{1h} = \frac{1}{s+s^{-1}}(s^{-1}$  $+$  $-(\delta^{-1}s^{-1}+\delta^{-1}\alpha^{-1})$)( );

$e_{2h} = \frac{1}{s+s^{-1}}(s$  $-$  $+(-\delta^{-1}s+\delta^{-1}\alpha^{-1})$)( );

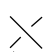
$e_{3h} = \delta^{-1}$)(.

With respect to the $\otimes$ operation and the basis elements e_{1h}, e_{2h} and e_{3h}, similar properties of the basis elements given in Proposition 1 still hold, which we state as a corollary below.

Corollary 1. (1) $e_{1h}\otimes e_{2h} = e_{2h}\otimes e_{1h} = 0$, $e_{1h}\otimes e_{3h} = e_{3h}\otimes e_{1h} = 0$, $e_{2h}\otimes e_{3h} = e_{3h}\otimes e_{2h} = 0$;

(2) $e_{1h}\otimes e_{1h} = e_{1h}$, $e_{2h}\otimes e_{2h} = e_{2h}$, $e_{3h}\otimes e_{3h} = e_{3h}$;

(3))($= e_{1h} + e_{2h} + e_{3h}$;

(4) *let* $\sigma_h =$  . *Then* $\sigma_h\otimes e_{1h} = e_{1h}\otimes\sigma_h = se_{1h}$, $\sigma_h\otimes e_{2h} = e_{2h}\otimes\sigma_h = -s^{-1}e_{2h}$, $\sigma_h\otimes e_{3h} = e_{3h}\otimes\sigma_h = \alpha^{-1}e_{3h}$.

It follows that $\sigma_h{}^n_\otimes \otimes e_{1h} = e_{1h}\otimes\sigma_h{}^n_\otimes = s^n e_{1h}$, $\sigma_h{}^n_\otimes\otimes e_{2h} = e_{2h}\otimes\sigma_h{}^n_\otimes = (-s^{-1})^n e_{2h}$ and $\sigma_h{}^n_\otimes\otimes e_{3h} = e_{3h}\otimes\sigma_h{}^n_\otimes = \alpha^{-n}e_{3h}$, where $\sigma_h{}^n_\otimes$ represents n copies of σ_h multiplied through the "$\otimes$" operation.

The following are additional properties of the basis elements e_{1v}, e_{2v}, e_{3v}, e_{1h}, e_{2h}, e_{3h} with respect to the $\otimes$ operation.

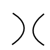
Let M be the 3×3 matrix given by

$$M = \begin{pmatrix} \frac{1}{s+s^{-1}}(s^{-1} - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1}) & \frac{1}{s+s^{-1}}(-s^{-1} - \delta^{-1}s + \delta^{-1}\alpha^{-1}) & \delta^{-1} \\ \frac{1}{s+s^{-1}}(-s - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1}) & \frac{1}{s+s^{-1}}(s - \delta^{-1}s + \delta^{-1}\alpha^{-1}) & \delta^{-1} \\ \frac{1}{s+s^{-1}}(s^{-1}\delta + \alpha - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1}) & \frac{1}{s+s^{-1}}(s\delta - \alpha - \delta^{-1}s + \delta^{-1}\alpha^{-1}) & \delta^{-1} \end{pmatrix}$$

Proposition 2. *For* $1 \leq i,\ j \leq 3$, $e_{i_h} \otimes e_{j_v} = m_{ij} e_{i_h}$.

Proof. Here we prove the case when $i = j = 1$ as an example; the rest can be proved in a similar fashion.

$e_{1_h} \otimes e_{1_v}$

$= \frac{1}{s+s^{-1}} e_{1_h} \otimes (s^{-1}$  $+$  $- (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})$  $)$

$= \frac{1}{s+s^{-1}} (s^{-1} e_{1_h} \otimes$  $+ e_{1_h} \otimes {\sigma_h}^{-1}$

$- (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1}) e_{1_h} \otimes$  $)$

$$= \frac{1}{s+s^{-1}} \left(0 + s^{-1}e_{1_h} - (\delta^{-1}s^{-1} + \delta^{-1}\alpha^{-1})e_{1_h}\right)$$

$$= \frac{1}{s+s^{-1}} (s^{-1} - \delta^{-1}s^{-1} - \delta^{-1}\alpha^{-1}) e_{1_h} = m_{11} e_{1_h}. \qquad \square$$

Remark. Notice that the matrix M is the base change matrix between the basis $\{e_{1_h}, e_{2_h}, e_{3_h}\}$ and $\{e_{1_v}, e_{2_v}, e_{3_v}\}$, i.e.,

$$(e_{1_v}, e_{2_v}, e_{3_v}) = (e_{1_h}, e_{2_h}, e_{3_h})M, \quad (e_{1_h}, e_{2_h}, e_{3_h}) = (e_{1_v}, e_{2_v}, e_{3_v})M.$$

It follows that $M^2 = I$, the 3×3 identity matrix.

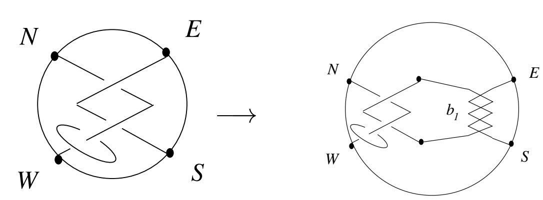
Remark. If we change $\otimes$ to $\odot$ and exchange the subscripts v and h in Proposition 2, the identities still hold. We state these in the next corollary.

Corollary 2. *For* $1 \leq i, j \leq 3$, $e_{i_v} \odot e_{j_h} = m_{ij} e_{i_v}$.

3. Linear skein maps on $K(B^3, NESW)$ and their matrices. A wiring of a space F into another space F' is a choice of inclusion of F into F' and a choice of a set of fixed curves and arcs in $F' - F$. The wiring of F into F' induces a well-defined linear map from the skein space $K(F)$ to $K(F')$ [**11**]. In this section we'll consider four wirings of B^3 into itself, three of which induce linear skein maps $K(B^3, NESW) \to K(B^3, NESW)$, while the fourth one induces a linear skein map $K(B^3, NESW) \to K(B^3)$. Since $K(B^3, NESW)$ and $K(B^3)$ are vector spaces over $\mathbf{Q}(\alpha, s)$, these linear maps are linear transformations of vector spaces. In the following, we choose $\{e_{1h}, e_{2h}, e_{3h}\}$ as the basis of $K(B^3, NESW)$ and represent these linear transformations by matrices with respect to this basis.

3.1. The linear map $B_1(b_1)$ and the matrix $B_1'(b_1)$. Let b_1 be a nonzero integer. Then the linear map $B_1(b_1) : K(B^3, NESW) \to K(B^3, NESW)$ is induced by the following wiring, also called $B_1(b_1)$, for convenience

$B_1(b_1)$:



where b_1 indicates the number of crossings; it is positive if the crossings form lefthand twists, it is negative if the crossings form righthand twists.

Lemma 1. $B_1(b_1)(xe_{1h} + ye_{2h} + ze_{3h}) = x(m_{11}s^{b_1} + m_{12}(-s^{-1})^{b_1} + m_{13}\alpha^{-b_1})e_{1h} + y(m_{21}s^{b_1} + m_{22}(-s^{-1})^{b_1} + m_{23}\alpha^{-b_1})e_{2h} + z(m_{31}s^{b_1} + m_{32}(-s^{-1})^{b_1} + m_{33}\alpha^{-b_1})e_{3h}$.

Proof. Let $s_1 \in K(B^3, NESW) \to K(B^3, NESW)$. Then $B_1(b_1)(s_1) = s_1 \otimes (\sigma_{\odot}^{b_1})$. Note that

$$\begin{aligned}
\sigma_{\odot}^{b_1} &= \sigma_{\odot}^{b_1} \odot \big)\big(= \sigma_{\odot}^{b_1} \odot (e_{1v} + e_{2v} + e_{3v}) \\
&= \sigma_{\odot}^{b_1} \odot e_{1v} + \sigma_{\odot}^{b_1} \odot e_{2v} + \sigma_{\odot}^{b_1} \odot e_{3v} \\
&= s^{b_1} e_{1v} + (-s^{-1})^{b_1} e_{2v} + \alpha^{-b_1} e_{3v}
\end{aligned}$$

from Proposition 1.

Now

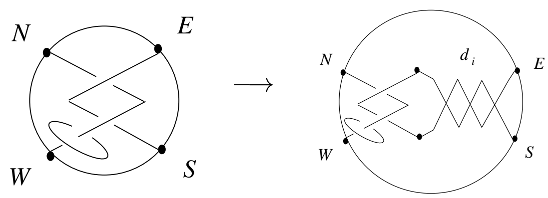
$$\begin{aligned}
&B_1(b_1)(xe_{1h} + ye_{2h} + ze_{3h}) \\
&= (xe_{1h} + ye_{2h} + ze_{3h}) \otimes \sigma_{\odot}^{b_1} \\
&= (xe_{1h} + ye_{2h} + ze_{3h}) \otimes \big(s^{b_1} e_{1v} + (-s^{-1})^{b_1} e_{2v} + \alpha^{-b_1} e_{3v}\big) \\
&= xe_{1h} \otimes \big(s^{b_1} e_{1v} + (-s^{-1})^{b_1} e_{2v} + \alpha^{-b_1} e_{3v}\big) + ye_{2h} \\
&\otimes \big(s^{b_1} e_{1v} + (-s^{-1})^{b_1} e_{2v} + \alpha^{-b_1} e_{3v}\big) + ze_{3h} \\
&\otimes \big(s^{b_1} e_{1v} + (-s^{-1})^{b_1} e_{2v} + \alpha^{-b_1} e_{3v}\big) \\
&= xs^{b_1} e_{1h} \otimes e_{1v} + x(-s^{-1})^{b_1} e_{1h} \otimes e_{2v} + x\alpha^{-b_1} e_{1h} \otimes e_{3v} + ys^{b_1} e_{2h} \\
&\otimes e_{1v} + y(-s^{-1})^{b_1} e_{2h} \otimes e_{2v} + y\alpha^{-b_1} e_{2h} \otimes e_{3v} + zs^{b_1} e_{3h} \\
&\otimes e_{1v} + z(-s^{-1})^{b_1} e_{3h} \otimes e_{2v} + z\alpha^{-b_1} e_{3h} \otimes e_{3v} \\
&= xs^{b_1} m_{11} e_{1h} + x(-s^{-1})^{b_1} m_{12} e_{1h} + x\alpha^{-b_1} m_{13} e_{1h} + ys^{b_1} m_{21} e_{2h} \\
&\quad + y(-s^{-1})^{b_1} m_{22} e_{2h} + y\alpha^{-b_1} m_{23} e_{2h} + zs^{b_1} m_{31} e_{3h} + z(-s^{-1})^{b_1} m_{32} e_{3h} \\
&\quad + z\alpha^{-b_1} m_{33} e_{3h} \\
&= x(m_{11} s^{b_1} + m_{12}(-s^{-1})^{b_1} + m_{13}) e_{1h} \\
&\quad + y(m_{21} s^{b_1} + m_{22}(-s^{-1})^{b_1} + m_{23}\alpha^{-b_1}) e_{2h} \\
&\quad + z(m_{31} s^{b_1} + m_{32}(-s^{-1})^{b_1} + m_{33}\alpha^{-b_1}) e_{3h}. \qquad \square
\end{aligned}$$

We define the corresponding matrix $B_1'(b_1) = (b_{ij})$, $1 \leq i, j \leq 3$, by

$$b_{ij} = \begin{cases} m_{11} s^{b_1} + m_{12}(-s^{-1})^{b_1} + m_{13}\alpha^{-b_1} & \text{if } i = j = 1 \\ m_{21} s^{b_1} + m_{22}(-s^{-1})^{b_1} + m_{23}\alpha^{-b_1} & \text{if } i = j = 2 \\ m_{31} s^{b_1} + m_{32}(-s^{-1})^{b_1} + m_{33}\alpha^{-b_1} & \text{if } i = j = 3 \\ 0 & \text{otherwise.} \end{cases}$$

3.2. The linear map $D(d_i)$ and the matrix $D'(d_i)$. Let d_i be a nonzero integer; the linear map $D(d_i) : K(B^3, NESW) \to K(B^3, NESW)$ is induced by the wiring

$D(d_i)$:



.

Similarly d_i indicates the number of crossings: it is positive if the crossings form lefthand twists; it is negative if the crossings form righthand twists.

Let $D'(d_i)$ be the matrix defined as

$$D'(d_i) = \begin{pmatrix} s^{d_i} & 0 & 0 \\ 0 & (-s^{-1})^{d_i} & 0 \\ 0 & 0 & \alpha^{-d_i} \end{pmatrix}.$$

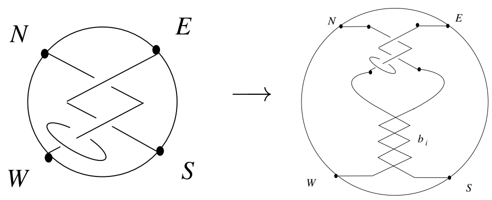
Lemma 2.

$$\begin{aligned} D(d_i)(xe_{1h} + ye_{2h} + ze_{3h}) &= xs^{d_i}e_{1h} + y(-s^{-1})^{d_i}e_{2h} + z\alpha^{-d_i}e_{3h} \\ &= (e_{1h}\ e_{2h}\ e_{3h})D'(d_i)\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

Proof. Note that $\sigma_{h\otimes}^{d_i} = \sigma_{h\otimes}^{d_i} \otimes ($  $) = \sigma_{h\otimes}^{d_i} \otimes (e_{1h} + e_{2h} + e_{3h}) = s^{d_i}e_{1h} + (-s^{-1})^{d_i}e_{2h} + \alpha^{-d_i}e_{3h}$ by the idempotent properties of the basis elements. Now by substitution, $D(d_i)(xe_{1h} + ye_{2h} + ze_{3h}) = (xe_{1h} + ye_{2h} + ze_{3h}) \otimes (\sigma_{h\otimes}^{d_i}) = (xe_{1h} + ye_{2h} + ze_{3h}) \otimes (s^{d_i}e_{1h} + (-s^{-1})^{d_i}e_{2h} + \alpha^{-d_i}e_{3h}) = xs^{d_i}e_{1h} + y(-s^{-1})^{d_i}e_{2h} + z\alpha^{-d_i}e_{3h}$. ◻

3.3. The linear map $B(b_i)$ and the matrix $B'(b_i)$. Let b_i be a nonzero integer. The linear map $B(b_i) : K(B^3, NESW) \to K(B^3, NESW)$ is induced by the wiring

$B(b_i)$:



where b_i indicates the number of crossings. It is positive if the crossings form lefthand twists; it is negative if the crossings form righthand twists.

Let $B'(b_i)$ be the matrix defined by

$$B'(b_i) = M \begin{pmatrix} s^{b_i} & 0 & 0 \\ 0 & (-s^{-1})^{b_i} & 0 \\ 0 & 0 & \alpha^{-b_i} \end{pmatrix} M.$$

Lemma 3. $B(b_i)(xe_{1h}+ye_{2h}+ze_{3h}) = (e_{1h}\ e_{2h}\ e_{3h})MB'(b_i)M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, *where M is the base change matrix between the basis $\{e_{1h}, e_{2h}, e_{3h}\}$ and $\{e_{1v}, e_{2v}, e_{3v}\}$.*

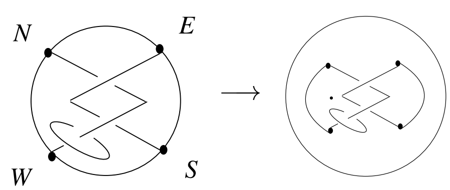
Proof. Note $\sigma_{\odot}^{b_i} = \sigma_{\odot}^{b_i} \odot$)($= \sigma_{\odot}^{b_i} \odot (e_{1v} + e_{2v} + e_{3v}) = s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}$,

$$\begin{aligned}
&B(b_i)(xe_{1h} + ye_{2h} + ze_{3h}) \\
&= (xe_{1h} + ye_{2h} + ze_{3h}) \odot (\sigma_{\odot}^{b_i}) \\
&= (xe_{1h} + ye_{2h} + ze_{3h}) \odot \left(s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}\right) \\
&= (xe_{1h} + ye_{2h} + ze_{3h}) \odot \left(s^{b_1}e_{1v} + (-s^{-1})^{b_1}e_{2v} + \alpha^{-b_1}e_{3v}\right) \\
&= (xe_{1h} + ye_{2h} + ze_{3h}) \odot s^{b_1}e_{1v} + (xe_{1h} + ye_{2h} + ze_{3h})
\end{aligned}$$

$$
\begin{aligned}
&\odot (-s^{-1})^{b_1} e_{2_v} + (xe_{1_h} + ye_{2_h} + ze_{3_h}) \odot \alpha^{-b_1} e_{3_v} \\
&= (xm_{11} + ym_{12} + zm_{13}) s^{b_1} e_{1_v} + (xm_{21} + ym_{22} + zm_{23})(-s^{-1})^{b_1} e_{2_v} \\
&\quad + (xm_{31} + ym_{32} + zm_{33}) \alpha^{-b_1} e_{3_v} \\
&= (e_{1_v}\ e_{2_v}\ e_{3_v}) \begin{pmatrix} s^{b_i} & 0 & 0 \\ 0 & (-s^{-1})^{b_i} & 0 \\ 0 & 0 & \alpha^{-b_i} \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= (e_{1_h}\ e_{2_h}\ e_{3_h}) M \begin{pmatrix} s^{b_i} & 0 & 0 \\ 0 & (-s^{-1})^{b_i} & 0 \\ 0 & 0 & \alpha^{-b_i} \end{pmatrix} M \begin{pmatrix} x \\ y \\ z \end{pmatrix},
\end{aligned}
$$

as $(e_{1_v}\ e_{2_v}\ e_{3_v}) = (e_{1_h}\ e_{2_h}\ e_{3_h})M$. □

3.4. The closure-map C and the matrix C'. The linear map $C : K(B^3, NESW) \to K(B^3)$ is induced by the closure wiring:



Lemma 4.

$$C(xe_{1_h} + ye_{2_h} + ze_{3_h}) = z\delta\varnothing,$$

where $\varnothing$ represents the empty link which generates $K(B^3)$.

Proof. The closure of e_{1_h} is zero, and the closure of e_{2_h} is also zero by the orthogonal properties. The closure of e_{3_h} can be simplified as $\delta^{-1}\delta^2\varnothing = \delta\varnothing$. □

We therefore define the matrix $C' = (0, 0, \delta)$.

4. The Kauffman polynomials of the 2-bridge knots. The 2-bridge knot with continuous fraction notation $[b_1, d_1, b_2, d_2, \ldots, d_n, b_{n+1}]$ is an image of the compositions of wiring maps defined in last section. We summarize our main results in:

Theorem 1. *Let $F(D) = [b_1, d_1, b_2, d_2, \ldots, d_n, b_{n+1}]$ be the 2-bridge knot given in Figure 1. Then the Kauffman polynomial of D is*

$$\langle D \rangle = (0,0,\delta) B'(b_{n+1}) D'(d_n) \cdots B'(b_{i+1}) D'(d_i) \cdots B'(b_2) D'(d_1) B'_1(b_1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where $B'_1(b_1)$, $B'(b_i)$ and $D'(d_i)$ are matrices defined in the previous section.

Proof. Using the linear maps defined in the previous section and their compositions, the 2-bridge knot $D = C \circ B(b_{n+1}) \circ D(d_n) \circ \cdots \circ B(b_{i+1}) \circ D(d_i) \circ \cdots \circ B(b_2) \circ D(d_1) \circ B_1(b_1)($  $)$.

As each of these maps is a linear transformation between vector spaces, it can be represented by its matrix with respect to the basis $\{e_{1h}, e_{2h}, e_{3h}\}$. Note that  $= e_{1h} + e_{2h} + e_{3h}$, so

$$\begin{aligned} D &= C \circ B(b_{n+1}) \circ D(d_n) \circ \cdots \circ B(b_{i+1}) \circ D(d_i) \\ &\quad \circ \cdots \circ B(b_2) \circ D(d_1) \circ B_1(b_1)(e_{1h} + e_{2h} + e_{3h}), \\ &= C \circ B(b_{n+1}) \circ D(d_n) \circ \cdots \circ B(b_{i+1}) \circ D(d_i) \\ &\quad \circ \cdots \circ B(b_2) \circ D(d_1) \left(\begin{pmatrix} e_{1h} & e_{2h} & e_{2h} \end{pmatrix} B'_1(b_1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \\ &= C \left(\begin{pmatrix} e_{1h} & e_{2h} & e_{2h} \end{pmatrix} B'(b_{n+1}) D'(d_n) \cdots B'(b_{i+1}) D'(d_i) \right. \\ &\qquad \left. \cdots B'(b_2) D'(d_1) B'_1(b_1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \\ &= (0, 0, \delta\varnothing) B'(b_{n+1}) D'(d_n) \cdots B'(b_{i+1}) D'(d_i) \\ &\quad \cdots B'(b_2) D'(d_1) B'_1(b_1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{aligned}$$

by Lemma 4.

Take the Kauffman polynomial. We have

$$\langle D\rangle = (0,0,\delta)B'(b_{i+1})D'(d_n)\cdots B'(b_{i+1})D'(d_i) \cdots B'(b_2)D'(d_1)B'_1(b_1)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

since the Kauffman polynomial of the empty link $\varnothing$ is $\langle\varnothing\rangle = 1$. □

5. An example–The Kauffman polynomial of the Whitehead link. Here we demonstrate how to calculate the Kauffman polynomial of the Whitehead link using linear maps and matrices. We choose diagram W with the continued fraction notation $F(W) = [-2, 1, -2]$ for the Whitehead link. One can easily write down the corresponding matrices.

According to Theorem 1, the Kauffman polynomial of W is

$$\begin{aligned}\langle W\rangle &= (0,0,\delta)B'(-2)D'(1)B'_1(-2)\begin{pmatrix}1\\1\\1\end{pmatrix}\\ &= \frac{1}{\alpha^3 s^4(-1+s^2)^2}(-\alpha^2+\alpha^4-\alpha^3 s+\alpha^5 s+s^2+\alpha^2 s^2\\ &\quad -\alpha^4 s^2-\alpha^6 s^2+\alpha s^3+2\alpha^3 s^3-2\alpha^5 s^3-\alpha^7 s^3\\ &\quad -2s^4+\alpha^4 s^4+2\alpha^6 s^4-\alpha s^5-3\alpha^3 s^5+\alpha^5 s^5+3\alpha^7 s^5\\ &\quad +3s^6-2\alpha^2 s^6-\alpha^4 s^6-2\alpha^6 s^6+\alpha s^7+3\alpha^3 s^7\\ &\quad -\alpha^5 s^7-3\alpha^7 s^7-2s^8+\alpha^4 s^8+2\alpha^6 s^8-\alpha s^9\\ &\quad -2\alpha^3 s^9+2\alpha^5 s^9+\alpha^7 s^9+s^{10}+\alpha^2 s^{10}\\ &\quad -\alpha^4 s^{10}-\alpha^6 s^{10}+\alpha^3 s^{11}-\alpha^5 s^{11}-\alpha^2 s^{12}+\alpha^4 s^{12}).\end{aligned}$$

Our calculations are carried out using Mathematica.

Remark. An unoriented 2-bridge link may correspond to possibly two different oriented 2-bridge links. If one is interested in the Kauffman polynomial of an oriented 2-bridge link, one can find it using the writhe adjusted formula $\langle L\rangle = \alpha^{-w(L_1)}\langle L_1\rangle$ according to the particular orientation given on the diagram L_1 to obtain L.

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