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MULTIPLICITY OF SOLUTIONS FOR ELLIPTIC SYSTEMS WITH TOPOLOGICAL METHODS

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ABSTRACT. We study the existence and multiplicity of solutions of Dirichlet boundary value problem for nonlinear elliptic systems of the form $-\Delta u = f(x,u,v)$, $-\Delta v = g(x,u,v)$ in Ω, where Ω is a bounded open set in R^n with smooth boundary $\partial\Omega$. To study the system we use the topological methods.

1. Introduction. In applications of differential equations, critical points correspond to weak solutions of the equation. Many nonlinear problems in physical science can be reduced to finding critical points of the corresponding functionals. Indeed, this fact makes critical point theory which is an important existence tool in studying nonlinear differential equations.

The elliptic system has an extensive practical background. It can be used to describe the multiplicate chemical reaction catalyzed by catalyst grains under constant or variant temperature, and it can be a simple model of tubular chemical reaction; more naturally, it can be a correspondence of the stable station of a dynamical system determined by the reaction-diffusion system (see [**3**]).

The system of nonlinear elliptic equations present some new and interesting phenomena, which are not presented in the study of a single equation. In this paper we study existence and multiplicity of solutions for nonlinear elliptic systems of the form

$$\begin{aligned} -\Delta u &= f(x,u,v) \quad \text{in } \Omega, \\ -\Delta v &= g(x,u,v) \quad \text{in } \Omega, \end{aligned}$$

where $\Omega \subset R^n$ is a bounded smooth domain, subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$. To study the system we use topological methods.

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In Section 2, we state the critical point theorem on product space. In Section 3, we prove the existence of nontrivial solutions for the system, in the cases where $f = b_1 u + h_1(x, u, v)$, $g = b_2 v^+ + h_2(x, u, v)$ and h_i, $i = 1, 2$ are real valued Caratheodory functions and b_1, b_2 are real numbers. In Section 4, we show that there exist four nontrivial solutions for the system, in the case where $f = \lambda((u + v + \phi_1])^+ - \phi_1)$, $g = \mu((u + v + \phi_1])^+ - \phi_1)$.

2. Critical point theorems on product space. Let λ_k denote the eigenvalues and e_k the corresponding eigenfunctions, suitably normalized with respect to the $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, with Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, $\lambda_i \to +\infty$ and that $e_1 > 0$ for all $x \in \Omega$. To introduce a variation of the linking theorem on product space, we define the following sets. Let M be a Hilbert space and V a C^2 complete connected Finsler manifold. Suppose $M = M_1 \oplus M_2$ where M_1 are finite-dimensional subspaces of M.

Let $0 < \delta < R$, $e_1 \in M_1$ moreover, consider

$$\begin{aligned} Q_R &= \{se_1 + u : u \in M_2, s \ge 0 \| se_1 + u\| \le R\}, \\ S_\delta &= B_\delta \cap M_1, \end{aligned}$$

then ∂Q_R links ∂S_δ.

We recall a theorem of existence of two critical levels for a functional which is a linking theorem on the product space.

Theorem 2.1. *Suppose*

$$\sup_{\partial S_\delta \times V} I < \inf_{\partial Q_R \times V} I$$
$$\inf_{Q_R \times V} I > -\infty, \qquad \sup_{S_\delta \times V} I < +\infty,$$

and that I satisfies $(PS)_c^*$ with respect to X, for every

$$c \in \left[\inf_{Q_R \times V} I, \sup_{S_\delta \times V} I \right].$$

Then I admits at least two distinct critical values c_1, c_2 such that

$$\inf_{Q_R\times V} I \le c_1 \le \sup_{\partial S_\delta\times V} I < \inf_{\partial Q_R\times V} I \le c_2 \le \sup_{S_\delta\times V} I,$$

and at least $2+2$ cuplength(V) distinct critical points.

3. An application to nonlinear elliptic systems. In this section we will discuss systems of the form

$$\begin{aligned} -\Delta u &= b_1 u + h_1(x,u,v),\\ -\Delta v &= b_2 v^+ + h_2(x,u,v), \end{aligned} \quad u = 0 = v \text{ on } \partial\Omega, \tag{3.1}$$

where $\Omega \subset R^n$ is a bounded smooth domain, subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$, h_i, $i = 1,2$ are real valued Caratheodory functions and b_1, b_2 are real numbers.

We shall assume that there exists a function $H : \Omega \times R^1 \times R^1 \to R^1$ such that

$$\begin{aligned} \nabla H(x,u,v) &= \left(\frac{\partial}{\partial u}H(x,u,v), \frac{\partial}{\partial v}H(x,u,v)\right)\\ &= (h_1(x,u,v), h_2(x,u,v)); \end{aligned}$$

without loss of generality, we set

$$H(x,u,v) = \int_{(0,0)}^{(u,v)} h_1(x,u,v)du + h_2(x,u,v)\,dv.$$

We consider the following assumptions:

(H_1) There exist $\alpha > 2$ and $R > 0$ such that, for all $uv \neq 0$ and $\|u\| + \|v\| > R$ almost every $x \in \Omega$,

$$h_1(x,u,v)u + h_2(x,u,v)v \ge \alpha H(x,u,v) > 0.$$

(H_2) The following hold

$$\lim_{u\to 0} h_1(x,u,v) = 0, \qquad \lim_{u\to 0} h_2(x,u,v) = 0$$

uniformly in $x \in \Omega$ on compact sets of R^2.

(H_3) When $|v| \to 0$,

$$\frac{H(0,v)}{v^2} \longrightarrow 0.$$

Remark 3.1. (i) Hypothesis (H_2) implies that (3.1) possesses the "trivial solution" $u \equiv 0$.

(ii) Integrating condition (H_1) shows that there exist constants $a_1, a_2 > 0$ such that $H(x,u,v) \geq |u|^\alpha + |v|^\alpha$ for all $x \in \overline{\Omega}$ and $U \in H \times H$. Thus since $\alpha > 2$, $H(x,u,v)$ grows at a "superquadratic" rate and by (H_1), H grows at a "superlinear" rate as $|U| \to \infty$.

We shall work in the functional space $H \times H$ where $H := W_0^{1,2}(\Omega)$. We shall endow $H \times H$ with the Hilbert structure induced by the inner product

$$(U,V)_{H\times H} = \int_\Omega \nabla u(x) \nabla \phi(x)\, dx + \int_\Omega \nabla v(x) \nabla \varphi(x)\, dx,$$

where $U = (u,v), V = (\phi, \varphi)$. We denote the corresponding norm by $\|\cdot\|$. We define the energy functional associated to (3.1) as

$$\begin{aligned} I(u,v) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx - \frac{b_1}{2}\int_\Omega u^2\, dx \\ &\quad - \frac{b_2}{2}\int_\Omega (v^+)^2\, dx - \int_\Omega H(x,u,v)\, dx. \end{aligned}$$

It is easy to see that $I \in C^1(H \times H, R)$, and thus it makes sense to lock for solutions to (3.1) in the weak sense as critical points for I, i.e., $U = (u,v) \in H \times H$ such that $I'(u,v) = 0$, where

$$\begin{aligned} I'(U)V &= \int_\Omega (\nabla u(x)\nabla\phi(x) + \nabla v(x)\nabla\varphi(x))\ dx - b_1\int_\Omega u(x)\phi(x)\, dx \\ &\quad - b_2\int_\Omega v^+(x)\varphi(x)\, dx - \int_\Omega (h_1\phi(x) + h_2\varphi(x))\, dx, \\ I'(U)V &= \langle \nabla I(u,v), (\phi,\varphi)\rangle_{H\times H}, \quad \text{for all } V = (\phi,\varphi) \in H. \end{aligned}$$

Lemma 3.2. *If $b_1, b_2 > 0$, then functional $I(u,v)$ satisfies the Palais Smale condition.*

Proof. Let $\{U_n\} = \{(u_n, v_n)\}$ be a sequence in $E = H \times H$ such that $|I(U_n)| \leq C$ and $I'(U_n) \to 0$ as $n \to \infty$. First we prove that $\{U_n\} = \{(u_n, v_n)\}$ is bounded. Choose $\beta \in (\alpha^{-1}, 2^{-1})$. For large n, by condition H_1 we have

(3.2)
$$\begin{aligned} C_0 + o(1)\|U_n\| &\geq I(U_n) - \beta\langle \nabla I(U_n), U_n\rangle_{H\times H} \\ &= \left(\frac{1}{2} - \beta\right)\|U_n\|^2 - \left(\frac{1}{2} - \beta\right) \\ &\quad \times \left(b_1 \int {u_n}^2 dx - b_2 \int {v_n}^{+2}\, dx\right) \\ &\quad - \int_\Omega H(U_n)\, dx + \beta \int_\Omega h_1(U_n) u_n\, dx \\ &\quad + \beta \int_\Omega h_2(U_n) v_n\, dx \\ &\geq \left(\frac{1}{2} - \beta\right)\|U_n\|^2 + (\beta\alpha - 1)\int H(U_n)\, dx \\ &\quad - \left(\frac{1}{2} - \beta\right)\max\{b_1, b_2\}{\|U_n\|^2}_{L^2(\Omega)} \\ &\geq \left(\frac{1}{2} - \beta\right)\|U_n\|^2 + (\beta\alpha - 1)a_1{\|U_n\|^\alpha}_{L^\alpha(\Omega)} \\ &\quad - \left(\frac{1}{2} - \beta\right)\max\{b_1, b_2\}{\|U_n\|^2}_{L^2(\Omega)}. \end{aligned}$$

By standard inequalities, and since $\alpha > 2$, for $\varepsilon > 0$,

$$\|u_n\|_{L^2(\Omega)} \leq a_3\|u_n\|_{L^\alpha(\Omega)} \leq a_4 K(\varepsilon) + \varepsilon\|u_n\|^\alpha_{L^\alpha(\Omega)},$$

where $K(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Choose small ε. Then the L^α term in (3.2) absorbs the L^2 term. Consequently, we obtain

$$C_0 + o(1)\|U_n\| \geq \left(\frac{1}{2} - \beta\right)\|U_n\|^2,$$

where $o(1) \to 0$ as $n \to \infty$ and C is positive constant independent of constant C_0. This implies that $\{U_n\}$ is bounded in E. Hence, there exists a subsequence $\{U_{kj}\}_{j=1}^\infty$ and $U \in E$, with $U_{kj} \to U$ weakly in E, and $\{U_{kj}\}_{j=1}^{\infty}$ in $L^p \times L^p$ for $1 \leq p < 2^*$, by the Rellich-Kondrachov compactness theorem. We can compute that $\|U_{kj}\| \to \|U\|$, so $U_{kj} \to U$ in $H \times H$ and thus I satisfies (PS) condition. □

Let $W = H_0^1 \times \{0\} = \text{span}\,\{e_i^-, i \in N\}$, $Z = \{0\} \times H_0^1 = \text{span}\,\{e_i^+, i \in N\}$, where $e_i^{\pm}$ are the eigenfunctions associated to $\lambda_i^{\pm} = \pm\lambda_i(-\Delta)$ (where $\lambda_i(-\Delta)$ denotes the ith eigenvalue of the Laplace operator on H_0^1, with associated the eigenfunction e_i) and $e_i^+ = (0, e_i)$, $e_i^- = (e_i, 0)$. From now on, $\|\cdot\|$ will denote the H_0^1 norm.

Lemma 3.3. *If* (H_1) *and* (H_2) *hold, then for* $\lambda_1 < b_1$, $\lambda_1 < b_2$, *there exists an* $R > 0$ *such that*

$$\sup_{W \oplus Re_1^+} I < 0.$$

Proof. In fact, by (H_1) and (H_2), for every $u \in W$

$$\begin{aligned} I(u, \mathrm{Re}_1^+) &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla \mathrm{Re}_1^+|^2\right) dx - \frac{b_1}{2}\int_\Omega u^2\,dx \\ &\quad - \frac{b_2}{2}\int_\Omega (\mathrm{Re}_1^+)^2\,dx - \int_\Omega H(x, u, \mathrm{Re}_1^+)\,dx, \\ &\leq \frac{1}{2}\|u\|^2 - \frac{b_1}{2\lambda_1}\|u\|^2 + \frac{1}{2}\lambda_1 R^2 \\ &\quad - \frac{b_2}{2} - \int_\Omega H(x, u, Re_1)\,dx, \\ &\leq \frac{1}{2}\left(1 - \frac{b_1}{\lambda_1}\right)\|u\|^2 + \frac{1}{2}R^2(\lambda_1 - b_2), \end{aligned}$$

and hence there exists an $R^\star > 0$ such that for $R > R^\star$ we have $\sup_{W \oplus Re_1^+} I < 0$. □

Lemma 3.4. *If* (H_3) *holds, then for* $b_2 < \lambda_2$, *there exists a* $\rho > 0$ *such that*

$$0 < \inf_{\partial B_\rho(Z \ominus e_1^+)} I.$$

Proof. Given $\varepsilon < (\lambda_2 - b_2)/2\lambda_2$, by ($H_3$) there exists a $\rho > 0$ such that, for every $v \in Z \ominus e_1^+$ with $\|v\| < \rho$ we have

$$\begin{aligned} I(0,v) &= \frac{1}{2}\int_\Omega \left(|\nabla v|^2 - \frac{b_2}{2}\int_\Omega (v^+)^2\right) dx - \int_\Omega H(x,0,v)\,dx, \\ &\geq \frac{1}{2}\|v\|^2 - \frac{b_2}{2\lambda_2}\|v\|^2 - \varepsilon\|v\|^2, \\ &\geq \frac{1}{2}(1 - \frac{b_2}{\lambda_2} - \varepsilon)\|v\|^2 \\ &> 0. \end{aligned}$$

Thus we have $0 < \inf_{\partial B_\rho(Z \ominus e_1^+)} I$. □

Theorem 3.5. *Suppose* (H_1), (H_2), (H_3) *holds and* $b_1 > \lambda_1$, $\lambda_1 < b_2 < \lambda_2$. *Then problem* (3.1) *has at least two nontrivial solutions.*

Proof. We can apply Theorem 2.1 with $B_\rho \times V = B_\rho(Z \ominus e_1^+)$, $Q_R \times V = W \oplus Re_1^+$. Indeed since $b_1 > \lambda_1$ and $\lambda_1 < b_2 < \lambda_2$, by Lemmas 3.3 and 3.4, we can find $R > 0$, $\rho > 0$ such that $R > \rho > 0$ and

$$\sup_{\partial Q_R \times V} I < 0 < \inf_{\partial B_\rho \times V} I.$$

By Theorem 2.1, $I(u,v)$ has at least two nonzero critical value c_1, c_2

$$c_1 < \sup_{\partial Q_R \times V} I < 0 < \inf_{\partial B_\rho \times V} I < c_2.$$

Therefore, (3.1) has at least two nontrivial solutions. □

4. The system with nonlinearity crossing eigenvalues. In this section, we consider the existence and multiplicity of solutions for elliptic systems of the form

$$\begin{aligned} &\Delta u + \lambda[(u + v + \phi_1)^+ - \phi_1] = 0 \text{ in } \Omega, \\ &\Delta v + \mu[(u + v + \phi_1)^+ - \phi_1] = 0 \text{ in } \Omega, \\ &u = 0 = v \text{ on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $\Omega \subset R^n$ is a smooth open bounded set and $(u,v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$.

Theorem 4.1. *If* $0 < \lambda + \mu < \lambda_1$, *then problem* (4.1) *has only the trivial solution.* □

Proof. From problem (4.1) we compute that

$$\begin{aligned} \Delta u &= \frac{\lambda}{\mu}\Delta v \text{ in } \Omega, \\ u &= \frac{\lambda}{\mu} v \text{ on } \partial\Omega. \end{aligned} \tag{4.2}$$

The problem

$$\Delta u = 0 \text{ in } \Omega, \qquad u = \frac{\lambda}{\mu} v \text{ on } \partial\Omega$$

has only the trivial solution, so the solution (u, v) of problem (4.1) satisfies $u = (\lambda/\mu)v$. On the other hand, from problem (4.1), we get the equation

$$\begin{aligned} \Delta(u+v) + (\lambda+\mu)[(u+v+\phi_1)^+ - \phi_1] &= 0 \text{ in } \Omega, \\ u + v &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.3}$$

Let $w = u + v$. Then the above equation is equivalent to

$$\begin{aligned} \Delta w + (\lambda+\mu)[(w+\phi_1)^+ - \phi_1] &= 0 \text{ in } \Omega, \\ w &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.4}$$

Letting $-\Delta = L$, $k = (\lambda_1/2)$, we can rewrite (4.4) as follows.

$$\begin{aligned} Lw - kw &= (\lambda+\mu)[(w+\phi_1)^+ - \phi_1] - kw, \\ w &= (L-k)^{-1}((\lambda+\mu)[(w+\phi_1)^+ - \phi_1] - kw). \end{aligned} \tag{4.5}$$

Let $g(w) = (\lambda+\mu)[(w+\phi_1)^+ - \phi_1] - kw$. Then we have

$$\begin{aligned} &|g(w_2) - g(w_1)| \\ &\qquad = |(\lambda+\mu)[(w_2+\phi_1)^+ - (w_1+\phi_1)^+] - k(w_2 - w_1)|, \\ &\qquad \le |(\lambda+\mu) - k||w_2 - w_1|, \end{aligned}$$

and so we get

$$\|g(w_2) - g(w_1)\| \le |(\lambda+\mu) - k|\|w_2 - w_1\|,$$

where $\|\cdot\|$ is the L^2 norm in H. The operator $(L-k)^{-1}$ is a self-adjoint compact linear operator. The norm of $(L-k)^{-1}$ in H is

$$\|(L-k)^{-1}\| = \frac{1}{|\lambda_1 - k|}.$$

Since $0 < \lambda + \mu < \lambda_1$, the righthand side of equation (4.5) defines a Lipschitz mapping with Lipschitz constant less than 1. Therefore, by the contraction mapping principle, there exists a unique solution of (4.3). So equation (4.1) has only the trivial solution in H. □

The number of solutions of system (4.1) is equivalent to the number of solutions of the following single equation,

$$\begin{aligned} \Delta w + (\lambda+\mu)[(w+\phi_1)^+ - \phi_1] = 0 \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.6}$$

So we consider the semi-linear elliptic Dirichlet boundary value problem

$$\begin{aligned} \Delta w + b[(w+\phi_1)^+ - \phi_1] = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.7}$$

We rewrite (4.7) as

$$\Delta w - \lambda_1\phi_1 + b(w+\phi_1)^+ = b\phi_1 - \lambda_1\phi_1.$$

To investigate the multiplicity of the equation we consider the following equation

$$\Delta u + bu^+ = f. \tag{4.8}$$

Here we suppose that $\lambda_2 < b < \lambda_3$ and $f = s_1\phi_1 + s_2\phi_2$.

Remark 4.2. If $f = s\phi_1 (s < 0)$, then (4.8) has no solution and if $f = s\phi_1 (s = 0)$, then (4.8) has exactly one solution.

For the proof we rewrite (4.8) as

$$(\Delta + \lambda_1)u + (b - \lambda_1)u^+ + \lambda_1 u^- = s\phi_1.$$

Multiplying by ϕ_1 and integrating over Ω, we have

$$\int_\Omega [(b-\lambda_1)u^+]\phi_1 + \int_\Omega \lambda_1 u^- \phi_1 = s\int_\Omega \phi_1^2 = s. \tag{4.9}$$

Here, we used the self-adjointness of operator Δ and the orthogonality of eigenfunctions. Therefore, the lefthand side of (4.9) is always greater than or equal to zero, and there is no solution of (4.8). Also, if $s = 0$, then the only possibility is that $u \equiv 0$.

Let V be the two-dimensional subspace of H spanned by $\{\phi_1, \phi_2\}$, and let W be the orthogonal complement of V in H. Let P be an orthogonal projection from H into V. Then every element $u \in H$ is expressed by $u = v + w$, where $v = Pu$, $w = (I - P)u$. Hence, equation (4.8) is equivalent to the system

$$\Delta w + (I-P)(b(v+w)^+) = 0, \tag{4.10}$$

$$\Delta v + Pb((v+w)^+) = s_1\phi_1 + s_1\phi_2. \tag{4.11}$$

Here we consider the equation (4.10) and (4.11) as a system with two unknowns, v and w.

Lemma 4.3. *For a fixed $v \in V$, the equation (4.10) has a unique solution $w_\varepsilon = \theta_\varepsilon(v)$. Furthermore, $\theta(v)$is Lipschitz continuous(with respect to the L^2 norm) in terms of v.*

Proof. We use the contraction mapping theorem. Let $\delta = b/2$. Rewriting equation (4.10) as

$$(-\Delta - \delta)w = (I-P)(b(v+w)^+ - \delta(v+w)),$$

or equivalently as

$$w = (-\Delta - \delta)^{-1}(I-P)g_v(w), \tag{4.12}$$

where $g_v(w) = b(v+w)^+ - \delta(v+w)$, since

$$|g_v(w_1) - g_v(w_2)| \le |b-\delta||w_1 - w_2|,$$

we have

$$\|g_v(w_1) - g_v(w_2)\| \le |b-\delta|\|w_1 - w_2\|.$$

The operator $(-\Delta - \delta)^{-1}(I-P)$ is a self-adjoint compact linear map from $(I-P)H$ into itself. Its eigenvalues in W are $(\lambda_n - \delta)^{-1}$, where $\lambda_n \geq \lambda_3$. Therefore, its L^2 norm is $1/(\lambda_3 - \delta)$. Since $|b-\delta| < \lambda_3 - \delta$, it follows that for fixed $v \in V$, the righthand side of (4.12) defines a Lipschitz mapping of W into itself with Lipschitz constant $\gamma < 1$. Hence, by the contraction mapping principle, for each $v \in V$, there is a unique $w \in W$ which satisfies (4.10). Also, it follows, by the standard argument principle, that $\theta(v)$ is Lipschitz continuous in v. □

By this lemma, the study of the multiplicity of solutions of equation (4.11) is reduced to the study of the multiplicity of solutions of an equivalent problem

$$\Delta v + P(b(v + \theta(v))^+ = f,$$

defined on the two-dimensional subspace V.

If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$. For example, if we take $v \geq 0$ and $\theta(v) \equiv 0$, then equation (4.10) will be reduced to

$$\Delta 0 + (I-P)(bv^+) = 0,$$

which holds, because $v^+ = v \in V$ and $(I-P)v = 0$.

Since subspace V is spanned by $\{\phi_1, \phi_2\}$, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_1 \geq 0, |c_2| \leq kc_1\},$$

so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_1 \leq 0, |c_2| \leq kc_1\},$$

so that $v \leq 0$ for all $v \in C_3$. Thus, we do not know $\theta(v)$ for all $v \in V$, but we know that $\theta(v) \equiv 0$ for all $v \in C_1 \cup C_2$. Now we define a map $\Phi : V \to V$ given by $\Phi(v) = \Delta v + P(b(v + \theta(v))^+), v \in V$. Then Φ is continuous on V. Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_1 + d_2\phi_2 | d_1 \geq 0, |d_2| \leq k\left(\frac{b-\lambda_2}{b-\lambda_1}\right)d_1 \right\},$$

maps C_2 onto the cone

$$R_3 = \left\{ d_1\phi_1 + d_2\phi_2 | d_1 \geq 0, |d_2| \leq k\left(\frac{\lambda_2}{\lambda_1}\right)d_1 \right\}.$$

We note that $R_1 \subset R_3$ since $\lambda_1 < \lambda_2 < b < \lambda_3$. We investigate the images of the cones C_2, C_4 under Φ, where $C_2 = \{c_1\phi_1 + c_2\phi_2 | c_2 \geq 0, k|c_1| \leq c_2\}$, $C_4 = \{c_1\phi_1 + c_2\phi_2 | c_2 \leq 0, k|c_1| \leq |c_2|\}$. The image of C_2 under Φ is a cone containing

$$R_2 = \left\{ d_1\phi_1 + d_2\phi_2 \mid d_1 \geq 0, -k\left(\frac{\lambda_2}{\lambda_1}\right)d_1 \leq d_2 \leq k\left(\frac{\lambda_2 - b}{\lambda_1 - b}\right)d_1 \right\},$$

and the image of C_4 under Φ is a cone containing

$$R_4 = \left\{ d_1\phi_1 + d_2\phi_2 \mid d_1 \geq 0, -k\left(\frac{\lambda_2 - b}{\lambda_1 - b}\right)d_1 \leq d_2 \leq k\left(\frac{\lambda_2}{\lambda_1}\right)d_1 \right\}.$$

We note that all the cones R_2, R_3 and R_4 contain R_1. Also R_3, R_2 contain the cone $R_2 \setminus R_1$, and R_3, R_4 contain the cone $R_4 \setminus R_1$. If a solution of (4.8) is in C_1, then it is positive. If it is in C_3, then it is negative. If it is in the interior of $C_2 \cap C_4$, then it has both signs.

Therefore, if f is in the interior of R_1, then equation (4.8) has at least four solutions. Therefore, we have the following theorem (cf. [**6**]).

Theorem 4.4. *Suppose that $\lambda_2 < b < \lambda_3$, and f is in the interior of R_1. Then equation* (4.8) *has at least four solutions.*

From the above theorem we have the following corollary.

Corollary 4.5. *Suppose $\lambda_2 < b < \lambda_3$ and $s > 0$. Then problem* (4.1) *has at least four solutions.*

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