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BOUNDS FOR STRENGTHENED HARDY AND POLYA-KNOPP'S DIFFERENCES

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ABSTRACT. In this paper we prove an improvement and reverse of strengthened Hardy-Knopp type inequality and its dual inequality

1. Introduction. In [**2**] Hardy proved the following integral inequality (see also [**3**, Chapter 9, Theorem 328]); if $p > 1$, $f(x) \geq 0$ and $F(x) = \int_0^x f(t)\,dt$, then

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx. \tag{1}$$

On the other hand, the following related exponential integral inequality, the so called exponential integral inequality (or Polya Knopp's inequality) [**5, 6**]

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)\,dt\right) dx < e\int_0^\infty f(x)\,dx \tag{2}$$

is valid for positive functions $f \in L^1(0,\infty)$. Inequalities (1) and (2) are closely related, since (2) can be obtained from (1) by rewriting it with the function f replaced by $f^{1/p}$ and letting $p \to \infty$. Therefore, Polya-Knopp's inequality may be considered as a limiting relation of Hardy's inequality. In [**4**] Kaijser et al. pointed out both (1) and (2) are just special cases of the much more general Hardy-Knopp-type inequality for positive function f

$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leq \int_0^\infty \phi(f(x))\frac{dx}{x}, \tag{3}$$

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where ϕ is a convex function on $(0,\infty)$. In [**1**] Pečarić, Čižmešija and Persson proved the strengthened Hardy-Knopp-type inequality that generalized inequality (3) given by Theorem 1.1. They also formulated its dual result given by Theorem 1.2.

Theorem 1.1. *Suppose $0 < b \le \infty$, let $u : (0,b) \to \mathbf{R}$ be a nonnegative function such that the function $x \mapsto (u(x)/x)$ is locally integrable in $(0,b)$, and the function v is defined by*

$$v(t) = t\int_t^b \frac{u(x)}{x^2}\,dx, \quad t \in (0,b).$$

If the real-valued function ϕ is convex on (a,c), where $-\infty \le a < c \le \infty$, then the inequality

$$\int_0^b u(x)\phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \le \int_0^b v(x)\phi(f(x))\frac{dx}{x} \tag{4}$$

holds for all integrable functions $f : (0,b) \to \mathbf{R}$, such that $f(x) \in (a,c)$ for all $x \in (0,b)$.

Theorem 1.2. *For $0 \le b \le \infty$, let $u : (b,\infty) \to \mathbf{R}$ be a nonnegative locally integrable function in (b,∞), and the function v is defined by*

$$v(t) = \frac{1}{t}\int_b^t u(x)\,dx, \quad t \in (b,\infty).$$

If the real-valued function ϕ is convex on (a,c), where $-\infty \le a < c \le \infty$, then the inequality

$$\int_b^\infty u(x)\phi\left(x\int_x^\infty f(t)\,\frac{dt}{t^2}\right)\frac{dx}{x} \le \int_b^\infty v(x)\phi(f(x))\,\frac{dx}{x} \tag{5}$$

holds for all integrable functions $f : (b,\infty) \to \mathbf{R}$, such that $f(x) \in (a,\, c)$ for all $x \in (b,\infty)$.

As a special case the following extensions of (1) and (2) and of their dual inequalities were obtained (see [**1**]):

$$\int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx < \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right] x^{p-k} f^p(x)\,dx,$$

whenever $k > 1$ and $0 < \int_0^b x^{p-k} f^p(x)\,dx < \infty$; and

$$\int_b^\infty x^{-k}\left(\int_x^\infty f(t)\,dt\right)^p dx < \left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{(1-k)/p}\right] x^{p-k} f^p(x)\,dx,$$

whenever $k < 1$ and $0 < \int_b^\infty x^{p-k} f^p(x)\,dx < \infty$; and

$$\int_0^b x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\,dt\right] dx < e^{\gamma/\alpha}\int_0^b \left[1-\left(\frac{x}{b}\right)^{\alpha}\right] x^{\gamma-1} f(x)\,dx,$$

whenever $\alpha > 0$ and $0 < \int_0^b x^{\gamma-1} f(x)\,dx < \infty$; and

$$\int_b^\infty x^{\gamma-1} \exp\left[-\frac{\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}\log f(t)\,dt\right] dx < e^{\gamma/\alpha}\int_b^\infty \left[1-\left(\frac{b}{x}\right)^{-\alpha}\right] x^{\gamma-1} f(x)\,dx,$$

whenever $\alpha < 0$ and $0 < \int_b^\infty x^{\gamma-1} f(x)\,dx < \infty$, where $\alpha, \gamma, b, p, k \in \mathbf{R}$ such that $b > 0$, $\alpha \neq 0$, $p > 1$, $k \neq 1$ and f is a nontrivial, nonnegative function.

Now we state and prove some improvements and reverses of these results.

2. Log-convexity of Hardy-Polya-Knopp differences.

Lemma 2.1. *Let us define the function*

$$\varphi_s(x) = \begin{cases} x^s/s(s-1) & s \neq 0, 1; \\ -\log x & s = 0; \\ x\log x & s = 1. \end{cases}$$

Then $\varphi_s''(x) = x^{s-2}$, *that is,* $\varphi_s(x)$ *is convex for* $x > 0$.

Lemma 2.2. *Let us define another function,*

$$\psi_s(x) = \begin{cases} (1/s^2)e^{sx} & s \neq 0; \\ (1/2)x^2 & s = 0. \end{cases}$$

Then $\psi_s''(x) = e^{sx}$, that is, $\psi_s(x)$ is convex.

The following lemma is equivalent to the definition of the convex function (see [**7**, page 2]).

Lemma 2.3. *If ϕ is continuous and convex for all s_1, s_2 and s_3 of an open interval I for which $s_1 < s_2 < s_3$, then*

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0.$$

We quote here another useful lemma from log-convexity theory.

Lemma 2.4 [**8**]. *A positive function f is* log-*convex in the Jensen sense on an open interval I, that is, for each $s, t \in I$,*

$$f(s)f(t) \geq f^2\left(\frac{s+t}{2}\right)$$

if and only if the relation

$$u^2 f(s) + 2uwf\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0$$

holds for each real u, w and $s, t \in I$.

First, consider Hardy differences and their dual.

Theorem 2.5. *Let the conditions of Theorem* 1.1 *be satisfied, and let φ_s be given by Lemma* 2.1. *Let F be defined by*

$$F(s) = \int_0^b v(x)\varphi_s(f(x))\frac{dx}{x} - \int_0^b u(x)\varphi_s\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x}. \tag{6}$$

Then F is log*-convex, i.e., the following inequality is valid*

$$[F(p)]^{r-s} \leq [F(r)]^{p-s}[F(s)]^{r-p}, \tag{7}$$

for $-\infty < s < p < r < \infty$.

Proof. Let us consider the function ϕ defined by

$$\begin{aligned} \phi(x) &= u^2\varphi_s(x) + 2uw\varphi_r(x) + w^2\varphi_p(x), \\ &\text{where } r = \frac{s+p}{2}; \quad u, w \in \mathbf{R}, \\ \phi''(x) &= u^2x^{s-2} + 2uwx^{r-2} + w^2x^{p-2} \\ &= (ux^{(s/2)-1} + wx^{(p/2)-1})^2 \geq 0. \end{aligned}$$

ϕ is convex for $x \in \mathbf{R}^+$; therefore (4) is equivalent to

$$u^2F(s) + 2uwF(r) + w^2F(p) \geq 0,$$

i.e., by Lemma 2.4,

$$F^2(r) \leq F(s)F(p).$$

So F is log-convex in the Jensen sense. Since

$$\lim_{s\to 0} F(s) = F(0) \text{ and } \lim_{s\to 1} F(s) = F(1),$$

F is continuous for $s \in R$ and therefore $\log F$ is convex. Lemma 2.3 for $-\infty < s < p < r < \infty$ yields:

$$(r-s)\log F(p) \leq (r-p)\log F(s) + (p-s)\log F(r),$$

which is equivalent to (7). □

A similar consequence of Theorem 1.2 is:

Theorem 2.6. *Let the conditions of Theorem* 1.2 *be satisfied, and let φ_s be given by Lemma* 2.1*. Let $\widetilde{F}$ be defined by*

$$\widetilde{F}(s) = \int_b^\infty v(x)\varphi_s(f(x))\,\frac{dx}{x} - \int_b^\infty u(x)\varphi_s\left(x\int_x^\infty \frac{f(t)}{t^2}\,dt\right)\frac{dx}{x}. \tag{8}$$

Then $\widetilde{F}$ *is* log-*convex, i.e., the following inequality is valid.*

$$\left[\widetilde{F}(p)\right]^{r-s} \leq \left[\widetilde{F}(r)\right]^{p-s} \left[\widetilde{F}(s)\right]^{r-p}, \tag{9}$$

for $-\infty < s < p < r < \infty$.

If we use ψ_s for φ_s, we get the following:

Theorem 2.7. *Let the conditions of Theorem* 1.1 *be satisfied, and let* ψ_s *be given by Lemma* 2.2. *Let* G *be defined by*

$$G(s) = \int_0^b v(x)\psi_s(f(x))\frac{dx}{x} - \int_0^b u(x)\psi_s\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x}. \tag{10}$$

Then G *is* log-*convex, i.e., the following inequality is valid.*

$$[G(p)]^{r-s} \leq [G(r)]^{p-s}[G(s)]^{r-p}, \tag{11}$$

for $-\infty < s < p < r < \infty$.

Theorem 2.8. *Let the conditions of Theorem* 1.2 *be satisfied, and let* ψ_s *be given by Lemma* 2.2. *Let* $\widetilde{G}$ *be defined by*

$$\begin{aligned} \widetilde{G}(s) = &\int_b^\infty v(x)\psi_s(f(x))\frac{dx}{x} \\ &- \int_b^\infty u(x)\psi_s\left(x\int_x^\infty \frac{f(t)}{t^2}\,dt\right)\frac{dx}{x}. \end{aligned} \tag{12}$$

Then $\widetilde{G}$ *is* log-*convex, i.e., the following inequality is valid.*

$$\left[\widetilde{G}(p)\right]^{r-s} \leq \left[\widetilde{G}(r)\right]^{p-s} \left[\widetilde{G}(s)\right]^{r-p}, \tag{13}$$

for $-\infty < s < p < r < \infty$.

3. Improvements and reverses of Hardy's inequality. We state and prove an improvement and reverse of strengthened classical Hardy's inequality and it dual.

Theorem 3.1. *Let $k, b \in \mathbf{R}$ be such that $k \neq 1$ and $b > 0$, let f be a nontrivial and nonnegative function, and let $p \in \mathbf{R} \backslash \{0,1\}$.*

(i) *If $(k-1)/p > 0$, then*

$$\frac{1}{p(p-1)}\left\{\left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right] x^{p-k} f^p(x)\,dx - \int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx\right\} \leq \left(\frac{p}{k-1}\right)^p [H(s)]^{(r-p)/(r-s)} [H(r)]^{(p-s)/(r-s)} \tag{14}$$

for $-\infty < s < p < r < \infty$; and

$$\frac{1}{p(p-1)}\left\{\left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right] x^{p-k} f^p(x)\,dx - \int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^p dx\right\} \geq \left(\frac{p}{k-1}\right)^p [H(s)]^{(r-p)/(r-s)} [H(r)]^{(p-s)/(r-s)} \tag{15}$$

for $-\infty < p < r < s < \infty$ and $-\infty < r < s < p < \infty$, where

$$\begin{aligned} H(r) = &\int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right] \varphi_r\left(x^{(p-k+1)/p} f(x)\right) \frac{dx}{x} \\ &- \int_0^b \varphi_r\left(\frac{k-1}{p} x^{(-k+1)/p} \int_0^x f(t)\,dt\right) \frac{dx}{x}. \end{aligned} \tag{16}$$

(ii) *If $(1-k)/p > 0$, then*

$$\frac{1}{p(p-1)}\left\{\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{(1-k)/p}\right] x^{p-k} f^p(x)\,dx - \int_b^\infty x^{-k}\left(\int_x^\infty f(t)\,dt\right)^p dx\right\} \leq \left(\frac{p}{1-k}\right)^p \left[\widetilde{H}(s)\right]^{(r-p)/(r-s)} \left[\widetilde{H}(r)\right]^{(p-s)/(r-s)} \tag{17}$$

for $-\infty < s < p < r < \infty$; *and*

$$\frac{1}{p(p-1)}\left\{\left(\frac{p}{1-k}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{(1-k)/p}\right] x^{p-k} f^p(x)\,dx - \int_b^\infty x^{-k}\left(\int_x^\infty f(t)\,dt\right)^p dx\right\} \geq \left(\frac{p}{1-k}\right)^p \left[\widetilde{H}(s)\right]^{(r-p)/(r-s)} \left[\widetilde{H}(r)\right]^{(p-s)/(r-s)} \tag{18}$$

for $-\infty < p < r < s < \infty$ *and* $-\infty < r < s < p < \infty$, *where*

$$\begin{aligned}\widetilde{H}(r) = &\int_b^\infty \left[1-\left(\frac{b}{x}\right)^{(1-k)/p}\right] \varphi_r\left(x^{(p-k+1)/p} f(x)\right) \frac{dx}{x}\\ &- \int_b^\infty \varphi_r\left(\frac{1-k}{p} x^{(-k+1)/p} \int_x^\infty f(t)\,dt\right)\frac{dx}{x}.\end{aligned} \tag{19}$$

Proof. The proof follows from Theorems 2.5 and 2.6 by choosing the weight function $u(x) = 1$ (so that $v(x) = 1-(x/b)$ and $v(x) = 1-(b/x)$).

Consider the case when $k > 1$ first. Let $\alpha > 0$; by replacing the parameter b by $a(= b^\alpha)$ and choosing for f the function $x \mapsto f(x^{\alpha^{-1}})x^{\alpha^{-1}-1}$ (6) becomes

$$\begin{aligned}F_\alpha(s) = &\int_0^a \left(1-\frac{x}{a}\right) \varphi_s\left(f\left(x^{\alpha^{-1}}\right) x^{\alpha^{-1}-1}\right) \frac{dx}{x}\\ &- \int_0^a \varphi_s\left(\frac{1}{x}\int_0^x f\left(t^{\alpha^{-1}}\right) t^{\alpha^{-1}-1}\,dt\right)\frac{dx}{x},\end{aligned} \tag{20}$$

while (7) becomes

$$[F_\alpha(p)]^{r-s} \leq [F_\alpha(r)]^{p-s}\,[F_\alpha(s)]^{r-p}, \tag{21}$$

i.e., $F_\alpha(s)$ is log-convex. Of course, we can give simpler form for F_α. By the substitutions $l = t^{\alpha^{-1}}$ and $y = x^{\alpha^{-1}}$, respectively, we have

$$F_\alpha(s) = \alpha\left\{\int_0^{a^{\alpha^{-1}}} \left(1-\left(\frac{y}{b}\right)^\alpha\right) \varphi_s\left(f(y)y^{1-\alpha}\right)\frac{dy}{y} - \int_0^{a^{\alpha^{-1}}} \varphi_s\left(\alpha\, y^{-\alpha}\int_0^y f(l)\,dl\right)\frac{dy}{y}\right\}$$

i.e.,

$$F_\alpha(s) = \alpha\biggl\{\int_0^b \left(1-\left(\frac{x}{b}\right)^\alpha\right)\varphi_s\left(f(x)x^{1-\alpha}\right)\,\frac{dx}{x} - \int_0^b \varphi_s\left(\alpha\, x^{-\alpha}\int_0^x f(l)\,dl\right)\frac{dx}{x}\biggr\}.$$

For $\alpha = (k-1)/p$, we have

$$F_{(k-1)/p}(s) = \frac{k-1}{p}\biggl\{\int_0^b \left(1-\left(\frac{x}{b}\right)^{(k-1)/p}\right)\varphi_s\Bigl(f(x)x^{(p-k+1)/p}\Bigr)\frac{dx}{x} - \int_0^b \varphi_s\left(\frac{k-1}{p}\,x^{(1-k)/p}\int_0^x f(l)\,dl\right)\frac{dx}{x}\biggr\}.$$

From here (21) reduces to

$$\text{(22)}\quad \int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right]\varphi_p\left(f(x)x^{(p-k+1)/p}\right)\,\frac{dx}{x} - \int_0^b \varphi_p\left(\frac{k-1}{p}x^{(1-k)/p}\int_0^x f(l)\,dl\right)\frac{dx}{x} \le [H(s)]^{(p-r)/(s-r)}[H(r)]^{(s-p)/(s-r)}.$$

For $p \in \mathbf{R}\backslash\{0,1\}$, we get (14).

If in (21) $s \to r$, $p \to s$, $r \to p$ and $s \to p$, $p \to r$, $r \to s$, then for $\alpha = (k-1)/p$, we have

$$\text{(23)}\quad \int_0^b \left[1-\left(\frac{x}{b}\right)^{(k-1)/p}\right]\varphi_p\left(f(x)x^{(p-k+1)/p}\right)\,\frac{dx}{x} - \int_0^b \varphi_p\left(\frac{k-1}{p}x^{(1-k)/p}\int_0^x f(l)\,dl\right)\frac{dx}{x} \ge [H(s)]^{(p-r)/(s-r)}[H(r)]^{(s-p)/(s-r)}.$$

And from here for $p \in \mathbf{R}\backslash\{0,1\}$, we get (15).

Now, suppose that $k < 1$. We choose, again, the weight function $u(x) = 1$ and $v(x) = 1 - b/x$. Let $\beta > 0$. Now by replacing the parameter b by $a(=\ b^{\beta})$ and by choosing for f the function $x \mapsto f(x^{\beta^{-1}})\,x^{\beta^{-1}+1}$ (8) becomes

$$
\begin{aligned}
\widetilde{F}_{\beta}(s) &= \int_a^{\infty} \left(1 - \frac{a}{x}\right) \varphi_s \left(f(x^{\beta^{-1}})\,x^{\beta^{-1}+1}\right) \frac{dx}{x} \\
&\quad - \int_a^{\infty} \varphi_s \left(x \int_x^{\infty} f(t^{\beta^{-1}})\,t^{\beta^{-1}-1}\,dt\right) \frac{dx}{x},
\end{aligned} \tag{24}
$$

while (9) becomes

$$
\left[\widetilde{F}_{\beta}(p)\right]^{r-s} \le \left[\widetilde{F}_{\beta}(r)\right]^{p-s} \left[\widetilde{F}_{\beta}(s)\right]^{r-p}. \tag{25}
$$

i.e., $\widetilde{F}_{\beta}(s)$ is log-convex. Of course, we can give simpler form for $\widetilde{F}_{\beta}$. By the substitutions $l = t^{\beta^{-1}}$ and $y = x^{\beta^{-1}}$, respectively, we have

$$
\begin{aligned}
\widetilde{F}_{\beta}(s) = \beta \int_{a^{\beta^{-1}}}^{\infty} &\left[1 - \frac{a}{y^{\beta}}\right] \varphi_s \left(y^{1+\beta} f(y)\right) \frac{dy}{y} \\
&- \beta \int_{a^{\beta^{-1}}}^{\infty} \varphi_s \left(\beta y^{\beta} \int_y^{\infty} f(l)\,dl\right) \frac{dy}{y}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\widetilde{F}_{\beta}(s) = \beta \Bigg\{ &\int_b^{\infty} \left[1 - \left(\frac{b}{x}\right)^{\beta}\right] \varphi_s \left(x^{1+\beta} f(x)\right) \frac{dx}{x} \\
&- \int_b^{\infty} \varphi_s \left(\beta x^{\beta} \int_x^{\infty} f(l)\,dl\right) \frac{dx}{x} \Bigg\}.
\end{aligned}
$$

For $\beta = (1-k)/p$, we have

$$
\begin{aligned}
\widetilde{F}_{(1-k)/p}(s) = \frac{1-k}{p} \Bigg\{ &\int_b^{\infty} \left[1 - \left(\frac{b}{x}\right)^{(1-k)/p}\right] \varphi_s \left(x^{(p-k+1)/p} f(x)\right) \frac{dx}{x} \\
&- \int_b^{\infty} \varphi_s \left(\frac{1-k}{p} x^{(1-k)/p} \int_x^{\infty} f(l)\,dl\right) \frac{dx}{x} \Bigg\}.
\end{aligned}
$$

From here (25) reduces to

$$
\begin{aligned}
\int_b^{\infty} &\left[1 - \left(\frac{b}{x}\right)^{(1-k)/p}\right] \varphi_p \left(f(x) x^{(p-k+1)/p}\right) \frac{dx}{x} \\
&- \int_b^{\infty} \varphi_p \left(\frac{1-k}{p}\, x^{(1-k)/p} \int_x^{\infty} f(l)\,dl\right) \frac{dx}{x} \\
&\le [\widetilde{H}(s)]^{(p-r)/(s-r)} [\widetilde{H}(r)]^{(s-p)/(s-r)}.
\end{aligned} \tag{26}
$$

For $p \in \mathbf{R}\backslash\{0,1\}$, we get (17).

If in (25) $s \to r$, $p \to s$, $r \to p$; and $s \to p$, $p \to r$, $r \to s$, then for $\beta = (1-k)/p$, we have

$$\begin{aligned}(27)\quad &\int_b^\infty \left[1-\left(\frac{b}{x}\right)^{(1-k)/p}\right] \varphi_p\left(f(x)x^{(p-k+1)/p}\right) \frac{dx}{x} \\ &\qquad - \int_b^\infty \varphi_p\left(\frac{1-k}{p}\, x^{(1-k)/p} \int_x^\infty f(l)\, dl\right) \frac{dx}{x} \\ &\qquad \geq [\widetilde{H}(s)]^{(p-r)/(s-r)} [\widetilde{H}(r)]^{(s-p)/(s-r)}.\end{aligned}$$

And from here for $p \in \mathbf{R}\backslash\{0,1\}$, we get (18). □

Remark 3.2. In fact we have proved the more general results. Namely, (22) and (26) are valid for $-\infty < s < p < r < \infty$; the inequalities (23) and (27) are valid for $-\infty < r < s < p < \infty$ and $-\infty < p < r < s < \infty$.

4. Improvements and reverses of Polya-Knopp inequality. We state and prove an improvement and reverse of the Polya-Knopp inequality and of its dual.

Theorem 4.1. *Let $\alpha, \gamma, b \in \mathbf{R}$ be such that $\alpha \neq 0$ and $b > 0$, and let f be a positive function,*

(i) *if $\alpha > 0$, then*

$$\begin{aligned}(28)\quad &e^{\gamma/\alpha} \int_0^b \left[1-\left(\frac{x}{b}\right)^\alpha\right] x^{\gamma-1} f(x)\, dx \\ &\qquad - \int_0^b x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\, dt\right] dx \\ &\qquad \leq e^{\gamma/\alpha}[P(r)]^{(1-s)/(r-s)}[P(s)]^{(r-1)/(r-s)}\end{aligned}$$

for $-\infty < s < 1 < r < \infty$; and

$$\begin{aligned}(29)\quad &e^{\gamma/\alpha} \int_0^b \left[1-\left(\frac{x}{b}\right)^\alpha\right] x^{\gamma-1} f(x)\, dx \\ &\qquad - \int_0^b x^{\gamma-1} \exp\left[\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\log f(t)\, dt\right] dx \\ &\qquad \geq e^{\gamma/\alpha}[P(r)]^{(1-s)/(r-s)}[P(s)]^{(r-1)/(r-s)}\end{aligned}$$

for $-\infty < 1 < r < s < \infty$ *and* $-\infty < r < s < 1 < \infty$*, where*

$$
\begin{aligned}
P(s) = &\int_0^b \left[1 - \left(\frac{x}{b}\right)^{\alpha}\right] \psi_s\left(\log(x^{\gamma} f(x))\right) \frac{dx}{x} \\
&- \int_0^b \psi_s\biggl(\alpha x^{-\alpha} \int_0^x l^{\alpha-1} \log\left(l^{\gamma} f(l)\right) dl\biggr) \frac{dx}{x}.
\end{aligned} \tag{30}
$$

(ii) *If* $\alpha < 0$*, then*

$$
\begin{aligned}
e^{\gamma/\alpha} &\int_b^{\infty} \left[1 - \left(\frac{b}{x}\right)^{-\alpha}\right] x^{\gamma-1} f(x)\, dx \\
&- \int_b^{\infty} x^{\gamma-1} \exp\left[-\frac{\alpha}{x^{\alpha}} \int_x^{\infty} t^{\alpha-1} \log f(t)\, dt\right] dx \\
&\le e^{\gamma/\alpha} \left[\widetilde{P}(r)\right]^{(1-s)/(r-s)} \left[\widetilde{P}(s)\right]^{(r-1)/(r-s)}
\end{aligned} \tag{31}
$$

for $-\infty < s < 1 < r < \infty$*; and*

$$
\begin{aligned}
e^{\gamma/\alpha} &\int_b^{\infty} \left[1 - \left(\frac{b}{x}\right)^{-\alpha}\right] x^{\gamma-1} f(x)\, dx \\
&- \int_b^{\infty} x^{\gamma-1} \exp\left[-\frac{\alpha}{x^{\alpha}} \int_x^{\infty} t^{\alpha-1} \log f(t)\, dt\right] dx \\
&\ge e^{\gamma/\alpha} \left[\widetilde{P}(r)\right]^{(1-s)/(r-s)} \left[\widetilde{P}(s)\right]^{(r-1)/(r-s)}
\end{aligned} \tag{32}
$$

for $-\infty < 1 < r < s < \infty$ *and* $-\infty < r < s < 1 < \infty$*, where*

$$
\begin{aligned}
\widetilde{P}(s) = &\int_b^{\infty} \left[1 - \left(\frac{b}{x}\right)^{-\alpha}\right] \psi_s\left(\log(x^{\gamma} f(x))\right) \frac{dx}{x} \\
&- \int_b^{\infty} \psi_s\biggl(-\alpha x^{-\alpha} \int_x^{\infty} l^{\alpha-1} \log\left(l^{\gamma} f(l)\right) dl\biggr) \frac{dx}{x}.
\end{aligned} \tag{33}
$$

Proof. The proof follows from Theorems 2.7 and 2.8 by choosing the weight function $u(x) = 1$ (so that $v(x) = 1-(x/b)$ and $v(x) = 1-(b/x)$).

Let $\alpha > 0$. By replacing the parameter b by $a(= b^\alpha)$ and choosing for the function f, $x \mapsto \log(x^{\gamma/\alpha} f(x^{1/\alpha}))$. Then (10) becomes

$$\begin{aligned} G_\alpha(s) = &\int_0^a \left(1 - \frac{x}{a}\right) \psi_s \left(\log\left(x^{\gamma/\alpha} f\left(x^{1/\alpha}\right)\right)\right) \frac{dx}{x} \\ &- \int_0^a \psi_s \left(\frac{1}{x} \int_0^x \log\left(t^{\gamma/\alpha} f\left(t^{1/\alpha}\right)\right) dt\right) \frac{dx}{x}, \end{aligned} \tag{34}$$

while (11) becomes

$$[G_\alpha(p)]^{r-s} \le [G_\alpha(r)]^{p-s} [G_\alpha(s)]^{r-p}, \tag{35}$$

i.e., $G_\alpha(s)$ is log-convex. Of course we can give simpler form for G_α. By the substitutions $l = t^{\alpha^{-1}}$ and $y = x^{\alpha^{-1}}$, respectively, we have

$$\begin{aligned} G_\alpha(s) = \alpha &\int_0^{a^{\alpha^{-1}}} \left[1 - \frac{y^\alpha}{a}\right] \psi_s \left(\log(y^\gamma f(y))\right) \frac{dy}{y} \\ &- \alpha \int_0^{a^{\alpha^{-1}}} \psi_s \left(\alpha y^{-\alpha} \int_0^y l^{\alpha-1} \log(l^\gamma f(l))\, dl\right) \frac{dy}{y} \end{aligned}$$

i.e.,

$$\begin{aligned} G_\alpha(s) = \alpha \Bigg\{ &\int_0^b \left[1 - \left(\frac{x}{b}\right)^\alpha\right] \psi_s \left(\log(x^\gamma f(x))\right) \frac{dx}{x} \\ &- \int_0^b \psi_s \left(\alpha x^{-\alpha} \int_0^x l^{\alpha-1} \log(l^\gamma f(l))\, dl\right) \frac{dx}{x} \Bigg\}. \end{aligned}$$

From here, (35) is equivalent to

$$\begin{aligned} &\int_0^b \left[1 - \left(\frac{x}{b}\right)^\alpha\right] \psi_p \left(\log(x^\gamma f(x))\right) \frac{dx}{x} \\ &\qquad - \int_0^b \psi_p \left(\alpha x^{-\alpha} \int_0^x l^{\alpha-1} \log(l^\gamma f(l))\, dl\right) \frac{dx}{x} \\ &\qquad \le [P(r)]^{(p-s)/(r-s)} [P(s)]^{(r-p)/(r-s)}. \end{aligned} \tag{36}$$

And, from here for $p = 1$, we get (28).

If in (35) $s \to r$, $p \to s$, $r \to p$ and $s \to p$, $p \to r$, $r \to s$, then we have

$$\begin{aligned}
(37) \quad & \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\alpha}\right] \psi_p \left(\log(x^{\gamma} f(x))\right) \frac{dx}{x} \\
& \qquad - \int_0^b \psi_p \left(\alpha x^{-\alpha} \int_0^x l^{\alpha-1} \log(l^{\gamma} f(l))\, dl \right) \frac{dx}{x} \\
& \qquad \geq [P(r)]^{(p-s)/(r-s)} [P(s)]^{(r-p)/(r-s)}.
\end{aligned}$$

And from here, for $p = 1$, we get (29).

For the case when $\alpha < 0$, we make substitution $x \mapsto \log(x^{-\gamma/\alpha} f(x^{-1/\alpha}))$ for the function f and replace parameter b by $a (= b^{-\alpha})$. Then (12) becomes

$$\begin{aligned}
\widetilde{G}_{\alpha}(s) = & \int_a^{\infty} \left(1 - \frac{a}{x}\right) \psi_s \left(\log(x^{-\gamma/\alpha} f(x^{-1/\alpha}))\right) \frac{dx}{x} \\
& - \int_a^{\infty} \psi_s \left(x \int_x^{\infty} \log\left(t^{-\gamma/\alpha} f\left(t^{-1/\alpha}\right)\right) \frac{dt}{t^2} \right) \frac{dx}{x},
\end{aligned} \tag{38}$$

while (13) becomes

$$\left[\widetilde{G}_{\alpha}(p)\right]^{r-s} \leq \left[\widetilde{G}_{\alpha}(r)\right]^{p-s} \left[\widetilde{G}_{\alpha}(s)\right]^{r-p}, \tag{39}$$

i.e., $\widetilde{G}_{\alpha}(s)$ is log-convex. Of course we can give the simpler form for $\widetilde{G}_{\alpha}$. By the substitutions $l = t^{-\alpha^{-1}}$ and $y = x^{-\alpha^{-1}}$, respectively, we have

$$\begin{aligned}
\widetilde{G}_{\alpha}(s) = & -\alpha \int_{a^{-\alpha^{-1}}}^{\infty} (1 - a\, y^{\alpha})\, \psi_s \left(\log(y^{\gamma} f(y))\right) \frac{dy}{y} \\
& + \alpha \int_{a^{-\alpha^{-1}}}^{\infty} \psi_s \left(-\alpha\, y^{-\alpha} \int_y^{\infty} l^{\alpha-1} \log(l^{\gamma} f(l))\, dl \right) \frac{dy}{y},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\widetilde{G}_{\alpha}(s) = -\alpha \Bigg\{ & \int_b^{\infty} \left[1 - \left(\frac{x}{b}\right)^{\alpha}\right] \psi_s \left(\log(x^{\gamma} f(x))\right) \frac{dx}{x} \\
& - \int_b^{\infty} \psi_s \left(-\alpha x^{-\alpha} \int_x^{\infty} l^{\alpha-1} \log(l^{\gamma} f(l))\, dl \right) \frac{dx}{x} \Bigg\}.
\end{aligned}$$

From here (39) is equivalent to

$$
\begin{aligned}
(40)\quad \int_b^\infty \left[1-\left(\frac{x}{b}\right)^\alpha\right]\psi_p\left(\log(x^\gamma f(x))\right)\,\frac{dx}{x}&\\
-\int_b^\infty \psi_p\bigg(-\alpha x^{-\alpha}\int_x^\infty l^{\alpha-1}\log(l^\gamma f(l))\,dl\bigg)\,\frac{dx}{x}&\\
\le \left[\widetilde{P}(r)\right]^{(p-s)/(r-s)}\left[\widetilde{P}(s)\right]^{(r-p)/(r-s)}&.
\end{aligned}
$$

From here for $p=1$, we get (31).

If in (39) $s\to r$, $p\to s$, $r\to p$ and $s\to p$, $p\to r$, $r\to s$, we have

$$
\begin{aligned}
(41)\quad \int_b^\infty \left[1-\left(\frac{x}{b}\right)^\alpha\right]\psi_p\left(\log(x^\gamma f(x))\right)\,\frac{dx}{x}&\\
-\int_b^\infty \psi_p\left(-\alpha x^{-\alpha}\int_x^\infty l^{\alpha-1}\log(l^\gamma f(l))\,dl\right)\,\frac{dx}{x}&\\
\ge \left[\widetilde{P}(r)\right]^{(p-s)/(r-s)}\left[\widetilde{P}(s)\right]^{(r-p)/(r-s)}&.
\end{aligned}
$$

And from here, for $p=1$, we get (32). □

Remark 4.2. In fact, we have proved the more general results. Namely (36) and (40) are valid for $-\infty<s<p<r<\infty$; the inequalities (37) and (41) are valid for $-\infty<r<s<p<\infty$ and $-\infty<p<r<s<\infty$.

REFERENCES

1. A. Čižmešija, J. Pečarić and L.E. Persson, *On strengthened Hardy and Pólya-Knopp's inequalities*, J. Approx. Theory **125** (2003), 74–84.

2. G.H. Hardy, *Notes on some points in the integral calculus*, Messenger Math. **54** (1925), 150–156.

3. G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1959.

4. S. Kaijser, L.E. Persson and A. Öberg, *On Carleman and Knopp's inequalities*, J. Approx. Theory **117** (2002), 140–151.

5. A. Kufner and L.E. Persson, *The Hardy inequality–About its history and current status*, Research Report **6**, Dept. Math., Lulea University of Technology, 2002 (ISSN 1400-4003).

6. ———, *Weighted inequalities of Hardy type*, World Scientific Publishing Co., New Jersey, 2003.

7. J. Pečarić, Frank Proschan and Y.L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, Inc., New York, 1992.

8. Slavko Simić, *On logarithmic convexity for differences of power means*, J. Inequal. Appl. **2007**, Article ID 37359, 8 pages.

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