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RANKS OF CROSS COMMUTATORS ON BACKWARD SHIFT INVARIANT SUBSPACES OVER THE BIDISK

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ABSTRACT. On backward shift invariant subspaces M over the bidisk, it is proved that if M is Hilbert-Schmidt, then

$$\text{rank}\,[R_z, R_w^*] - 1 \leq \text{rank}\,[S_z, S_w^*] \leq \text{rank}\,[R_z, R_w^*] + 1.$$

1. Introduction. Let Γ^2 be the two-dimensional unit torus. We write z, w for variables in $\Gamma^2 = \Gamma_z \times \Gamma_w$. Let $L^2 = L^2(\Gamma^2)$ be the space of square integrable functions on Γ^2 with the inner product

$$\langle f, g\rangle = \int_0^{2\pi}\int_0^{2\pi} (f\overline{g})(e^{is}, e^{it})\frac{dsdt}{(2\pi)^2}.$$

Let $H^2 = H^2(\Gamma^2)$ be the Hardy space over Γ^2, and let $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ be the Hardy spaces on the unit circle Γ with variables z and w, respectively. We think of $H^2(\Gamma_z)$ and $H^2(\Gamma_w)$ as closed subspaces of H^2. For a function ψ in $L^\infty(\Gamma^2)$, the Toeplitz operator T_ψ on H^2 is defined by $T_\psi f = P(\psi f)$, where P is the orthogonal projection from L^2 onto H^2. It is known that $T_\psi^* = T_{\overline{\psi}}$, and $T_{\varphi(z)}^* T_{\psi(w)} = T_{\psi(w)} T_{\varphi(z)}^*$ for every $\varphi(z), \psi(w) \in H^\infty(\Gamma)$. A nonzero closed subspace M of H^2 with $M \neq H^2$ is called invariant if $T_z M \subset M$ and $T_w M \subset M$. A function f in H^2 is called inner if $|f| = 1$ almost everywhere on Γ^2. A well-known theorem due to Beurling [**2**] says that an invariant subspace M of $H^2(\Gamma_z)$ has a form $M = q(z)H^2(\Gamma_z)$ for an inner function $q(z)$. In the two variables case, the structure of invariant subspaces of H^2 is extremely complicated, see [**3, 12**].

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We write

$$N = H^2 \ominus M := \{f \in H^2 : f \perp M\}.$$

Then $T_z^* N \subset N$ and $T_w^* N \subset N$, and N is called a backward shift invariant subspace of H^2.

Let $\psi \in L^\infty$ and R_ψ denote the operator on M defined by $R_\psi f = P_M(\psi f)$, where P_M is the orthogonal projection on L^2 with range M. One has $R_\psi^* = R_{\overline{\psi}}$ and $R_z = T_z|_M$. Let $[R_z, R_w^*]$ denote the cross commutator of R_z and R_w, that is, $[R_z, R_w^*] = R_z R_w^* - R_w^* R_z$. On a backward shift invariant subspace N, one has the operator S_ψ on N defined by $S_\psi f = P_N(\psi f)$. It is easy to see that $S_\psi^* = S_{\overline{\psi}}$ and $S_z^* = T_z^*|_N$.

In [**10**], Mandrekar proved that $[R_z, R_w^*] = 0$ if and only if M is of Beurling type, that is, $M = qH^2$ for some inner function q on Γ^2. This is a nice characterization of Beurling type invariant subspaces of H^2 and is our starting point of the study. In [**9**], it is proved that $[S_z, S_w^*] = 0$ if and only if M has one of the following forms;

$$M = q_1(z)H^2, \qquad M = q_2(w)H^2, \qquad M = q_1(z)H^2 + q_2(w)H^2$$

for nonconstant one variable inner functions $q_1(z)$ and $q_2(w)$.

In [**13, 14, 15**], Yang pointed out that $[R_z, R_w^*]$ and $[S_z, S_w^*]$ are Hilbert-Schmidt operators under some mild condition. So, to study the structure of invariant subspaces, it is important to study the cases when $\operatorname{rank}[R_z, R_w^*] < \infty$ and $\operatorname{rank}[S_z, S_w^*] < \infty$. The authors studied these subjects in [**5, 6, 7, 8**]. As a consequence, we had the following conjecture in [**7**].

Conjecture. $\operatorname{rank}[R_z, R_w^*] - 1 \le \operatorname{rank}[S_z, S_w^*] \le \operatorname{rank}[R_z, R_w^*] + 1.$

This conjecture is deeply connected with Guo and Yang's works [**4, 16, 17**]. For an invariant subspace M, in [**4**] they defined the core operator C on H^2 associated with M, and they showed

$$C = 1 - R_z R_z^* - R_w R_w^* + R_z R_w R_z^* R_w^*. \tag{1.1}$$

An invariant subspace M is said to be Hilbert-Schimdt if its core operator C is Hilbert-Schimdt. As pointed out in [**16**], almost all known examples of invariant subspaces are Hilbert-Schimdt. In [**17**, Proposition 2.6], Yang showed that if M is Hilbert-Schimdt, then

$$\operatorname{rank}[R_w^*, R_w][R_z^*, R_z] = \operatorname{rank}[R_z, R_w^*] + 1.$$

In Section 2, we prove that

$$\operatorname{rank}[R_w^*, R_w][R_z^*, R_z] - 2 \leq \operatorname{rank}[S_z, S_w^*] \leq \operatorname{rank}[R_w^*, R_w][R_z^*, R_z].$$

Combining the above results, we get

$$\operatorname{rank}[R_z, R_w^*] - 1 \leq \operatorname{rank}[S_z, S_w^*] \leq \operatorname{rank}[R_z, R_w^*] + 1.$$

if M is Hilbert-Schmidt. We also show that if $\operatorname{rank}[R_w^*, R_w][R_z^*, R_z] < \infty$, then $\operatorname{rank}[R_z, R_w^*] < \infty$ and M is Hilbert-Schmidt.

In Section 3, we give some examples concerned with the above inequalities.

For closed subspaces M_1 and M_2, when $M_1 \subset M_2$, we write $M_2 \ominus M_1 = \{f \in M_2 : f \perp M_1\}$, and when $M_1 \perp M_2$, $M_1 \oplus M_2$ means the orthogonal sum. For a subset E of H^2, we denote by $\overline{E}$ the norm closure of E.

2. Rank of cross commutators.

Lemma 2.1. $[S_z, S_w^*] = P_N T_w^* P_M T_z|_N$.

Proof. Since $T_z T_w^* = T_w^* T_z$, we have

$$\begin{aligned}[S_z, S_w^*] &= P_N T_z T_w^*|_N - P_N T_w^* P_N T_z|_N \\ &= P_N\big(T_z T_w^* - T_w^*(I - P_M)T_z\big)|_N \\ &= P_N T_w^* P_M T_z|_N. \qquad \square\end{aligned}$$

Lemma 2.2. $\overline{P_M T_z N} = (M \ominus zM) \ominus (M \cap H^2(\Gamma_w))$.

Proof. Let $f \in N$ and $g \in zM$. Then $T_z^* g \in M$ and $\langle P_M T_z f, g\rangle = \langle f, T_z^* g\rangle = 0$. This shows that $P_M T_z N \subset M \ominus zM$. Since $T_z N \perp M \cap H^2(\Gamma_w)$,

$$\overline{P_M T_z N} \subset (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)).$$

Let $f \in M \ominus zM$. We have $T_z^* f \in N$. If $f \perp P_M T_z N$, then $T_z^* f \perp N$. Hence $T_z^* f = 0$. This shows that $f \in M \cap H^2(\Gamma_w)$. Thus we get the assertion. □

Lemma 2.3.

$$\operatorname{rank}[S_z, S_w^*] = \dim T_w^* P_{M\ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big).$$

Proof. By Lemmas 2.1 and 2.2,

$$\overline{[S_z, S_w^*]N} = \overline{P_N T_w^*\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big)}.$$

Hence,

$$\operatorname{rank}[S_z, S_w^*] = \dim P_N T_w^*\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big).$$

Let $F \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w))$. By the Wold decomposition theorem,

$$M = \sum_{n=0}^{\infty} \oplus (M \ominus wM) w^n.$$

Then we can write

$$F = \sum_{n=0}^{\infty} \oplus F_n w^n, \quad F_n \in M \ominus wM,$$

so

$$T_w^* F = T_w^* F_0 + \sum_{n=1}^{\infty} \oplus F_n w^{n-1}.$$

Since $T_w^*(M \ominus wM) \subset N$, we have

$$T_w^* F_0 \in N \quad \text{and} \quad \sum_{n=1}^{\infty} \oplus F_n w^{n-1} \in M.$$

Thus,

$$P_N T_w^* F = T_w^* F_0 = T_w^* P_{M\ominus wM} F,$$

and we get the assertion. □

One easily sees the following.

Lemma 2.4.

$$[R_w^*, R_w][R_z^*, R_z] = (I - R_w R_w^*)(I - R_z R_z^*) = P_{M\ominus wM} P_{M\ominus zM}.$$

Theorem 2.5.

$$\operatorname{rank}[R_w^*, R_w][R_z^*, R_z] - 2 \leq \operatorname{rank}[S_z, S_w^*] \leq \operatorname{rank}[R_w^*, R_w][R_z^*, R_z].$$

Proof. We write $M \cap H^2(\Gamma_w) = q_1(w)H^2(\Gamma_w)$, where $q_1(w)$ is either 0 or an inner function. We have

$$M \ominus zM = \big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big) \oplus (M \cap H^2(\Gamma_w)).$$

Hence,

$$\begin{aligned} P_{M\ominus wM}(M \ominus zM) = P_{M\ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big) \\ + \mathbf{C} \cdot P_{M\ominus wM}(q_1(w)) \end{aligned}$$

and

$$\begin{aligned} T_w^* P_{M\ominus wM}(M \ominus zM) = T_w^* P_{M\ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big) \\ + \mathbf{C} \cdot T_w^* P_{M\ominus wM}(q_1(w)). \end{aligned}$$

Therefore, by Lemma 2.3,

$$\begin{aligned} \dim T_w^* P_{M\ominus wM}(M \ominus zM) - 1 &\leq \operatorname{rank}[S_z, S_w^*] \\ &\leq \dim T_w^* P_{M\ominus wM}(M \ominus zM). \end{aligned}$$

If there are no $f \in P_{M\ominus wM}(M \ominus zM)$ with $f \neq 0$ satisfying $T_w^* f = 0$, then we have

$$\dim P_{M\ominus wM}(M \ominus zM) - 1 \leq \operatorname{rank}[S_z, S_w^*] \leq \dim P_{M\ominus wM}(M \ominus zM).$$

Suppose that there is an $f \in P_{M\ominus wM}(M \ominus zM)$ with $f \neq 0$ satisfying $T_w^* f = 0$. Then $f = f(z) \in M \cap H^2(\Gamma_z)$. By the Beurling theorem,

$M \cap H^2(\Gamma_z) = q_2(z)H^2(\Gamma_z)$ for an inner function $q_2(z)$. There exists a $G \in M \ominus zM$ such that $f(z) = P_{M\ominus wM}G$. We have $G = f(z) + wG_1$ for some $G_1 \in M$, and

$$f(z) = G - wG_1 \in (M \ominus zM) + wM.$$

Note that $M \ominus zM \perp zq_2(z)H^2(\Gamma_z)$ and $wM \perp zq_2(z)H^2(\Gamma_z)$. Hence, $f(z) \perp zq_2(z)H^2(\Gamma_z)$. Therefore, $f(z) = cq_2(z)$ for some $c \in \mathbf{C}$, and

$$\dim T_w^* P_{M\ominus wM}(M \ominus zM) = \dim P_{M\ominus wM}(M \ominus zM) - 1.$$

Thus, we get

$$\begin{aligned}\dim P_{M\ominus wM}(M \ominus zM) - 2 &\le \operatorname{rank}[S_z, S_w^*] \\ &\le \dim P_{M\ominus wM}(M \ominus zM).\end{aligned}$$

Combining this with Lemma 2.4, we get the assertion. □

As mentioned in the introduction, in [**17**] Yang showed that if M is Hilbert-Schimdt, then

$$\operatorname{rank}[R_w^*, R_w][R_z^*, R_z] = \operatorname{rank}[R_z, R_w^*] + 1,$$

so that we get the following.

Corollary 2.6. *If an invariant subspace M of H^2 is Hilbert-Schmidt, then*

$$\operatorname{rank}[R_z, R_w^*] - 1 \le \operatorname{rank}[S_z, S_w^*] \le \operatorname{rank}[R_z, R_w^*] + 1.$$

Here we have a question whether it holds that $\operatorname{rank}[R_w^*, R_w][R_z^*, R_z] < \infty$ if and only if $\operatorname{rank}[R_z, R_w^*] < \infty$. We have the following.

Theorem 2.7. *If* $\operatorname{rank}[R_w^*, R_w][R_z^*, R_z] < \infty$, *then* $\operatorname{rank}[R_z, R_w^*] < \infty$ *and M is Hilbert-Schmidt.*

Proof. Note that $\operatorname{rank}[R_z, R_w^*] = \operatorname{rank}[R_z^*, R_w]$. It is not difficult to see that $[R_z^*, R_w] = 0$ on zM. Let $F \in M \ominus zM$. Then $T_z^*F \in N$ and

$$[R_z^*, R_w]F = R_z^* R_w F = P_M(\overline{z}wF) = \overline{z}P_{zM}(wF).$$

Hence,

$$[R_z^*, R_w]M = \overline{z} P_{zM}(w(M \ominus zM)). \tag{2.1}$$

Suppose that $n = \operatorname{rank}[R_w^*, R_w][R_z^*, R_z] < \infty$. We write

$$[R_w^*, R_w][R_z^*, R_z]M = \mathbf{C} \cdot A_1 \oplus \mathbf{C} \cdot A_2 \oplus \cdots \oplus \mathbf{C} \cdot A_n,$$

where $A_j \in M \ominus wM$ and $\|A_j\| = 1$ for every $1 \leq j \leq n$. Since $R_w^*(M \ominus zM) \subset M \ominus zM$, we can write

$$F = f_1(w)A_1 \oplus f_2(w)A_2 \oplus \cdots \oplus f_n(w)A_n,$$

where $f_j(w) \in H^2(\Gamma_w)$ for $1 \leq j \leq n$. Write

$$wF = F_1 \oplus F_2 \in (M \ominus zM) \oplus zM. \tag{2.2}$$

Also we can write

$$F_1 = g_1(w)A_1 \oplus g_2(w)A_2 \oplus \cdots \oplus g_n(w)A_n,$$

where $g_j(w) \in H^2(\Gamma_w)$ for $1 \leq j \leq n$. Hence,

$$F_2 = (wf_1(w) - g_1(w))A_1 \oplus \cdots \oplus (wf_n(w) - g_n(w))A_n.$$

Therefore,

$$P_{zM}(w(M \ominus zM)) \subset zM \cap \big(H^2(\Gamma_w)A_1 \oplus \cdots \oplus H^2(\Gamma_w)A_n\big). \tag{2.3}$$

We write

$$K = zM \cap \big(H^2(\Gamma_w)A_1 \oplus \cdots \oplus H^2(\Gamma_w)A_n\big).$$

Note that K is a T_w-invariant closed subspace of $H^2(\Gamma_w)A_1 \oplus \cdots \oplus H^2(\Gamma_w)A_n$. By (2.2), $F = R_w^*F_1 + R_w^*F_2$. Since $F, R_w^*F_1 \in M \ominus zM$, we have $R_w^*F_2 \in M \ominus zM$. This, (2.2), and (2.3) show that

$$P_{zM}(w(M \ominus zM)) \subset K \ominus wK.$$

It is known that $\dim K \ominus wK \leq n$, see [**1, 11**]. Therefore, by (2.1), $\operatorname{rank}[R_z^*, R_w] \leq n < \infty$.

In the proof of Theorem 2.7 of [**17**], Yang showed that for the core operator C defined by (1.1), we have

$$\operatorname{rank} C = \operatorname{rank}\,[R_w^*, R_w][R_z^*, R_z] + \operatorname{rank}\,[R_w^*, R_z].$$

Hence C is of finite rank, so M is Hilbert-Schmidt. □

Corollary 2.8. *If* $\operatorname{rank}\,[R_w^*, R_w][R_z^*, R_z] < \infty$, *then*

$$\operatorname{rank}\,[R_z, R_w^*] - 1 \le \operatorname{rank}\,[S_z, S_w^*] \le \operatorname{rank}\,[R_z, R_w^*] + 1.$$

Suppose that $M \cap H^2(\Gamma_w) = q_1(w)H^2(\Gamma_w)$ for an inner function $q_1(w)$. By the proof of Theorem 2.5, to give a more exact relation between $\operatorname{rank}\,[S_z, S_w^*]$ and $\operatorname{rank}\,[R_w^*, R_w][R_z^*, R_z]$ we need to know whether

$$T_w^* P_{M\ominus wM}(q_1(w)) \in T_w^* P_{M\ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big)$$

holds or not. Here we give some properties concerning with $T_w^* P_{M\ominus wM} \times (q_1(w))$.

Proposition 2.9. *Suppose that* $M \cap H^2(\Gamma_w) = q_1(w)H^2(\Gamma_w)$ *for an inner function* $q_1(w)$. *Then we have the following.*

(i) $T_w^* P_{M\ominus wM}(q_1(w)) \neq 0$, *that is*, $P_{M\ominus wM}(q_1(w)) \notin H^2(\Gamma_z)$.

(ii) *If there is an* $f(z)$ *in* $H^2(\Gamma_z) \cap P_{M\ominus wM}(M \ominus zM)$ *but which is not contained in* $P_{M\ominus wM}((M \ominus zM) \ominus (M \cap H^2(\Gamma_w)))$, *then*

$$T_w^* P_{M\ominus wM}(q_1(w)) \in T_w^* P_{M\ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big)$$

and

$$\operatorname{rank}\,[S_z, S_w^*] = \operatorname{rank}\,[R_w^*, R_w][R_z^*, R_z] - 1.$$

(iii) *If*

$$P_{M\ominus wM}(q_1(w)) \notin P_{M\ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big)$$

and

$$T_w^* P_{M \ominus wM}(q_1(w)) \in T_w^* P_{M \ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big),$$

then $H^2(\Gamma_z) \cap P_{M \ominus wM}(M \ominus zM) \neq \{0\}$.

Proof. (i) Suppose that $P_{M \ominus wM}(q_1(w)) \in H^2(\Gamma_z)$. Then we can write $q_1(w) = f(z) + wF$, where $f(z) \in H^2(\Gamma_z)$ and $F \in M$. We have $T_w^* q_1(w) = F \in M$. Since $M \neq H^2$, $q_1(w)$ is nonconstant, and $T_w^* q_1(w) \notin q_1(w) H^2(\Gamma_w) = M \cap H^2(\Gamma_w)$. This is a contradiction.

(ii) We can write

$$f(z) = c P_{M \ominus wM}(q_1(w)) + f_1,$$

where

$$f_1 \in P_{M \ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big) \quad \text{and} \quad c \neq 0.$$

Since $T_w^* f(z) = 0$, we get the assertion.

(iii) There exists an $F \in P_{M \ominus wM}((M \ominus zM) \ominus (M \cap H^2(\Gamma_w)))$ satisfying $T_w^* P_{M \ominus wM}(q_1(w)) = T_w^* F$. Then

$$P_{M \ominus wM}(q_1(w)) - F \in H^2(\Gamma_z) \cap P_{M \ominus wM}(M \ominus zM)$$

and $P_{M \ominus wM}(q_1(w)) - F \neq 0$. □

3. Examples. In [**4**], it is proved that if M has finite codimension in H^2, then M is Hilbert-Schmidt. Also if M is Hilbert-Schmidt, then φM is Hilbert-Schmidt for every inner function φ.

Example 3.1. (i) Let $M_1 = zH^2 + wH^2$ and $N_1 = H^2 \ominus M_1 = \mathbf{C} \cdot 1$. Then M_1 is Hilbert-Schmidt, $[S_z, S_w^*] N_1 = \{0\}$ and $P_{M_1 \ominus wM_1}(M_1 \ominus zM_1) = \mathbf{C} \cdot z + \mathbf{C} \cdot w$. Hence,

$$\operatorname{rank}[R_z, R_w^*] - 1 = \operatorname{rank}[S_z, S_w^*] = 0.$$

(ii) Let $M_2 = wM_1$ and $N_2 = H^2 \ominus M_2$. Then M_2 is Hilbert-Schmidt, $[S_z, S_w^*] N_2 = \mathbf{C} \cdot z$, and $P_{M_2 \ominus wM_2}(M_2 \ominus zM_2) = \mathbf{C} \cdot zw + \mathbf{C} \cdot w^2$. Hence,

$$\operatorname{rank}[R_z, R_w^*] = \operatorname{rank}[S_z, S_w^*] = 1.$$

(iii) Let $M_3 = zM_2$ and $N_3 = H^2 \ominus M_3$. Then M_3 is Hilbert-Schmidt, $[S_z, S_w^*]N_3 = \mathbf{C}\cdot z^2 + \mathbf{C}\cdot zw$, and $P_{M_3 \ominus wM_3}(M_3 \ominus zM_3) = \mathbf{C}\cdot z^2 w + \mathbf{C}\cdot zw^2$. Hence,

$$\operatorname{rank}[R_z, R_w^*] + 1 = \operatorname{rank}[S_z, S_w^*] = 2. \qquad \square$$

We give examples M for which $M \cap H^2(\Gamma_w) = q_1(w)H^2(\Gamma_w)$ and $T_w^* P_{M \ominus wM}(q_1(w))$ is (not) contained in

$$T_w^* P_{M \ominus wM}\big((M \ominus zM) \ominus (M \cap H^2(\Gamma_w))\big).$$

Example 3.2. (i) For each positive integer n with $n \geq 2$, let M be the invariant subspace of H^2 generated by w^n and $\sum_{j=1}^n z^{n-j} w^{j-1}$, that is,

$$\begin{aligned} M = \mathbf{C}\cdot\left(\sum_{j=1}^{n} z^{n-j} w^{j-1}\right) &\oplus \mathbf{C}\cdot\left(\sum_{j=1}^{n-1} z^{n-j} w^{j}\right) \oplus \cdots \\ &\oplus \mathbf{C}\cdot\left(\sum_{j=1}^{2} z^{n-j} w^{j+n-3}\right) \oplus \mathbf{C}\cdot z^{n-1} w^{n-1} \oplus w^n H^2 \\ &\oplus z^n H^2(\Gamma_z) \oplus z^n w H^2(\Gamma_z) \oplus \cdots \oplus z^n w^{n-1} H^2(\Gamma_z). \end{aligned}$$

Since M has finite codimension in H^2, M is Hilbert-Schmidt. Let $q_1(w)$ be an inner function given in the proof of Theorem 2.5 for M, then $q_1(w) = w^n$. We have

$$\begin{aligned} M \ominus zM = \mathbf{C}\cdot\left(\sum_{j=1}^{n} z^{n-j} w^{j-1}\right) &\oplus \mathbf{C}\cdot\left(\left(\sum_{j=1}^{n-1} z^{n-j} w^{j}\right) - (n-1) z^n\right) \\ &\oplus \mathbf{C}\cdot\left(\left(\sum_{j=1}^{n-2} z^{n-j} w^{j+1}\right) - (n-2) z^n w\right) \oplus \cdots \\ &\oplus \mathbf{C}\cdot\left(z^{n-1} w^{n-1} - z^n w^{n-2}\right) \oplus w^n H^2(\Gamma_w) \end{aligned}$$

and

$$\begin{aligned} M \ominus wM = \mathbf{C} \cdot \left(\sum_{j=1}^{n} z^{n-j} w^{j-1} \right) & \\ \oplus \mathbf{C} \cdot \left(\left(\sum_{j=1}^{n-1} z^{n-j} w^{j} \right) - (n-1) w^n \right) & \\ \oplus \mathbf{C} \cdot \left(\left(\sum_{j=1}^{n-2} z^{n-j} w^{j+1} \right) - (n-2) z w^n \right) \oplus \cdots & \\ \oplus \mathbf{C} \cdot \left(z^{n-1} w^{n-1} - z^{n-2} w^n \right) \oplus z^n H^2(\Gamma_z). & \end{aligned}$$

Hence,

$$P_{M \ominus wM}(q_1) = \frac{1}{n} \left((n-1) w^n - \sum_{j=1}^{n-1} z^{n-j} w^j \right)$$

and

$$T_w^* P_{M \ominus wM}(q_1) = \frac{1}{n} \left((n-1) w^{n-1} - \sum_{j=1}^{n-1} z^{n-j} w^{j-1} \right).$$

We have

$$\left(\sum_{j=1}^{n-1} z^{n-j} w^j \right) - (n-1) z^n \in (M \ominus zM) \ominus (M \cap H^2(\Gamma_w)),$$

$$\begin{aligned} P_{M \ominus wM} & \left(\left(\sum_{j=1}^{n-1} z^{n-j} w^j \right) - (n-1) z^n \right) \\ &= \frac{1}{n} \left(\sum_{j=1}^{n-1} z^{n-j} w^j \right) - \frac{n-1}{n} w^n - (n-1) z^n, \end{aligned}$$

and

$$\begin{aligned} T_w^* P_{M \ominus wM} & \left(\left(\sum_{j=1}^{n-1} z^{n-j} w^j \right) - (n-1) z^n \right) \\ &= \frac{1}{n} \left(-(n-1) w^{n-1} + \sum_{j=1}^{n-1} z^{n-j} w^{j-1} \right). \end{aligned}$$

Therefore,

$$P_{M\ominus wM}(q_1) \notin P_{M\ominus wM}\big((M\ominus zM)\ominus(M\cap H^2(\Gamma_w))\big),$$
$$T_w^* P_{M\ominus wM}(q_1) \in T_w^* P_{M\ominus wM}\big((M\ominus zM)\ominus(M\cap H^2(\Gamma_w))\big),$$

and

$$H^2(\Gamma_z)\cap P_{M\ominus wM}(M\ominus zM) \neq \{0\}.$$

In this case, we have $\operatorname{rank}[S_z, S_w^*] = n$ and

$$\operatorname{rank}[R_z, R_w^*] + 1 = \operatorname{rank}[R_w^*, R_w][R_z^*, R_z] = n+1.$$

(ii) Let $M' = wM \oplus z^n H^2(\Gamma_z)$. In this case, $q_1(w) = w^{n+1}$ and

$$P_{M'\ominus wM'}(q_1) = \frac{1}{n}\bigg((n-1)w^{n+1} - \sum_{j=1}^{n-1} z^{n-j}w^{j+1}\bigg).$$

We also have

$$\bigg(\sum_{j=1}^{n-1} z^{n-j}w^{j+1}\bigg) - (n-1)z^n w \in (M'\ominus zM')\ominus(M'\cap H^2(\Gamma_w))$$

and

$$P_{M'\ominus wM'}\bigg(\bigg(\sum_{j=1}^{n-1} z^{n-j}w^{j+1}\bigg) - (n-1)z^n w\bigg)$$
$$= \frac{1}{n}\bigg(\sum_{j=1}^{n-1} z^{n-j}w^{j+1}\bigg) - \frac{n-1}{n}w^{n+1} = -P_{M'\ominus wM'}(q_1).$$

Hence,

$$P_{M'\ominus wM'}(q_1) \in P_{M'\ominus wM'}\big((M'\ominus zM')\ominus(M'\cap H^2(\Gamma_w))\big).$$

Note that $\operatorname{rank}[S_z, S_w^*] = n$. Since M' is Hilbert-Schmidt, we have

$$\operatorname{rank}[R_z, R_w^*] + 1 = \operatorname{rank}[R_w^*, R_w][R_z^*, R_z] = n+1.$$

(iii) Let $M'' = zM \oplus w^n H^2(\Gamma_w)$. In this case, $q_1(w) = w^n$ and $P_{M''\ominus wM''}(q_1) = q_1$. We have

$$T_w^* q_1 = w^{n-1} \notin T_w^* P_{M''\ominus wM''}\big((M'' \ominus zM'') \ominus (M'' \cap H^2(\Gamma_w))\big).$$

Note that $H^2(\Gamma_z) \cap P_{M''\ominus wM''}(M'' \ominus zM'') = \{0\}$, $\operatorname{rank}[S_z, S_w^*] = n$, and

$$\operatorname{rank}[R_z, R_w^*] + 1 = \operatorname{rank}[R_w^*, R_w][R_z^*, R_z] = n + 1. \qquad \square$$

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