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EXISTENCE OF NONSTATIONARY PERIODIC SOLUTIONS OF Γ-SYMMETRIC ASYMPTOTICALLY LINEAR AUTONOMOUS NEWTONIAN SYSTEMS WITH DEGENERACY

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ABSTRACT. For a finite group Γ, we consider a Γ symmetric autonomous Newtonian system, which is asymptotically linear at ∞ and has 0 and ∞ as isolated degenerate critical points of the corresponding energy function. By means of the equivariant degree theory for gradient G-maps with $G = \Gamma \times S^1$, we associate to the system a topological invariant $\deg_\infty - \deg_0$, which is computable up to an unknown factor due to the degeneracy of the system. Under certain assumptions, this invariant still contains enough information about the symmetric structure of the set of periodic solutions, including the existence, multiplicity and symmetric classification. Numerical examples are provided for Γ being the dihedral groups D_6, D_8, D_{10}, D_{12}.

1. Introduction. Consider a finite group Γ, which is a symmetry group of certain regular polygon or polyhedron in $\mathbf{R}^n$, and define a Γ-action on $V := \mathbf{R}^n$ by permuting the coordinates of the vectors $x \in V$. In particular, V is an *orthogonal* Γ-representation with respect to the usual Euclidean metric. The goal of this paper is to study, in the presence of Γ-*symmetry*, the existence of nonstationary periodic solutions $x : \mathbf{R} \to V$ of the following autonomous Newtonian system

$$(1.1)\qquad \begin{cases} \ddot{x} = -\nabla\varphi(x), \\ x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi), \end{cases}$$

where $\varphi : V \to \mathbf{R}$ is a C^2-differentiable Γ-invariant function such that $(\nabla\varphi)^{-1}(0) = \{0\}$ and $\nabla\varphi$ is asymptotically linear at infinity, i.e., there

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exists a symmetric Γ-equivariant linear map $B : V \to V$ such that

$$\nabla\varphi(x) = Bx + o(\|x\|) \text{ as } \|x\| \to \infty.$$

Moreover, we assume that system (1.1) satisfies the following *degeneracy* assumption

$$\big(\sigma(\nabla^2\varphi(0)) \cup \sigma(B)\big) \cap \{l^2 : l = 0, 1, 2, \dots\} \neq \varnothing.$$

In the nonsymmetric case, i.e., $\Gamma = \{1\}$, problem (1.1) has been investigated by many authors (cf. [**1, 3, 6, 10**], for example. More precisely, the existence problem of nonstationary T-periodic solutions of the system

$$\ddot{x} = -\nabla\varphi(x), \tag{1.2}$$

has been studied for some $T > 0$ and a C^2-differentiable function $\varphi : \mathbf{R}^n \to \mathbf{R}$. A nonstationary T-periodic solution of (1.2) was treated as a critical point of a certain associated S^1-invariant functional, which was defined on an appropriate functional space admitting a natural S^1-action given by the shift in time. Under the assumption that φ has only finitely many critical points with possibly a degenerate one at ∞ (having zero as its Hesse matrix), using a Morse index argument, Benci and Fortunato proved the existence of nonstationary T-periodic solutions and provided a lower estimate on the number of solutions for T large enough, cf. [**3**]. The existence result for system (1.1), allowing finitely many degenerate critical points with possibly at ∞, was also proved in [**6**] by means of the S^1-equivariant degree for gradient maps.

In the symmetric case, system (1.1) was studied in [**11**], where φ was assumed to have only *nondegenerate* critical points at 0 and ∞. The problem (1.1) was reformulated as a variational problem in a $\Gamma \times S^1$-invariant functional space. By associating topological invariants (using the $\Gamma \times S^1$-degree for gradient maps) to the potential functional at 0 and ∞, respectively, the result was the existence of nonstationary periodic solutions of (1.1), as well as a lower estimate of the number of nonstationary periodic solutions with their different symmetries. The main computational tool was based on the usage of a specially developed Maple© package.[1]

The goal of this paper is to obtain similar results as obtained in [**11**], for system (1.1) allowing 0 and ∞ to be isolated *degenerate* critical points of φ. We apply the same scheme as in [**11**] and associate to the system (1.1) a topological invariant $\deg_{\infty} - \deg_0$. Due to the degeneracy assumption, the value of $\deg_p$, for $p \in \{0, \infty\}$, can be only computed up to an unknown factor. However, with additional assumptions on the kernel subspaces, by analyzing the appearances of certain *maximal* orbit types $(H^{\varphi,l})$ in $\deg_p$, the invariant $\deg_{\infty} - \deg_0$ can still provide a sufficient amount of information about the symmetric structure of the set of nonstationary periodic solutions, including the existence, multiplicity and symmetric classification.

The rest of this paper is organized as follows. In Section 2, we include several notions and results from the equivariant topology, which are needed later (in Section 3) to describe the main properties of the equivariant gradient degree and the computational formulae for the degree of $\Gamma \times S^1$-equivariant gradient linear maps, based upon the multiplicativity property and the notion of the so-called *basic degree* (cf. [**2, 7, 8**]). In Section 3 we also formulate the so-called splitting lemma (cf. [**6**]), which is essential for our study of the degenerate system (1.1). In Section 4, we study the existence problem in autonomous Newtonian systems with Γ-symmetries, which allows certain *degeneracy* at the origin and the infinity. By means of the equivariant gradient degree, we associate to the system (1.1) a topological invariant $\deg_{\infty} - \deg_0$ and compute its value up to an unknown factor. The main result is obtained in subsection 4.2, cf. Theorem 4.2, where we discuss the existence and nonexistence of certain maximal orbit types appearing in $\deg_p$ in several degenerate cases, which help us to achieve a symmetric classification of the periodic solution set including the existence, multiplicity and symmetry results. Finally, computational examples are provided with Γ being the dihedral groups D_6, D_8, D_{10} and D_{12}.

2. Preliminaries. In this section, we collect several basic notions and facts from equivariant topology and introduce a few notations used later.

2.1. Notations. Hereafter, Γ stands for a finite group (endowed with discrete topology). The group $S^1 := \{z \in \mathbf{C} : |z| = 1\}$ is considered here as the unit circle with the standard complex multiplication. Let G stand for (if not otherwise specified as $\Gamma \times S^1$) a general compact Lie group.

We write $H \subset G$ to indicate that H is a closed subgroup of G. Two closed subgroups H and K of G are called *conjugate* if there exists a $g \in G$ such that $K = gHg^{-1}$. Denote by (H) the *conjugacy class* of H in G, by $N(H)$ the *normalizer* of H in G, and by $W(H) = N(H)/H$ the *Weyl group* of H in G. The set $\Phi(G) := \{(H) : H \subset G\}$ admits a partial order given by: $(L) \leq (H)$ if and only if L is conjugate to a subgroup of H.

Let X be a G-invariant set and $x \in X$. We adopt the following notations:

$$\begin{aligned} G_x &:= \{g \in G : gx = x\}, \\ G(x) &:= \{gx : g \in G\}, \\ X^H &:= \{x \in X : H \subset G_x\}, \\ X_H &:= \{x \in X : H = G_x\}, \\ X^{(H)} &:= G(X^H), \quad X_{(H)} := G(X_H), \end{aligned}$$

where G_x is called the *isotropy subgroup* of x and $G(x)$ is the *orbit* of x. For $x \in X$, the conjugacy class (G_x) is called the *orbit type* of x. Note that $(G_{x_1}) = (G_{x_2})$ for every $x_1, x_2 \in G(x)$. Roughly speaking, (G_x) can be considered as the *symmetry* of the orbit $G(x)$. We denote by $\mathcal{J}(X)$ the set of all orbit types occurring in X, i.e.,

$$\mathcal{J}(X) := \{(H) \in \Phi(G) : \exists x \in X \text{ such that } H = G_x\}.$$

Let V be an orthogonal G-representation. For $r > 0$, denote by

$$B_r(V) := \{v \in V : \|v\| < r\},$$

and write $B(V) := B_1(V)$ for the unit ball in V. For an infinite-dimensional isometric Hilbert G-representation W, similar notations $B(W)$ and $B_r(W)$ will also be used.

2.2. Euler ring $U(G)$**.** As the equivariant degree defined for gradient G-maps takes values in the so-called *Euler ring* $U(G)$, we recall its definition and basic properties (cf. [**5**]). Motivated by applications, in particular, to study the existence problem of periodic solutions for the Γ-symmetric systems, we also present the general structure of the multiplication tables for $U(\Gamma \times S^1)$ with Γ being a finite group (cf. [**11**]).

Definition 2.1. Given a compact Lie group G, the *Euler ring* $U(G)$ is the free $\mathbf{Z}$-module generated by $\Phi(G)$, i.e., $U(G) = \mathbf{Z}[\Phi(G)]$, with the multiplication $\star : U(G) \times U(G) \to U(G)$ defined on generators (H), $(K) \in \Phi(G)$ by the formula

$$(H) \star (K) = \sum_{(L)\in\Phi(G)} n_L \cdot (L), \tag{2.3}$$

where $n_L = \chi_c((G/H \times G/K)_L/W(L))$, and χ_c stands for the Euler characteristic in Alexander-Spanier cohomology with compact support (cf. [**5, 9, 13**]).

Let $n := \dim G$. For $k = 0, 1, \dots, n$, denote by $\Phi_k(G) := \{(H) \in \Phi(G) : \dim W(H) = k\}$ and $A_k(G) := \mathbf{Z}[\Phi_k(G)]$. Then, the Euler ring $U(G)$, as a $\mathbf{Z}$-module, can be expressed by

$$U(G) = \bigoplus_{k=0}^{n} A_k(G),$$

where each $A_k(G)$ is viewed as a $\mathbf{Z}$-submodule of $U(G)$.

For a general compact Lie group G, the structure of the Euler ring $U(G)$ may be difficult to compute. However, being interested in studying Γ-symmetric problems, we consider a particular type of group G, namely $G = \Gamma \times S^1$ for Γ being a finite group. In this case, we have

$$\Phi_0(G) = \{(K \times S^1) \ : \ K \subset \Gamma\}.$$

Notice that $A_0(G)$ can be identified with the Burnside ring $A(\Gamma)$ of Γ. It can be verified (cf. [**2**]) that the elements of $\Phi_1(G)$ are the conjugacy classes $(\mathcal{H})$ of the so-called φ *twisted l-folded* subgroups of $\Gamma \times S^1$ (with $l = 0, 1, \dots$) given by

$$\mathcal{H} = H^{\varphi,l} := \big\{(\gamma, z) \in H \times S^1 : \varphi(\gamma) = z^l\big\},$$

where H is a subgroup of Γ and $\varphi : H \to S^1$ is a homomorphism. A φ twisted one-folded subgroup $H^{\varphi,1}$ is denoted by H^φ and is called a

TABLE 1. Multiplication table for $U(G)$, $G = \Gamma \times S^1$.

	$A_0(G)$	$A_1(G)$
$A_0(G)$	$A_0(G)$-multiplication	multiplication in $A_0(G)$-module $A_1(G)$
$A_1(G)$	multiplication in $A_0(G)$-module $A_1(G)$	0

twisted subgroup of $\Gamma \times S^1$. Moreover, it was shown in [**2**] that there exists an $A_0(G)$-module structure on $A_1(G)$.

The following $U(G)$-multiplication result was proved in [**11**].

Theorem 2.1. *Let $G = \Gamma \times S^1$ with Γ being a finite group. Then the multiplication table for the Euler ring $U(\Gamma \times S^1)$ is given by Table* 1.

Remark 2.1. For $G = \Gamma \times S^1$ with Γ being finite, both the $A_0(G)$-multiplication and the multiplication in the $A_0(G)$-module $A_1(G)$, as referred to in Theorem 2.1, can be effectively computed using explicit formulae and specially developed Maple© routines (cf. [**2, 11**] for example).

3. Equivariant degree for gradient G-maps. In this section, we recall several properties of the equivariant degree for gradient G-maps defined in [**7**]. Based on these properties, we present a simplified derivation of explicit computational formulae for gradient G-isomorphisms in the case $G = \Gamma \times S^1$ with Γ being finite (for more detailed derivation, we refer to [**11**]). In preparation for studying problems with degeneracy conditions, we include a result called the *splitting lemma* (cf. [**6**]). We also extend the computational formulae for G-equivariant gradient compact fields with degenerate critical points.

3.1. Definition and properties. Let G be a compact Lie group and V an orthogonal G-representation.

Definition 3.1. (i) A map $f : V \to V$ is called a *gradient G-map* if there exists a G-invariant function $\varphi : V \to \mathbf{R}$ of class C^1 such that

$f = \nabla\varphi$. Similarly, a map $h : [0,1] \times V \to V$ is called a *gradient G-homotopy* if there exists a G-invariant C^1-function $\psi : [0,1] \times V \to \mathbf{R}$ such that $h_t = \nabla_x \psi_t$, where $h_t(x) := h(t,x)$ and $\psi_t(x) := \psi(t,x)$ for all $(t,x) \in [0,1] \times V$ (here, ∇_x stands for the gradient with respect to x).

(ii) Let $\Omega \subset V$ be an open bounded G-invariant set. A gradient G-map $f : V \to V$ is called *Ω-admissible* if $f(x) \neq 0$ for all $x \in \partial\Omega$, and the pair (f, Ω) is called a *∇_G-admissible* pair. Two ∇_G-admissible pairs (f_0, Ω) and (f_1, Ω) are said to be *∇_G-homotopic*, if there exists a gradient G-homotopy $h : [0,1] \times V \to V$ such that $h(0,\cdot) = f_0$, $h(1,\cdot) = f_1$ with (h_t, Ω) being ∇_G-admissible for all $t \in (0,1)$.

It was established in [**7**] that to each ∇_G-admissible pair (f, Ω), one can associate an element $\nabla_G\text{-deg}\,(f,\Omega)$ in $U(G)$. This function ∇_G-deg satisfies all the properties expected from a reasonable degree theory and, in fact, it classifies the ∇_G-homotopy classes of gradient G-maps (cf. [**4**]). The important properties of this degree are listed in the following theorem.

Theorem 3.1 (cf. [**7**]). *Let G be a compact Lie group, V an orthogonal G-representation, $\Omega \subset V$ an open bounded G-invariant subset and $f : V \to V$ an Ω-admissible gradient G-map. There exists a function ∇_G-deg associating to each ∇_G-admissible pair (f,Ω) an element $\nabla_G\text{-deg}(f,\Omega) \in U(G)$ such that the following properties are satisfied*:

(P1) (Existence). *If $\nabla_G\text{-deg}\,(f,\Omega) = \sum_{(H)} n_H \cdot (H)$ is such that $n_{H_0} \neq 0$ for some $(H_0) \in \Phi(G)$, then there exists an $x_0 \in \Omega$ with $f(x_0) = 0$ and $H_0 \subset G_{x_0}$.*

(P2) (Additivity). *Suppose that Ω_1 and Ω_2 are two disjoint open G-invariant subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then,*

$$\nabla_G\text{-deg}\,(f,\Omega) = \nabla_G\text{-deg}\,(f,\Omega_1) + \nabla_G\text{-deg}\,(f,\Omega_2).$$

(P3) (Homotopy invariance). *If $h : [0,1] \times V \to V$ is a ∇_G-admissible homotopy, then*

$$\nabla_G\text{-deg}\,(h_t,\Omega) \equiv \text{constant},$$

with respect to t.

(P4) (Multiplicativity). *Let W be another orthogonal G-representation, and let $(\widetilde{f},\widetilde{\Omega})$ be a ∇_G-admissible pair with $\widetilde{\Omega} \subset W$ an open bounded G-invariant subset and $\widetilde{f} : W \to W$ a gradient G-map. Then,*

$$\nabla_G\text{-deg}\,(f \times \widetilde{f}, \Omega \times \widetilde{\Omega}) = \nabla_G\text{-deg}\,(f,\Omega) \star \nabla_G\text{-deg}\,(\widetilde{f},\widetilde{\Omega}),$$

where the multiplication '$\star$' is taken in $U(G)$.

(P5) (Suspension). *Suppose that X is another orthogonal G-representation, and let $\mathcal{O}$ be an open bounded G-invariant neighborhood of 0 in X. Then*

$$\nabla_G\text{-deg}\,(f \times \text{Id}, \Omega \times \mathcal{O}) = \nabla_G\text{-deg}\,(f,\Omega),$$

where Id is the identity map on X.

Remark 3.1. In the infinite-dimensional case, this gradient degree ∇_G-deg can be extended to the class of G-equivariant gradient compact fields, and all the properties listed above remain valid (cf. [**12**]). We apply the same notations for such an extension.

The simplest examples of gradient G-maps, which provide us with nontrivial gradient degrees, are the negative identity maps defined on irreducible G-representations.

Definition 3.2. Let G be a compact Lie group, and let $\mathcal{V}$ be an irreducible G-orthogonal representation. Consider the map $-\text{Id} : \mathcal{V} \to \mathcal{V}$ given by $x \mapsto -x$. We call the element

$$\text{Deg}_{\mathcal{V}} := \nabla_G\text{-deg}\,(-\text{Id}, B(\mathcal{V})) \in U(G) \tag{3.3}$$

the *gradient basic degree of* $\mathcal{V}$, where $B(\mathcal{V})$ is the unit ball in $\mathcal{V}$.

The concept of the gradient basic degree plays an important role in the computations of general gradient degrees. In many cases, it is possible to reduce the computations of gradient degrees for an arbitrary gradient G-map to the computations of gradient basic degrees. Though the values of gradient basic degrees are not completely clear for a general compact Lie group G, it turns out that, in the case $G = \Gamma \times S^1$ for a finite group Γ, they can be fully computed via the so-called *twisted* *basic degrees* in the language of *twisted primary equivariant degree* (cf. [**2**]). To avoid confusion of notation, we use $\deg_{\mathcal{V}}$ to denote the twisted basic degree of $\mathcal{V}$. The following identities describe the relation between $\mathrm{Deg}_{\mathcal{V}}$ and $\deg_{\mathcal{V}}$, in the case $G = \Gamma \times S^1$ with Γ being a finite group (cf. [**2, 11**])

$$\begin{cases} \mathrm{Deg}_{\mathcal{V}} = \deg_{\mathcal{V}} & \text{if } S^1 \text{ acts trivially on } \mathcal{V}, \\ \mathrm{Deg}_{\mathcal{V}} = (G) - \deg_{\mathcal{V}} & \text{if } S^1 \text{ acts nontrivially on } \mathcal{V}. \end{cases} \tag{3.4}$$

3.2. Computational formulae. Throughout this subsection, $G = \Gamma \times S^1$ with Γ being finite, and V is an orthogonal G-representation.

Viewed as an S^1-representation, V allows the following (G-invariant) decomposition

$$V = V^{S^1} \oplus V', \tag{3.5}$$

where V' is the orthogonal complement of the closed subrepresentation V^{S^1} in V. Let $Q : V \to V$ be a symmetric G-equivariant isomorphism. Consider the restricted maps with respect to the decomposition (3.5), i.e., $\overline{Q} := Q|_{V^{S^1}} : V^{S^1} \to V^{S^1}$ and $Q' := Q|_{V'} : V' \to V'$, which are clearly symmetric G-equivariant isomorphisms. By the multiplicativity property (P4) of the gradient degree, we have

$$\nabla_G\text{-deg}\,(Q, B(V)) = \nabla_G\text{-deg}\,(\overline{Q}, B(V^{S^1})) \star \nabla_G\text{-deg}\,(Q', B(V')).$$

To compute $\nabla_G\text{-deg}\,(\overline{Q}, B(V^{S^1}))$, we find the maximal subrepresentation $E_{\max}$ of V^{S^1} on which $\overline{Q}$ is negative definite. To this end, let $\sigma_-(\overline{Q})$ be the negative spectrum of $\overline{Q}$. Then, the subrepresentation $E_{\max}$ is precisely the direct sum of all the eigenspaces $E(\mu)$, for $\mu \in \sigma_-(\overline{Q})$. By the suspension property (P5) of the gradient degree, we have

$$\begin{aligned} \nabla_G\text{-deg}\,(\overline{Q}, B(V^{S^1})) &= \nabla_G\text{-deg}\,(\overline{Q}, B(E_{\max})) \\ &= \prod_{\mu \in \sigma_-(\overline{Q})} \nabla_G\text{-deg}\,(-\mathrm{Id}, B(E(\mu))), \end{aligned} \tag{3.6}$$

where the second equality uses the multiplicativity property (P4).

Let $\{\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_r\}$ be a complete list of the real irreducible Γ-representations.[2] Considered as a real Γ-representation, each $E(\mu)$ admits the following so-called *isotypical decomposition*

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_r(\mu),$$

where $E_i(\mu)$ is modeled on $\mathcal{V}_i$, meaning that $E_i(\mu)$ is isomorphic to the direct sum of k copies of $\mathcal{V}_i$ for some nonnegative integer k. The integer k is called the $\mathcal{V}_i$*-multiplicity* of μ and will be denoted from now on by $m_i(\mu)$. Therefore, by (3.6) and the multiplicativity property (P4), we have

$$\begin{aligned}
\nabla_G\text{-deg}\,(\overline{Q}, B(V^{S^1})) &= \prod_{\mu\in\sigma_-(\overline{Q})} \nabla_G\text{-deg}\,(-\mathrm{Id}, B(E(\mu))) \\
&= \prod_{\mu\in\sigma_-(\overline{Q})} \prod_{i=0}^{r} \nabla_G\text{-deg}\,(-\mathrm{Id}, B(E_i(\mu))) \\
&= \prod_{\mu\in\sigma_-(\overline{Q})} \prod_{i=0}^{r} \big(\nabla_G\text{-deg}\,(-\mathrm{Id}, B(\mathcal{V}_i))\big)^{m_i(\mu)} \\
&= \prod_{\mu\in\sigma_-(\overline{Q})} \prod_{i=0}^{r} \big(\mathrm{Deg}_{\mathcal{V}_i}\big)^{m_i(\mu)}.
\end{aligned} \tag{3.7}$$

Similarly, one can derive a computational formula for ∇_G-deg $(Q', B(V'))$. More precisely, since $(V')^{S^1} = \{0\}$, the space V' admits a complex structure induced by the S^1-action, so it is a *complex* Γ-representation. Assume that $\{\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_s\}$ is a complete list of the complex irreducible Γ-representations. Then, V' admits the following *complex* Γ-isotypical decomposition

$$V' = U_0 \oplus U_1 \oplus \cdots \oplus U_s,$$

where each U_j is modeled on $\mathcal{U}_j$. Since each subspace U_j is also S^1-invariant, we have the S^1-isotypical decomposition of U_j,

$$U_j = V_{j,1} \oplus V_{j,2} \oplus \cdots \oplus V_{j,l_j},$$

for some integer l_j, where the S^1-action on the component $V_{j,l}$ is defined by the l-folded complex multiplication

$$(\gamma, z)w := z^l \cdot (\gamma w), \text{ for } (\gamma, z) \in \Gamma \times S^1, \quad w \in V_{j,l}.$$

Consequently, we obtain the following G-isotypical decomposition of V'

$$V' = \bigoplus_{j,l} V_{j,l}.$$

Now, by applying the multiplicativity property (P4) of the gradient degree we have

$$\begin{aligned}\nabla_G\text{-deg}\,(Q', B(V')) &= \prod_{j,l} \nabla_G\text{-deg}\,(Q', B(V_{j,l})) \\ &= \prod_{j,l}\ \prod_{\xi\in\sigma_-(Q')} (\nabla_G\text{-deg}\,(Q', B(\mathcal{V}_{j,l})))^{m_{j,l}(\xi)} \qquad (3.8)\\ &= \prod_{\xi\in\sigma_-(Q')}\ \prod_{j,l} (\text{Deg}_{\mathcal{V}_{j,l}})^{m_{j,l}(\xi)}\end{aligned}$$

where $m_{j,l}(\xi)$ is called the $\mathcal{V}_{j,l}$-*multiplicity* of ξ given by $m_{j,l}(\xi) = \dim\,(E(\xi)\cap V_{j,l})/\dim \mathcal{U}_j$.

Therefore, by combining (3.7)–(3.8) with the identities (3.4), we have

Proposition 3.1 (cf. [**11**]). *Let $G = \Gamma\times S^1$ for a finite group Γ, and let V be an orthogonal G-representation. Suppose that $Q : V \to V$ is a linear symmetric G-equivariant isomorphism. Then,*

$$\begin{aligned}\nabla_G\text{-deg}\,(Q, B(V)) = \nabla_G&\text{-deg}\,(\overline{Q}, B(V^{S^1})) \\ &- \nabla_G\text{-deg}\,(\overline{Q}, B(V^{S^1})) \\ &\star \sum_{\xi\in\sigma_-(Q')}\ \sum_{j,l} m_{j,l}(\xi)\text{deg}_{\mathcal{V}_{j,l}},\end{aligned}$$

where $\nabla_G\text{-deg}\,(\overline{Q}, B(V^{S^1}))$ is given by

$$\nabla_G\text{-deg}\,(\overline{Q}, B(V^{S^1})) = \prod_{\mu\in\sigma_-(\overline{Q})}\ \prod_{i=0}^{r} (\text{deg}_{\mathcal{V}_i})^{m_i(\mu)},$$

and $\text{deg}_{\mathcal{V}_i}$, $\text{deg}_{\mathcal{V}_{j,l}}$ denote the twisted basic degrees of $\mathcal{V}_i$ and $\mathcal{V}_{j,l}$, respectively.

3.3. Splitting lemma. In order to be able to study the existence of nonstationary periodic solutions of a variational problem allowing certain degeneracy at critical points, we need the so-called *splitting lemma*, which was established in [**6**]. We also extend the computational formulae in Proposition 3.1 to the class of equivariant gradient compact fields defined on an (infinite-dimensional) isometric Hilbert G-representation.

Let G be a compact Lie group and W an (infinite-dimensional) isometric Hilbert G-representation. Consider a C^2-differentiable G-invariant map $\Phi : W \to \mathbf{R}$, which has the following form

$$\Phi(x) = \frac{1}{2}\langle x, x\rangle_W - g(x), \tag{3.9}$$

where $\langle \cdot, \cdot\rangle_W$ denotes the G-invariant inner product on W and $g : W \to \mathbf{R}$ is a G-invariant function satisfying

(A1) $\nabla g : W \to W$ is a G-equivariant compact map.

Moreover, we assume that

(A2) For $p \in \{0, \infty\}$, there exists a G-equivariant symmetric compact operator $L_p : W \to W$ and a G-invariant $\eta : W \to \mathbf{R}$ such that $\Phi(x) = (1/2)\langle(\mathrm{Id} - L_p)x, x\rangle_W + \eta_p(x)$ with $\nabla\eta_p : W \to W$ being a compact map and

$$\|\nabla^2\eta_p(x)\| \longrightarrow 0, \text{ as } \|x\| \longrightarrow p.$$

(A3) $0 \in \sigma(\mathrm{Id} - L_p)$, i.e., $p \in \{0, \infty\}$ is a degenerate critical point of Φ and

(A4) $p \in \{0, \infty\}$ is isolated as critical point of Φ.

Notice that (A3) implies that $p = 0$ is a critical point of Φ. We also treat $p = \infty$ as a critical point, with Hesse matrix $\mathrm{Id} - L_\infty$. We call ∞ an *isolated* critical point if $\nabla\Phi^{-1}(0)$ is bounded.

Notation 3.1. Denote by $Z_p := \mathrm{Ker}\,(\mathrm{Id} - L_p)$ and $\mathcal{W}_p := \mathrm{Im}\,(\mathrm{Id} - L_p)$. Since L_p is a compact operator, we have that $\mathrm{Id} - L_p$ is a Fredholm operator of index zero. Thus, Z_p and $\mathcal{W}_p$ are finite and infinite dimensional orthogonal G-representations, respectively. Also, $\mathrm{Id} - L_p$ being a symmetric linear operator implies that $W = Z_p \oplus \mathcal{W}_p$ and the operator $\mathcal{Q}_p := (\mathrm{Id} - L_p)|_{\mathcal{W}_p}$ is a G-isomorphism.

The following splitting lemma, which is a simplified version of the theorem proved in [**6**], is essential for computations of the equivariant degree of $\nabla\Phi$ at 0 and ∞.

Lemma 3.1 (Splitting lemma). *Suppose Φ is of the form* (3.9) *satisfying* (A1)–(A4). *Then, for each $p \in \{0,\infty\}$, there exist $\varepsilon_p > 0$ and a G-equivariant gradient homotopy $\nabla H_p : [0,1]\times W \to W$ such that*

(i) $\nabla H_0^{-1}(0) \cap (\mathrm{cl}\,(B_{\varepsilon_0}(W)) \times [0,1]) = \{0\}\times[0,1]$, *and* $\nabla H_\infty^{-1}(0) \subset \mathrm{cl}\,(B_{\varepsilon_\infty}(W))\times[0,1]$.

(ii) $\nabla H_p(t,\cdot) = \mathrm{Id} - \nabla g_p(t,\cdot)$ *for* $t\in[0,1]$, *where* $\nabla g_p : [0,1]\times W \to W$ *is a compact map.*

(iii) $\nabla H_p(0,\cdot) = \nabla\Phi$, *and*

(iv) *there exists a G-equivariant gradient mapping $\nabla\varphi_p : Z_p \to Z_p$ such that* $\nabla H_p(1,(v,w)) = (\nabla\varphi_p(v), \mathcal{Q}_p(w))$, *for* $(v,w)\in Z_p\oplus\mathcal{W}_p$.

Therefore, by the multiplicativity property of the gradient degree, we have (cf. Remark 3.1)

Corollary 3.1. *Suppose Φ is of the form* (3.9) *satisfying* (A1)–(A4). *Then, for $p\in\{0,\infty\}$, there exist $\varepsilon_p>0$ and a G-equivariant gradient map $\nabla\varphi_p : Z_p\to Z_p$ such that*

$$\nabla_G\text{-deg}(\nabla\Phi, B_{\varepsilon_p}(W)) = \nabla_G\text{-deg}(\nabla\varphi_p, B_{\varepsilon_p}(Z_p)) \star \nabla_G\text{-deg}(\mathcal{Q}_p, B(\mathcal{W}_p)),$$

where Z_p, $\mathcal{W}_p$ and $\mathcal{Q}_p$ are given by Notation 3.1.

Remark 3.2. Notice that in the case $G = \Gamma\times S^1$ (as usual, we assume Γ is finite), the computational formulae in Proposition 3.1 can be easily extended to the class of G-equivariant gradient compact fields. Indeed, it is well known that each compact operator has a spectrum either composed of 0 and a finite number of eigenvalues, or it is an infinite sequence of eigenvalues convergent to 0 (which is also in the spectrum). Moreover, every nonzero eigenvalue has a finite multiplicity. Consequently, by compactness assumption (A2), there are only finitely many eigenvalues μ of L_p such that $\mu > 1$, which implies that the negative spectrum of $\mathcal{Q}_p = \mathrm{Id} - L_p$ consists of only finitely many eigenvalues, each of which has a finite multiplicity. Therefore, by the suspension property of the gradient degree in the infinite-dimensional case, we have the following analog of Proposition 3.1, which can be used for the computations of $\nabla_G\text{-deg}\,(\mathcal{Q}_p, B(\mathcal{W}_p, p))$.

Proposition 3.2. *Let $G = \Gamma \times S^1$ for a finite group Γ, and let $\mathcal{W}$ be an isometric Hilbert G-representation. Suppose that $\mathcal{Q} : \mathcal{W} \to \mathcal{W}$ is a linear isomorphic G-equivariant gradient compact field. Then,*

$$\begin{aligned}\nabla_G\text{-deg}\,(\mathcal{Q}, B(\mathcal{W}))& \\ &= \nabla_G\text{-deg}\,(\overline{\mathcal{Q}}, B(\mathcal{W}^{S^1}))\\ &\quad - \nabla_G\text{-deg}\,(\overline{\mathcal{Q}}, B(\mathcal{W}^{S^1})) \star \sum_{\xi\in\sigma_-(\mathcal{Q}')}\sum_{j,l} m_{j,l}(\xi)\deg_{\mathcal{V}_{j,l}},\end{aligned}$$

where $\nabla_G\text{-deg}\,(\overline{\mathcal{Q}}, B(\mathcal{W}^{S^1}))$ is given by

$$\nabla_G\text{-}deg\,(\overline{\mathcal{Q}}, B(\mathcal{W}^{S^1})) = \prod_{\mu\in\sigma_-(\overline{\mathcal{Q}})}\prod_{i=0}^{r}(\deg_{\mathcal{V}_i})^{m_i(\mu)}.$$

4. Γ-symmetric autonomous Newtonian systems. Let V be an orthogonal Γ-representation. We are interested in studying non-stationary periodic solutions $x : \mathbf{R} \to V$ of the following autonomous Newtonian system:

$$\begin{cases}\ddot{x} = -\nabla\varphi(x),\\ x(0) = x(2\pi),\ \dot{x}(0) = \dot{x}(2\pi),\end{cases} \tag{4.10}$$

where $\varphi : V \to \mathbf{R}$ is a C^2-differentiable Γ-invariant function satisfying the following assumptions:

(H0) $p \in \{0, \infty\}$ are the only possible critical points of φ,

(H1) $\nabla\varphi$ is asymptotically linear at ∞, i.e., there exists a symmetric Γ-equivariant linear map $B : V \to V$ such that

$$\nabla\varphi(x) = Bx + o(\|x\|) \text{ as } \|x\| \to \infty.$$

Let $A := \nabla^2\varphi(0)$. By (H1), A and B are linearizations of $\nabla\varphi$ at 0 and at ∞, respectively.

Remark 4.1. Notice that if $(\sigma(A) \cup \sigma(B)) \cap \{l^2 : l = 0, 1, 2, \dots\} = \varnothing$, then the linearizations of (4.10) at $p = 0$ and $p = \infty$ have no nonzero solutions. This nondegenerate case was studied in [**11**]. In this paper, we are interested in the *degenerate* situations, i.e., $(\sigma(A) \cup \sigma(B)) \cap \{l^2 : l = 0, 1, 2, \dots\} \neq \varnothing$. For simplicity, assume that $\sigma(A)$, respectively $\sigma(B)$, has a nontrivial intersection with $\{l^2 : l = 0, 1, 2, \dots\}$, which contains only one element, namely,

(D)
$$\begin{cases} \sigma(A) \cap \{l^2 : l = 0, 1, 2, \dots\} = \{l_0^2\}, \\ \sigma(B) \cap \{l^2 : l = 0, 1, 2, \dots\} = \{l_\infty^2\}. \end{cases}$$

4.1. Reformulation in functional spaces. We use the standard identification $\mathbf{R}/2\pi\mathbf{Z} \simeq S^1$. Consider the Sobolev space $W := H^1(S^1; V)$ equipped with the usual inner product

$$\langle u, v \rangle_{H^1} := \int_0^{2\pi} \langle \dot{u}(t), \dot{v}(t) \rangle + \langle u(t), v(t) \rangle \, dt, \quad u, v \in W.$$

It is an isometric Hilbert G-representation (for $G = \Gamma \times S^1$) with the G-action defined by

$$\left((\gamma, e^{i\tau})u\right)(t) := \gamma u(t + \tau), \quad \gamma \in \Gamma,\ \tau \in \mathbf{R},\ u \in W.$$

Define $\Phi : W \to \mathbf{R}$ by

$$\Phi(u) := \int_0^{2\pi} \left(\frac{1}{2} \|\dot{u}(t)\|^2 - \varphi(u(t)) \right) dt, \tag{4.11}$$

where $\|\cdot\|$ stands for the Γ-orthogonal norm in V. It can be easily verified that

$$x \text{ is a solution of (4.10)} \iff \nabla\Phi(x) = 0, \quad x \in W.$$

To determine an explicit form of $\nabla\Phi$, define $\widetilde{\varphi} : V \to \mathbf{R}$ by

$$\widetilde{\varphi}(v) = \varphi(v) + \frac{1}{2}\|v\|^2, \quad v \in V, \tag{4.12}$$

and introduce the following maps

$$\begin{aligned} L : H^2(S^1;V) &\longrightarrow L^2(S^1;V), & Lu &= -\ddot{u}+u,\\ j : H^2(S^1;V) &\longrightarrow H^1(S^1;V), & ju &= u,\\ N_{\nabla\widetilde{\varphi}} : C(S^1;V) &\longrightarrow L^2(S^1;V), & N_{\nabla\widetilde{\varphi}}(u) &= \nabla\widetilde{\varphi}(u). \end{aligned}$$

Notice that j is a *compact* embedding. It follows that (cf. [**11**])

$$\nabla\Phi(u) = u - j\circ L^{-1}\circ N_{\nabla\widetilde{\varphi}}(u), \quad u\in W, \tag{4.13}$$

is a G-equivariant gradient compact field. Moreover, by (H0)–(H1) and (D), we are in the setting of subsection 3.3. Indeed,

$$\Phi(u) = \frac{1}{2}\langle u,u\rangle_{H^1} - \int_0^{2\pi}\widetilde{\varphi}(u(t))\,dt$$

satisfies (A1)–(A3) for

$$L_0 = j\circ L^{-1}\circ(A+\mathrm{Id}), \tag{4.14}$$

$$L_\infty = j\circ L^{-1}\circ(B+\mathrm{Id}). \tag{4.15}$$

Also, by (H0), the functional Φ satisfies (A4) in the case $l_0 = l_\infty = 0$ in (D) (see, for instance, [**6**, Lemma 5.2.1]). In the case $l_p \neq 0$ for some $p\in\{0,\infty\}$, we assume that

(H2) $p\in\{0,\infty\}$ is an isolated critical point of Φ whenever $l_p\neq 0$.

Remark 4.2. In general, it is possible that (H2) fails for some $p\in\{0,\infty\}$ with $l_p\neq 0$. However, by an equivariant implicit function theorem argument, it is shown in [**6**] that in the case where (H2) fails, there already exist infinitely many solutions of (4.10), and the minimal period of any solution sufficiently close to the point p is equal to $(2\pi)/l_p$ (cf. [**6**, Theorem 5.2.2]). In particular, (4.10) allows infinitely many nonstationary $(2\pi)/l_p$-periodic solutions automatically. In this paper, we exclude such a possibility by assuming (H2).

Therefore, by (H0)–(H2) and (D), there exist a sufficiently small $\varepsilon > 0$ and large $R > 0$ such that $\nabla_G\text{-deg}\,(\nabla\Phi, B_\varepsilon(W))$ and $\nabla_G\text{-deg}\,(\nabla\Phi, B_R(W))$ are well defined. Moreover, if

$$\nabla_G\text{-deg}\,(\nabla\Phi, B_R(W)) - \nabla_G\text{-deg}\,(\nabla\Phi, B_\varepsilon(W)) \neq 0,$$

then there exists a solution of (4.10) in $B_R(W) \setminus B_\varepsilon(W)$. In order to obtain a multiplicity and symmetric classification result for the nonstationary periodic solution set of the problem (4.10), we need the following important notion (cf. [**2**]).

Definition 4.1. An orbit type $(\mathcal{H})$ in W is called *dominating*, if $(\mathcal{H})$ is maximal with respect to the usual order relation (see subsection 2.1) in the class of all φ-twisted one-folded orbit types in W (in particular, $\mathcal{H} = H^\varphi$).

The following theorem, which can be easily established following the same idea as in Theorem 6.2.1 of [**11**], provides us with a sufficient condition for the existence of a nonstationary periodic solution of (4.10), as well as a lower estimate on the number of solutions with their different symmetries.

Theorem 4.1. *Let $\varphi : V \to \mathbf{R}$ be a Γ-invariant C^2-differentiable map satisfying* (H0)–(H2), (D), *and let $\Phi : W \to \mathbf{R}$ be given by* (4.11), *for $W = H^1(S^1; V)$. Then, there exist $\varepsilon, R > 0$ such that ∇_G-deg$\,(\nabla\Phi, B_R(W)), \nabla_G$-deg$\,(\nabla\Phi, B_\varepsilon(W)) \in U(G)$ are well defined. Moreover, suppose that*

$$\nabla_G\text{-deg}\,(\nabla\Phi, B_R(W)) - \nabla_G\text{-deg}\,(\nabla\Phi, B_\varepsilon(W)) = \sum_{(\mathcal{H})} n_{\mathcal{H}} \cdot (\mathcal{H}) \neq 0.$$

Then,

(i) *if $n_{\mathcal{H}_0} \neq 0$ for some $(\mathcal{H}_0)$, then there exists a nonstationary solution x_0 of* (4.10) *satisfying $G_{x_0} \supset \mathcal{H}_0$.*

(ii) *If such $(\mathcal{H}_0) = (H^{\varphi,l}) \in \Phi_1(G)$ for a dominating orbit type (H^φ) in W, then there exist at least $|\Gamma/H|$ different nonstationary solutions of* (4.10) *with the symmetries at least $(H^{\varphi,l})$.*

For convenience, denote by

$$\begin{aligned} \deg_\infty &:= \nabla_G\text{-deg}\,(\nabla\Phi, B_R(W)), \\ \deg_0 &:= \nabla_G\text{-deg}\,(\nabla\Phi, B_\varepsilon(W)), \end{aligned}$$

where R and ε are given by Theorem 4.1.

4.2. Computations of $\deg_\infty - \deg_0$. This is the main section of this paper. We extend here the computations of $\deg_\infty$ and $\deg_0$ under degeneracy assumption (D), so to apply Theorem 4.1 to obtain the existence and multiplicity result. Therefore, we analyze several possible cases where a nontrivial $(H^{\varphi,l})$-term occurs in $\deg_\infty - \deg_0$, for some dominating orbit type (H^φ). Note that, in general, the complete values of $\deg_0$ and $\deg_\infty$ are unknown due to degeneracy assumption (D). Thus, the coefficients $n_{\mathcal{H}}$ in $\deg_\infty - \deg_0$ cannot be determined with precise values. However, to take advantage of Theorem 4.1 (ii), one only needs to look for an $(H^{\varphi,l})$-term in $\deg_\infty - \deg_0$ with a nonzero coefficient. More precisely, we are interested in finding a nontrivial $(H^{\varphi,l})$-term in $\deg_\infty$, respectively $\deg_0$, which does not appear in $\deg_0$, respectively $\deg_\infty$.

Define the following two linear G-maps

$$\mathcal{A} : W \longrightarrow W, \qquad \mathcal{A} = \mathrm{Id} - j \circ L^{-1} \circ (A + \mathrm{Id}),$$
$$\mathcal{B} : W \longrightarrow W, \qquad \mathcal{B} = \mathrm{Id} - j \circ L^{-1} \circ (B + \mathrm{Id}).$$

It follows that $\mathcal{A}$ and $\mathcal{B}$ are linearized maps of $\nabla\Phi$ at 0 and at ∞, respectively (cf. (4.14)–(4.15)). Consider the S^1-isotypical decomposition of W,

$$W = W^{S^1} \oplus W' = W^{S^1} \oplus \overline{\bigoplus_{l=1}^{\infty} W_l}, \tag{4.16}$$

where $W^{S^1} \simeq V$ is the subspace composed of all V-valued constant functions in W, $W' = (W^{S^1})^\perp$ is the orthogonal complement of W^{S^1} and $W_l \simeq e^{ilt} \cdot V^c$ is equivalent to the complexification V^c of the Γ-representation V (cf. [**2**]).

Recall the operator L is defined by $L(u) = -\ddot{u} + u$, which is S^1-equivariant. Moreover, we have $L|_{W^{S^1}} = \mathrm{Id}$ and $L|_{W_l} = (l^2 + 1)\mathrm{Id}$. Therefore, by the definitions of $\mathcal{A}$ and $\mathcal{B}$, we have

$$\begin{aligned} \mathcal{A}|_{W^{S^1}} &= -A, \qquad \mathcal{A}|_{W_l} = \mathrm{Id} - \frac{1}{l^2+1}(A + \mathrm{Id}), \\ \mathcal{B}|_{W^{S^1}} &= -B, \qquad \mathcal{B}|_{W_l} = \mathrm{Id} - \frac{1}{l^2+1}(B + \mathrm{Id}). \end{aligned} \tag{4.17}$$

Remark 4.3. As it was pointed out in Remark 4.1, if $(\sigma(A)\cup\sigma(B))\cap\{l^2 : l=0,1,\dots\}=\varnothing$, then both $\mathcal{A}$ and $\mathcal{B}$ are isomorphisms. In this case, the degrees $\deg_0$ and $\deg_\infty$ can be fully computed (cf. [**11**]). To discuss the degenerate case under assumption (D), we distinguish two cases when $l_p = 0$ and when $l_p > 0$. Thus, we consider the following types of degeneracy

($\mathrm{H3}_0$) $\sigma(A)\cap\{l^2 : l=0,1,2,\dots\}=\{0\}$,

($\mathrm{H3}_l$) $\sigma(A)\cap\{l^2 : l=0,1,2,\dots\}=\{l_0^2\neq 0\}$,

($\mathrm{H4}_0$) $\sigma(B)\cap\{l^2 : l=0,1,2,\dots\}=\{0\}$,

($\mathrm{H4}_l$) $\sigma(B)\cap\{l^2 : l=0,1,2,\dots\}=\{l_\infty^2\neq 0\}$.

Notice that (cf. (4.17))

$$\begin{cases}\mathcal{A}\text{ is a }G\text{-isomorphism on }W^{S^1} \iff 0\ \notin\sigma(A)\\ \mathcal{A}\text{ is a }G\text{-isomorphism on }W_l \quad\iff l^2\notin\sigma(A),\end{cases}\tag{4.18}$$

and a similar relation holds for $\mathcal{B}$.

Since the computations of $\deg_\infty$ and $\deg_0$ are completely analogous, using the formulae stated in Corollary 3.1 and Proposition 3.2, we only discuss in details the computations of $\deg_0$, under the assumptions ($\mathrm{H3}_0$) and ($\mathrm{H3}_l$). A table summarizing the existence/nonexistence of a nontrivial $(H^{\varphi,l})$-term in $\deg_p$, is presented in Theorem 4.2, for $p\in\{0,\infty\}$. For completeness, we also include the nondegeneracy conditions:

(H3) $\sigma(A)\cap\{l^2 : l=0,1,2,\dots\}=\varnothing$,

(H4) $\sigma(B)\cap\{l^2 : l=0,1,2,\dots\}=\varnothing$.

By Corollary 3.1, there exist $\varepsilon>0$ and a G-equivariant gradient map $\nabla\varphi_0 : Z_0\to Z_0$ such that

$$\deg_0 = \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0))\star\nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0)),$$

where $\nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0))$ can be computed by (cf. Proposition 3.2)

$$\nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0)) = \prod_{\mu\in\sigma_-(\mathcal{A}|_{\mathcal{W}_0^{S^1}})}\prod_{i=0}^{r}(\deg_{\mathcal{V}_i})^{m_i(\mu)}$$

$$-\prod_{\mu\in\sigma_-(\mathcal{A}|_{\mathcal{W}_0^{S^1}})}\prod_{i=0}^{r}(\deg_{\mathcal{V}_i})^{m_i(\mu)}\star\sum_{\xi\in\sigma_-(\mathcal{A}|_{\mathcal{W}_0'})}\sum_{j,l}m_{j,l}(\xi)\deg_{\mathcal{V}_{j,l}}.$$

To simplify the notations, put

$$\deg^0_{\mathcal{A}} := \prod_{\mu\in\sigma_-(\mathcal{A}|_{\mathcal{W}_0^{S^1}})} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}, \tag{4.19}$$

$$\deg^1_{\mathcal{A}} := \deg^0_{\mathcal{A}} \star \sum_{\xi\in\sigma_-(\mathcal{A}|_{\mathcal{W}'_0})} \sum_{j,l} m_{j,l}(\xi)\deg_{\mathcal{V}_{j,l}}. \tag{4.20}$$

Then, we have

$$\deg_0 = \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \star (\deg^0_{\mathcal{A}} - \deg^1_{\mathcal{A}}). \tag{4.21}$$

We simplify the formulae (4.19)–(4.21), under different assumptions ($\mathrm{H3}_0$), ($\mathrm{H3}_l$) and (H3), respectively.

Case ($\mathbf{H3}_0$). Under the assumption ($\mathrm{H3}_0$), $\mathcal{A}|_{W_l}$ is a linear G-isomorphism of W_l for each $l \in \{1,2,\dots\}$, and $Z_0 = \mathrm{Ker}\mathcal{A} = \mathrm{Ker}A \subset W^{S^1}$ (cf. (4.18)). Thus,

$$\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \in A_0(G). \tag{4.22}$$

Therefore,

$$\begin{aligned}\deg_0 &= \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \star (\deg^0_{\mathcal{A}} - \deg^1_{\mathcal{A}})\\ &= \underbrace{\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \star \deg^0_{\mathcal{A}}}_{\in A_0(G)} - \underbrace{\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \star \deg^1_{\mathcal{A}}}_{\in A_1(G)},\end{aligned}$$

where $-\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \star \deg^1_{\mathcal{A}}$ is the part that may contribute a nontrivial $(H^{\varphi,l})$-term to $\deg_0$.

Since $\mathcal{W}_0^{S^1} = \mathrm{Im}\,(A)$, we have that $\sigma_-(\mathcal{A}|_{\mathcal{W}_0^{S^1}}) = \sigma_+(A)$ (cf. (4.17)). To interpret the formula (4.20), it is sufficient to observe that (cf. [**11**])

$$\xi \in \sigma_-(\mathcal{A}|_{\mathcal{W}'_0}) \Longleftrightarrow \xi = 1 - \frac{\mu+1}{l^2+1},$$
$$\mu > l^2, \text{ for } \mu \in \sigma(A),\ l \in \{1,2,\dots\},$$

and

$$m_{j,l}(\xi) = \widetilde{m}_j(\mu),$$

where $\widetilde{m}_j(\mu)$ is the $\mathcal{U}_j$-multiplicity of μ. More precisely, consider the "complexified" operator $A : V^c \to V^c$. For $\mu \in \sigma(A)$, the corresponding eigenspace $\widetilde{E}(\mu)$ in V^c is a complex Γ-representation admitting the isotypical decomposition

$$\widetilde{E}(\mu) = \widetilde{E}_0(\mu) \oplus \widetilde{E}_1(\mu) \oplus \cdots \oplus \widetilde{E}_s(\mu),$$

where $\widetilde{E}_j(\mu)$ is modeled on $\mathcal{U}_j$ and $\widetilde{m}_j(\mu)$ is defined by $\dim \widetilde{E}_j(\mu)/\dim \mathcal{U}_j$.

Put $\widetilde{m}_j^k(A) := \sum_{k^2<\mu<(k+1)^2} \widetilde{m}_j(\mu)$. It can be directly verified that (cf. [**11**])

$$\sum_{\xi\in\sigma_-(\mathcal{A}')} \sum_{j,l} m_{j,l}(\xi)\deg_{\mathcal{V}_{j,l}} = \sum_{j=0}^{s}\sum_{k=1}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}}.$$

Therefore, the formulae (4.19)–(4.20) reduce to

$$\deg^0_{\mathcal{A}} = \prod_{\mu\in\sigma_+(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)},$$

$$\deg^1_{\mathcal{A}} = \deg^0_{\mathcal{A}} \star \sum_{j=0}^{s}\sum_{k=1}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}}.$$

Let $(H^{\varphi,l})$ be such that (H^{φ}) is a dominating orbit type in W. We introduce the following conditions:

(Y1) $\deg^1_{\mathcal{A}}$ contains a nontrivial $(H^{\varphi,l})$-term, and $Z_0 = \operatorname{Ker} A$ is such that

$$\begin{cases} (Z_0)^{\Gamma} = \{0\} \\ (\widetilde{H} \times S^1) \notin \mathcal{J}(Z_0) \text{ for any } (\widetilde{H}) \text{ such that } (H) \le (\widetilde{H}) < (\Gamma). \end{cases}$$

(N1) $\deg^1_{\mathcal{A}}$ does not contain a nontrivial $(H^{\varphi,l})$-term.

Proposition 4.1. *Let $\varphi : V \to \mathbf{R}$ be a Γ-invariant C^2-differentiable map satisfying* (H0), (H1) *and* ($\mathrm{H3}_0$). *Let $(H^{\varphi,l})$ be such that (H^{φ}) is a dominating orbit type in W. Then,*

(i) *Under assumption* (Y1), *there exists a* $(H^{\varphi,l})$*-term with a nonzero coefficient in* $\deg_0$;

(ii) *Under assumption* (N1), *there is no* $(H^{\varphi,l})$*-term with nonzero coefficient in* $\deg_0$.

Proof. (i) By $(Z_0)^\Gamma = \{0\}$ and $Z_0 \subset W^{S^1}$, we have that $(Z_0)^G = \{0\}$, and

$$\nabla_G\text{-deg}\,(\nabla\varphi, B_\varepsilon(Z_0)) = (G) + a_0 \in A_0(G), \tag{4.23}$$

for some $a_0 \in A_0(G)$ which does not contain nontrivial (G)-term. Substituting (4.23) in (4.21), we obtain

$$\begin{aligned}\deg_0 &= \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \star \nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0))\\ &= \big((G) + a_0\big) \star (\deg^0_{\mathcal{A}} - \deg^1_{\mathcal{A}})\\ &= \deg^0_{\mathcal{A}} - \deg^1_{\mathcal{A}} + a_0 \star \deg^0_{\mathcal{A}} - a_0 \star \deg^1_{\mathcal{A}}\\ &= \underbrace{\deg^0_{\mathcal{A}} + a_0 \star \deg^0_{\mathcal{A}}}_{\in A_0(G)} \underbrace{-\deg^1_{\mathcal{A}} - a_0 \star \deg^1_{\mathcal{A}}}_{\in A_1(G)}.\end{aligned}$$

Since $\deg^1_{\mathcal{A}}$ contains a nontrivial $(H^{\varphi,l})$-term, to conclude that $\deg_0$ also contains this $(H^{\varphi,l})$-term (with an opposite sign), it suffices to eliminate the possibility that

$$a_0 \star \deg^1_{\mathcal{A}} = -(H^{\varphi,l}) + \text{rest}.$$

By the maximality of (H^φ), this would only happen if a_0 contains a nontrivial $(\widetilde{H} \times S^1)$-term for some $(\widetilde{H}) \geq (H)$. Also notice that $(\widetilde{H}) < (\Gamma)$, since a_0 does not contain the (G)-term. By the assumption that such a $(\widetilde{H} \times S^1)$ does not occur in $\mathcal{J}(Z_0)$, it is impossible for a_0 to contain such a nontrivial $(\widetilde{H} \times S^1)$-term, so the statement follows.

(ii) It is clear that if $\deg^1_{\mathcal{A}}$ has no nontrivial $(H^{\varphi,l})$-term, $\deg_0$ does not permit one. □

Case $(\mathbf{H3}_l)$. Under the assumption (H3_l), $\mathcal{A}$ is a linear G-isomorphism when restricted to the S^1-isotypical components W^{S^1} and each W_l, for $l \neq l_0$ (cf. (4.16)). Indeed,

$$Z_0 = \operatorname{Ker}\mathcal{A} \subset W_{l_0}.$$

In particular, $(Z_0)^{S^1} = \{0\}$ and

$$\nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) = (G) + a_1, \text{ for } a_1 \in A_1(G). \tag{4.24}$$

Substituting (4.24) in (4.21), we obtain

$$\begin{aligned}\deg_0 &= \nabla_G\text{-deg}\,(\nabla\varphi_0, B_\varepsilon(Z_0)) \star \nabla_G\text{-deg}\,(\mathcal{A}|_{\mathcal{W}_0}, B(\mathcal{W}_0))\\ &= \big((G) + a_1\big) \star (\deg^0_{\mathcal{A}} - \deg^1_{\mathcal{A}})\\ &= \deg^0_{\mathcal{A}} - \deg^1_{\mathcal{A}} + a_1 \star \deg^0_{\mathcal{A}} - a_1 \star \deg^1_{\mathcal{A}}\\ &= \underbrace{\deg^0_{\mathcal{A}}}_{\in A_0(G)}\ \underbrace{-\deg^1_{\mathcal{A}} + a_1 \star \deg^0_{\mathcal{A}}}_{\in A_1(G)},\end{aligned}$$

where the last equality uses the fact that $a_1 \star \deg^1_{\mathcal{A}} = 0$, since $a_1, \deg^1_{\mathcal{A}} \in A_1(G)$ (cf. Theorem 2.1).

Moreover, we have

$$\begin{aligned}\deg^1_{\mathcal{A}} &= \deg^0_{\mathcal{A}} \star \sum_{\xi\in\sigma_-(\mathcal{A}')} \sum_{j=0}^{s} \sum_{l=1}^{\infty} m_{j,l}(\xi) \deg_{\mathcal{V}_{j,l}}\\ &= \deg^0_{\mathcal{A}} \star \Bigg(\sum_{j=0}^{s} \sum_{k=1}^{\infty} \widetilde{m}^k_j(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}}\\ &\qquad + \sum_{j=0}^{s} \widetilde{m}_j(l_0^2) \sum_{l=1}^{l_0-1} \deg_{\mathcal{V}_{j,l}} \Bigg),\end{aligned} \tag{4.25}$$

where it is clear that

$$\deg^0_{\mathcal{A}} = \prod_{\mu\in\sigma_+(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)}. \tag{4.26}$$

We introduce the following conditions:

(Y2) $\deg^1_{\mathcal{A}}$ contains a nontrivial $(H^{\varphi,l})$-term, and $(H^{\varphi,l}) \notin \mathcal{J}(Z_0)$.

(N2) $\deg^1_{\mathcal{A}}$ does not contain a nontrivial $(H^{\varphi,l})$-term and $(H^{\varphi,l}) \notin \mathcal{J}(Z_0)$.

Proposition 4.2. *Let $\varphi : V \to \mathbf{R}$ be a Γ-invariant C^2-differentiable map satisfying* (H0)–(H2) *and* (H3$_l$). *Let $(H^{\varphi,l})$ be such that (H^φ) is a dominating orbit type in W.*

(i) *Under assumption* (Y2), *there exists a* $(H^{\varphi,l})$*-term with nonzero coefficient in* $\deg_0$;

(ii) *Under assumption* (N2), *there is no* $(H^{\varphi,l})$*-term with nonzero coefficient in* $\deg_0$.

Proof. (i) By (Y2), $\deg^1_{\mathcal{A}}$ contains a nontrivial $(H^{\varphi,l})$-term. It is sufficient to show that $a_1 \star \deg^0_{\mathcal{A}}$ does not contain any $-(H^{\varphi,l})$-term so that a cancelation does not occur. But $(H^{\varphi,l}) \notin \mathcal{J}(Z_0)$, which implies that a_1 has no nontrivial $(H^{\varphi,l})$-term. Thus, by maximality of $(H^{\varphi,l})$, $a_1 \star \deg^0_{\mathcal{A}}$ contains no $(H^{\varphi,l})$-term. Therefore, it follows that there exists a $(H^{\varphi,l})$-term with a nonzero coefficient in $\deg_0$.

(ii) Similar proof as in (i). By (N2), $\deg^1_{\mathcal{A}}$ contains no nontrivial $(H^{\varphi,l})$-term. It is sufficient to show that $a_1 \star \deg^0_{\mathcal{A}}$ does not contain any $-(H^{\varphi,l})$-term, which is again the case by the condition $(H^{\varphi,l}) \notin \mathcal{J}(Z_0)$. □

Case (H3). Under nondegeneracy assumption (H3), $\mathcal{A}$ is a linear G-isomorphism of W. Thus, the complete value of $\deg_0$ can be obtained (cf. [**11**]). Then, it makes sense to formulate the following conditions:

(Y) $\deg_0$ contains a nontrivial $(H^{\varphi,l})$-term,

(N) $\deg_0$ does not contain any nontrivial $(H^{\varphi,l})$-term.

Theorem 4.2. *Let* $\varphi : V \to \mathbf{R}$ *be a* Γ*-invariant* C^2*-differentiable map satisfying* (H0)–(H2). *Let* $(H^{\varphi,l})$ *be such that* (H^{φ}) *is a dominating orbit type in* W. *Then, we have Table* 2 *summarizing the sufficient conditions of existence and nonexistence of a nontrivial* $(H^{\varphi,l})$*-term in* $\deg_p$, *for* $p \in \{0, \infty\}$ (*where the conditions* (Y1′), (Y′), (N1′), (N2′) *and* (N′) *of* $\mathcal{B}$ *are the counterparts of those of* $\mathcal{A}$).

Proof. Immediate consequence of Propositions 4.1–4.2. □

Corollary 4.1. *Let* $\varphi : V \to \mathbf{R}$ *be a* Γ*-invariant* C^2*-differentiable map satisfying* (H0)–(H2). *Let* $(H^{\varphi,l})$ *be such that* (H^{φ}) *is a dominating orbit type in* W. *Then, we have a nontrivial* $(H^{\varphi,l})$*-term in* $\deg_\infty - \deg_0$, *if the conditions in Table* 2 *are satisfied diagonally, i.e., one of the existence conditions for* $\deg_0$ *with one of the nonexistence conditions for* $\deg_\infty$ *or vice versa.*

TABLE 2. Existence/nonexistence of $(H^{\varphi,l})$-term in $\deg_p$.

	$\deg_0$	$\deg_\infty$
existence of $(H^{\varphi,l})$	$(H3_0)$+(Y1) or $(H3_l)$+(Y2) or (H3)+(Y)	$(H4_0)+(Y1')$ or $(H4_l)+(Y2')$ or $(H4)+(Y')$
nonexistence of $(H^{\varphi,l})$	$(H3_0)$+(N1) or $(H3_l)$+(N2) or (H3)+(N)	$(H4_0)+(N1')$ or $(H4_l)+(N2')$ or $(H4)+(N')$

4.3. Computational examples. We present computational examples for $\Gamma = D_n$ and $V = \mathbf{R}^n$ for $n = 6, 8, 10, 12$. Consider the potential $\varphi : V \to \mathbf{R}$ satisfying (H0)–(H1) with matrices A and B being of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & \cdots & 0 & d \\ d & c & d & 0 & \cdots & 0 & 0 \\ 0 & d & c & d & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d & 0 & 0 & 0 & \cdots & d & c \end{bmatrix}.$$

To obtain φ satisfying the above properties, one can define for example $\varphi : V \to \mathbf{R}$ by $\varphi(x) := (1/2)\langle Bx, x\rangle - (1/\sqrt{\langle (A-B)x, x\rangle + a})$, for certain $a > 0$. A similar computational example can be found in [**6**]. We also assume (H2) in all the computational examples. The degeneracy assumptions are listed in Table 3. For the notations used here, including complete lists of irreducible Γ-representations, computational data and usage of the Maple© routines, we refer to [**2**].

TABLE 3. Summary of the assumptions in the computational examples.

Γ	$\deg_0$	$\deg_\infty$
D_6	$(\mathrm{H3}_0)$+(Y1)	$(\mathrm{H4}_0)$+(N1′)
D_8	$(\mathrm{H3}_0)$+(Y1)	$(\mathrm{H4}_l)$+(N2′)
D_{10}	$(\mathrm{H3}_l)$+(N2)	$(\mathrm{H4}_0)$+(Y1′)
D_{12}	$(\mathrm{H3}_l)$+(N2)	$(\mathrm{H4}_l)$+(Y2′)

TABLE 4. Eigenvalues of A and B, $\Gamma = D_6$.

	c	d	μ_0	μ_1	μ_2	μ_4
A	8.8	4.4	17.6	13.2	4.4	0
B	1.1	1.1	3.3	2.2	0	-1.1

Dihedral symmetry D_6**.** Let $\Gamma = D_6$ and $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4$. Consider the potential $\varphi : V \to \mathbf{R}$ satisfying (H0)–(H2) with the matrices A and B being of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & 0 & d \\ d & c & d & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 \\ 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & d & c \end{bmatrix}.$$

It can easily be obtained that $\sigma(C) = \{\mu_0 = c + 2d, \mu_1 = c + d, \mu_2 = c - d, \mu_4 = c - 2d\}$, where each μ_i has its eigenspace $E(\mu_i) \simeq \mathcal{V}_i$. Take $c = 8.8$ and $d = 4.4$ for A and $c = d = 1.1$ for B, and list eigenvalues of A and B in Table 4. Notice that assumptions $(\mathrm{H3}_0)$ and $(\mathrm{H4}_0)$ are satisfied in this case. The dominating orbit types in W are (D_6), (D_6^d), $(\mathbf{Z}_6^{t_1})$, $(\mathbf{Z}_6^{t_2})$, $(D_2^{\hat{d}})$ and (D_2^{z}).

Using Table 4, we compute the numbers

$$\widetilde{m}_0^4(A) = 1, \quad \widetilde{m}_1^3(A) = 1, \quad \widetilde{m}_2^2(A) = 1,$$
$$\widetilde{m}_0^1(B) = 1, \quad \widetilde{m}_1^1(B) = 1.$$

The value of $\deg^1_{\mathcal{A}}$ is

$$
\begin{aligned}
\deg^1_{\mathcal{A}} &= \prod_{\mu\in\sigma_+(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} \star \sum_{j=0}^{s}\sum_{k=0}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \\
&= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \Bigg(1\cdot \Bigg(\sum_{l=1}^{4} \deg_{\mathcal{V}_{0,l}} \Bigg) \\
&\qquad\qquad + 1\cdot \Bigg(\sum_{l=1}^{3} \deg_{\mathcal{V}_{1,l}} \Bigg) + 1\cdot \Bigg(\sum_{l=1}^{2} \deg_{\mathcal{V}_{2,l}} \Bigg) \Bigg) \\
&= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \Big(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{2,1}} \Big) \\
&\quad + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \Big(\deg_{\mathcal{V}_{0,2}} + \deg_{\mathcal{V}_{1,2}} + \deg_{\mathcal{V}_{2,2}} \Big) \\
&\quad + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \big(\deg_{\mathcal{V}_{0,3}} + \deg_{\mathcal{V}_{1,3}} \big) + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \deg{}_{\mathcal{V}_{0,4}} \\
&= \Theta_1 \left[\texttt{showdegree[D6]}\ (1,1,1,0,0,0,1,1,1,0,0,0)\right] \\
&\quad + \Theta_2 \left[\texttt{showdegree[D6]}\ (1,1,1,0,0,0,1,1,1,0,0,0)\right] \\
&\quad + \Theta_3 \left[\texttt{showdegree[D6]}\ (1,1,1,0,0,0,1,1,0,0,0,0)\right] \\
&\quad + \Theta_4 \left[\texttt{showdegree[D6]}\ (1,1,1,0,0,0,1,0,0,0,0,0)\right] \\
&= -(D_6^d) - (\mathbf{Z}_6^{t_1}) - (\mathbf{Z}_6^{t_2}) + (D_2^z) + 3(D_2^d) + (D_2^{\widehat{d}}) + (D_2) \\
&\quad - 3(\widetilde{D}_1^z) - 2(\widetilde{D}_1) - 2(D_1^z) - 3(D_1) - 2(\mathbf{Z}_2^-) - (\mathbf{Z}_2) + 5(\mathbf{Z}_1) \\
&\quad - (D_6^{d,2}) - (\mathbf{Z}_6^{t_1,2}) - (\mathbf{Z}_6^{t_2,2}) + (D_2^{z,2}) + 3(D_2^{d,2}) + (D_2^{\widehat{d},2}) \\
&\quad + (D_2^2) - 3(\widetilde{D}_1^{z,2}) - 2(\widetilde{D}_1^2) - 2(D_1^{z,2}) - 3(D_1^2) - 2(\mathbf{Z}_2^{-,2}) \\
&\quad - (\mathbf{Z}_2^2) + 5(\mathbf{Z}_1^2) - (D_6^{d,3}) - (\mathbf{Z}_6^{t_1,3}) + 3(D_2^{d,3}) + (D_2^{\widehat{d},3}) \\
&\quad - 2(\widetilde{D}_1^{z,3}) - (\widetilde{D}_1^3) - (D_1^{z,3}) - 2(D_1^3) - 2(\mathbf{Z}_2^{-,3}) + 3(\mathbf{Z}_1^3) \\
&\quad - (D_6^{d,4}) + 2(D_2^{d,4}) - (\widetilde{D}_1^{z,4}) - (D_1^4) - (\mathbf{Z}_2^{-,4}) + (\mathbf{Z}_1^4).
\end{aligned}
$$

Since $Z_0 = \operatorname{Ker} A \simeq \mathcal{V}_4$, we have the set of all orbit types is $\mathcal{J}(\mathcal{V}_4) = \{(D_6 \times S^1), (D_3 \times S^1)\}$. By (Y1) and Proposition 4.1 (i), there exist the following nontrivial $(H^{\varphi,l})$-terms in $\deg_0$:

$$(4.27)\qquad \begin{gathered}(D_6^d),(\mathbf{Z}_6^{t_1}),(\mathbf{Z}_6^{t_2}),(D_2^z),(D_2^{\hat d}),(D_6^{d,2}),(\mathbf{Z}_6^{t_1,2}),(\mathbf{Z}_6^{t_2,2}),\\ (D_2^{z,2}),(D_2^{\hat d,2}),(D_6^{d,3}),(\mathbf{Z}_6^{t_1,3}),(D_2^{\hat d,3}),(D_6^{d,4}).\end{gathered}$$

On the other hand,

$$\begin{aligned}\deg{}^1_{\mathcal{B}} &= \prod_{\mu\in\sigma_+(B)}\prod_{i=0}^{r}(\deg_{\mathcal{V}_i})^{m_i(\mu)}\star\sum_{j=0}^{s}\sum_{k=0}^{\infty}\widetilde{m}_j^k(B)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}\\ &= \prod_{i=0,1}\deg_{\mathcal{V}_i}\star\big(1\cdot\deg_{\mathcal{V}_{0,1}}+1\cdot\deg_{\mathcal{V}_{1,1}}\big)\\ &= \Theta_1\,[\texttt{showdegree[D6]}\;(1,1,0,0,0,0,1,1,0,0,0,0)]\\ &= -(D_6^d)-(\mathbf{Z}_6^{t_1})-(D_2^d)-(D_2^{\hat d})+2(\widetilde{D}_1^z)+(\widetilde{D}_1)\\ &\quad+(D_1^z)+2(D_1)+(\mathbf{Z}_2^-)-3(\mathbf{Z}_1).\end{aligned}$$

By (N1′) and a similar statement as Proposition 4.1 (ii) for φ satisfying (H0), (H1) and ($H4_0$), we have that $\deg_\infty$ does not contain any nontrivial terms as listed in (4.27) except possibly for (D_6^d), $(\mathbf{Z}_6^{t_1})$ and (D_2^d). Therefore, the following orbit types will appear in the value $\deg_\infty - \deg{}_0$:

$$\begin{gathered}(D_6^{d,2}),(D_6^{d,3}),(D_6^{d,4}),(\mathbf{Z}_6^{t_1,2}),(\mathbf{Z}_6^{t_1,3}),(\mathbf{Z}_6^{t_2}),(\mathbf{Z}_6^{t_2,2}),\\ (D_2^z),(D_2^{z,2}),(D_2^{\hat d,2}),(D_2^{\hat d,3}).\end{gathered}$$

Conclusion. Under the assumptions (H0), (H1), ($H3_0$) and ($H4_0$), by Theorem 4.1, there exist at least 11 nonstationary solutions of (4.10). To be more specific, there are: 1 nonstationary solution with least symmetry $(D_6^{d,4})$, 2 nonstationary solutions with least symmetries $(\mathbf{Z}_6^{t_1,3})$, 2 nonstationary solutions with least symmetries $(\mathbf{Z}_6^{t_2,2})$, 3 nonstationary solutions with least symmetries $(D_2^{z,2})$ and 3 nonstationary solutions with least symmetries $(D_2^{\hat d,3})$.

Dihedral symmetry D_8. Let $\Gamma = D_8$ and $V = \mathcal{V}_0\oplus\mathcal{V}_1\oplus\mathcal{V}_2\oplus\mathcal{V}_3\oplus\mathcal{V}_5$. Consider the potential $\varphi : V \to \mathbf{R}$ satisfying (H0)–(H2) with matrices

A and B being of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & 0 & 0 & 0 & d \\ d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & 0 & 0 & d & c \end{bmatrix}.$$

It can easily be obtained that $\sigma(C) = \{\mu_0 = c + 2d, \mu_1 = c + \sqrt{2}d, \mu_2 = c, \mu_3 = c - \sqrt{2}d, \mu_5 = c - 2d\}$, where each μ_i has its eigenspace $E(\mu_i) \simeq \mathcal{V}_i$. Take $c = 4\sqrt{2}$, $d = 4$, for A and $c = 3$, $d = \sqrt{2}$, for B, and list eigenvalues of A and B in Table 5.[3] Notice that the assumptions (H3$_0$) and (H4$_l$) (for $l_\infty = 1$) are satisfied in this case. The dominating orbit types in W are (D_8), (D_8^d), $(\mathbf{Z}_8^{t_1})$, $(\mathbf{Z}_8^{t_2})$, $(\mathbf{Z}_8^{t_3})$, $(\widetilde{D}_4^d)$.

TABLE 5. Eigenvalues of A and B, $\Gamma = D_8$.

	c	d	μ_0	μ_1	μ_2	μ_3	μ_5
A	$4\sqrt{2}$	4	13.7	11.3	5.7	0	-2.3
B	3	$\sqrt{2}$	5.8	5	3	1	0.2

Using Table 5, we compute the numbers

$$\widetilde{m}_0^3(A) = 1, \quad \widetilde{m}_1^3(A) = 1, \quad \widetilde{m}_2^2(A) = 1,$$
$$\widetilde{m}_0^2(B) = 1, \quad \widetilde{m}_1^2(B) = 1, \quad \widetilde{m}_2^1(B) = 1, \quad \widetilde{m}_3(l_\infty^2) = 1.$$

Compute the value of $\deg_{\mathcal{A}}^1$ by

$$\begin{aligned} \deg_{\mathcal{A}}^1 &= \prod_{\mu \in \sigma_+(A)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} \star \sum_{j=0}^{s} \sum_{k=0}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \\ &= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \Bigg(1 \cdot \Bigg(\sum_{l=1}^{3} \deg_{\mathcal{V}_{0,l}}\Bigg) \\ &\qquad + 1 \cdot \Bigg(\sum_{l=1}^{3} \deg_{\mathcal{V}_{1,l}}\Bigg) + 1 \cdot \Bigg(\sum_{l=1}^{2} \deg_{\mathcal{V}_{2,l}}\Bigg)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \left(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{2,1}}\right) \\
&\quad + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \left(\deg_{\mathcal{V}_{0,2}} + \deg_{\mathcal{V}_{1,2}} + \deg_{\mathcal{V}_{2,2}}\right) \\
&\quad + \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \left(\deg_{\mathcal{V}_{0,3}} + \deg_{\mathcal{V}_{1,3}}\right) \\
&= \Theta_1 \left[\texttt{showdegree[D8]}\ (1,1,1,0,0,0,0,1,1,1,0,0,0,0)\right] \\
&\quad + \Theta_2 \left[\texttt{showdegree[D8]}\ (1,1,1,0,0,0,0,1,1,1,0,0,0,0)\right] \\
&\quad + \Theta_3 \left[\texttt{showdegree[D8]}\ (1,1,1,0,0,0,0,1,1,0,0,0,0,0)\right] \\
&= -(D_8) - (\widetilde{D}_4^d) - (D_4^d) - (\mathbf{Z}_8^{t_1}) - (\mathbf{Z}_8^{t_2}) \\
&\quad + (\widetilde{D}_2^z) + 2(\widetilde{D}_2^d) + 2(\widetilde{D}_2) + (D_2^z) + 2(D_2^d) \\
&\quad + 2(D_2) + (\mathbf{Z}_4^d) - 2(\widetilde{D}_1^z) - 3(\widetilde{D}_1) - 2(D_1^z) \\
&\quad - 3(D_1) - (\mathbf{Z}_2^-) - 3(\mathbf{Z}_2) + 5(\mathbf{Z}_1) - (D_8^2) \\
&\quad - (\widetilde{D}_4^{d,2}) - (D_4^{d,2}) - (\mathbf{Z}_8^{t_1,2}) - (\mathbf{Z}_8^{t_2,2}) \\
&\quad + (\widetilde{D}_2^{z,2}) + 2(\widetilde{D}_2^{d,2}) + 2(\widetilde{D}_2^2) + (D_2^{z,2}) \\
&\quad + 2(D_2^{d,2}) + 2(D_2^2) + (\mathbf{Z}_4^{d,2}) - 2(\widetilde{D}_1^{z,2}) \\
&\quad - 3(\widetilde{D}_1^2) - 2(D_1^{z,2}) - 3(D_1^2) - (\mathbf{Z}_2^{-,2}) \\
&\quad - 3(\mathbf{Z}_2^2) + 5(\mathbf{Z}_1^2) - (D_8^3) - (\mathbf{Z}_8^{t_1,3}) + (\widetilde{D}_2^{d,3}) \\
&\quad + (\widetilde{D}_2^3) + (D_2^{d,3}) + (D_2^3) - (\widetilde{D}_1^{z,3}) \\
&\quad - 2(\widetilde{D}_1^3) - (D_1^{z,3}) - 2(D_1^3) - (\mathbf{Z}_2^{-,3}) - (\mathbf{Z}_2^3) + 3(\mathbf{Z}_1^3).
\end{aligned}
$$

Since $Z_0 = \operatorname{Ker} A \simeq \mathcal{V}_3$, we have the set of all orbit types is $\mathcal{J}(\mathcal{V}_3) = \{(D_8 \times S^1), (D_1 \times S^1), (\widetilde{D}_1 \times S^1), (\mathbf{Z}_1 \times S^1)\}$. By (Y1) and Proposition 4.1 (i), there exist the following nontrivial $(H^{\varphi,l})$-terms in $\deg_0$:

(4.28)
$$(D_8), (\widetilde{D}_4^d), (\mathbf{Z}_8^{t_1}), (\mathbf{Z}_8^{t_2}), (D_8^2), (\widetilde{D}_4^{d,2}), (\mathbf{Z}_8^{t_1,2}), (\mathbf{Z}_8^{t_2,2}), (D_8^3), (\mathbf{Z}_8^{t_1,3}).$$

On the other hand,

$$\deg_{\mathcal{B}}^1 = \deg_{\mathcal{B}}^0 \star \left(\sum_{j=0}^{s} \sum_{k=1}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} + \sum_{j=0}^{s} \widetilde{m}_j(l_\infty^2) \sum_{l=1}^{l_\infty - 1} \deg_{\mathcal{V}_{j,l}} \right)$$

$$\overset{(l_\infty=1)}{=} \prod_{i=0,1,2,3,5} \deg_{\mathcal{V}_i} \star \big(1\cdot(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{0,2}})$$
$$+1\cdot(\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{1,2}})+1\cdot\deg_{\mathcal{V}_{2,1}}\big)$$
$$=\Theta_1\,[\texttt{showdegree[D8]}\,(1,1,1,1,0,1,0,1,1,1,0,0,0,0)]$$
$$+\Theta_2\,[\texttt{showdegree[D8]}\,(1,1,1,1,0,1,0,1,1,0,0,0,0,0)]$$
$$=-(D_8)-(\widetilde{D}_4^d)+(D_4^d)-(\mathbf{Z}_8^{t_1})-(\mathbf{Z}_8^{t_2})$$
$$+(\widetilde{D}_2^z)+(\widetilde{D}_2^d)+2(\widetilde{D}_2)-(D_2^z)-(D_2^d)-2(D_2)$$
$$+(\mathbf{Z}_4^{t_1})+(\mathbf{Z}_4^d)-(D_8^2)+(D_4^2)-(\mathbf{Z}_8^{t_1,2})$$
$$+(\widetilde{D}_2^{d,2})-(\widetilde{D}_2^2)-(D_2^{d,2})-(D_2^2)+(\mathbf{Z}_4^{t_1,2}).$$

By (N2′) and a similar statement as Proposition 4.2 (ii) for φ satisfying (H0)–(H2) and (H4$_l$), we have that $\deg_\infty$ does not contain any nontrivial terms as listed in (4.29) except possibly for (D_8), $(\widetilde{D}_4^d)$, $(\mathbf{Z}_8^{t_1})$, $(\mathbf{Z}_8^{t_2})$, (D_8^2) and $(\mathbf{Z}_8^{t_1,2})$. Moreover, since $Z_\infty \simeq \mathcal{V}_{3,1}$, we have that $\mathcal{J}(Z_\infty)=\{(D_8\times S^1),(\mathbf{Z}_8^{t_3}),(D_2^d),(\widetilde{D}_2^d),(\mathbf{Z}_2^d)\}$. Therefore, the following orbit types $(H^{\varphi,l})$ will appear in the value $\deg_\infty - \deg_0$:

$$(D_8^3),(\widetilde{D}_4^{d,2}),(\mathbf{Z}_8^{t_1,3}),(\mathbf{Z}_8^{t_2,2}). \tag{4.29}$$

Conclusion. Under the assumptions (H0)–(H2), (H3$_0$) and (H4$_l$) (with $l_\infty=1$), by Theorem 4.1, there exist at least 7 nonstationary solutions of (4.10). To be more specific, there are: 1 nonstationary solution with least symmetry (D_8^3), 2 nonstationary solutions with least symmetries $(\widetilde{D}_4^{d,2})$, 2 nonstationary solutions with least symmetries $(\mathbf{Z}_8^{t_2,2})$ and 2 nonstationary solutions with least symmetries $(\mathbf{Z}_8^{t_1,3})$.

TABLE 6. Eigenvalues of A and B, $\Gamma = D_{10}$.

	c	d	μ_0	μ_1	μ_2	μ_3	μ_4	μ_6
A	-2	3	4	2.9	-0.1	-3.9	-6.9	-8
B	4	$2(\cos(2\pi/5))^{-1}$	17	14.5	8	0	-6.5	-8.9

Dihedral symmetry D_{10}. Let $\Gamma = D_{10}$ and $V=\mathcal{V}_0\oplus\mathcal{V}_1\oplus\mathcal{V}_2\oplus\mathcal{V}_3\oplus\mathcal{V}_4\oplus\mathcal{V}_6$. Consider the potential $\varphi : V\to\mathbf{R}$ satisfying (H0)–(H2) with the matrices A and B being of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c \end{bmatrix}.$$

It can easily be obtained that $\sigma(C) = \{\mu_0 = c+2d, \mu_1 = c+2d\cos(\pi/5), \mu_2 = c + 2d\cos(2\pi/5), \mu_3 = c + 2d\cos(3\pi/5), \mu_4 = c + 2d\cos(4\pi/5), \mu_6 = c - 2d\}$, where each μ_i has its eigenspace $E(\mu_i) \simeq \mathcal{V}_i$. Take $c = -2$ and $d = 3$ for A and $c = 4$ and $d = 2(\cos(2\pi/5))^{-1}$ for B, and list eigenvalues of A and B in Table 6. Notice that the assumptions (H3$_l$) and (H4$_0$) are satisfied in this case (for $l_0 = 2$). The dominating orbit types in W are (D_{10}), (D_{10}^d), $(\mathbf{Z}_{10}^{t_1})$, $(\mathbf{Z}_{10}^{t_2})$, $(\mathbf{Z}_{10}^{t_3})$, $(\mathbf{Z}_{10}^{t_4})$, $(D_2^{\hat{d}})$ and (D_2^{z}).

Using Table 6, we compute the numbers

$$\widetilde{m}_0(l_0^2) = 1, \quad \widetilde{m}_1^1(A) = 1, \quad \widetilde{m}_0^4(B) = 1, \quad \widetilde{m}_1^3(B) = 1, \quad \widetilde{m}_2^2(B) = 1.$$

Compute the value of $\deg^1_{\mathcal{B}}$

$$\begin{aligned} \deg^1_{\mathcal{B}} &= \prod_{\mu\in\sigma_+(B)} \prod_{i=0}^{r} (\deg_{\mathcal{V}_i})^{m_i(\mu)} \star \sum_{j=0}^{s}\sum_{k=0}^{\infty} \widetilde{m}_j^k(B) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} \\ &= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \Bigg(1\cdot\Bigg(\sum_{l=1}^{4}\deg_{\mathcal{V}_{0,l}}\Bigg) \\ &\qquad\qquad + 1\cdot\Bigg(\sum_{l=1}^{3}\deg_{\mathcal{V}_{1,l}}\Bigg) + 1\cdot\Bigg(\sum_{l=1}^{2}\deg_{\mathcal{V}_{2,l}}\Bigg)\Bigg) \\ &= \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \big(\deg_{\mathcal{V}_{0,1}} + \deg_{\mathcal{V}_{1,1}} + \deg_{\mathcal{V}_{2,1}}\big) \end{aligned}$$

$$
\begin{aligned}
&+\prod_{i=0}^{2}\deg_{\mathcal{V}_i}\star\big(\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{1,2}}+\deg_{\mathcal{V}_{2,2}}\big)\\
&+\prod_{i=0}^{2}\deg_{\mathcal{V}_i}\star\big(\deg_{\mathcal{V}_{0,3}}+\deg_{\mathcal{V}_{1,3}}\big)\\
&+\prod_{i=0}^{2}\deg_{\mathcal{V}_i}\star\deg_{\mathcal{V}_{0,4}}\\
=&\ \Theta_1\left[\texttt{showdegree[D10]}\ (1,1,1,0,0,0,0,0,1,1,1,0,0,0,0,0)\right]\\
&+\Theta_2\left[\texttt{showdegree[D10]}\ (1,1,1,0,0,0,0,0,1,1,1,0,0,0,0,0)\right]\\
&+\Theta_3\left[\texttt{showdegree[D10]}\ (1,1,1,0,0,0,0,0,1,1,0,0,0,0,0,0)\right]\\
&+\Theta_4\left[\texttt{showdegree[D10]}\ (1,1,1,0,0,0,0,0,1,0,0,0,0,0,0,0)\right]\\
=&\ -(D_{10})-(\mathbf{Z}_{10}^{t_1})-(\mathbf{Z}_{10}^{t_2})+(D_2^{d})+(D_2^{\hat{d}})+(D_2^{z})\\
&+3(D_2)-2(\widetilde{D}_1^{z})-3(\widetilde{D}_1)-2(D_1^{z})\\
&-3(D_1)-(\mathbf{Z}_2^{-})-2(\mathbf{Z}_2)+5(\mathbf{Z}_1)-(D_{10}^{2})-(\mathbf{Z}_{10}^{t_1,2})\\
&-(\mathbf{Z}_{10}^{t_2,2})+(D_2^{d,2})+(D_2^{\hat{d},2})+(D_2^{z,2})\\
&+3(D_2^{2})-2(\widetilde{D}_1^{z,2})-3(\widetilde{D}_1^{2})-2(D_1^{z,2})-3(D_1^{2})-(\mathbf{Z}_2^{-,2})\\
&-2(\mathbf{Z}_2^{2})+5(\mathbf{Z}_1^{2})-(D_{10}^{3})-(\mathbf{Z}_{10}^{t_1,3})\\
&+(D_2^{d,3})+(D_2^{\hat{d},3})+2(D_2^{3})-(\widetilde{D}_1^{z,3})\\
&-2(\widetilde{D}_1^{3})-(D_1^{z,3})-2(D_1^{3})-(\mathbf{Z}_2^{-,3})-(\mathbf{Z}_2^{3})+3(\mathbf{Z}_1^{3})\\
&-(D_{10}^{4})+2(D_2^{4})-(\widetilde{D}_1^{4})-(D_1^{4})-(\mathbf{Z}_2^{4})+(\mathbf{Z}_1^{4}).
\end{aligned}
$$

Since $Z_\infty = \operatorname{Ker} B \simeq \mathcal{V}_3$, we have the set of all orbit types is $\mathcal{J}(\mathcal{V}_3) = \{(D_{10}\times S^1),(D_1\times S^1),(\widetilde{D}_1\times S^1),(\mathbf{Z}_1\times S^1)\}$. By (Y1′) and a similar statement as Proposition 4.1 (i), there exist the following nontrivial $(H^{\varphi,l})$-terms in $\deg_\infty$:

$$
\begin{aligned}
&(D_{10}),(\mathbf{Z}_{10}^{t_1}),(\mathbf{Z}_{10}^{t_2}),(D_2^{\hat{d}}),(D_2^{z}),(D_{10}^{2}),(\mathbf{Z}_{10}^{t_1,2}),(\mathbf{Z}_{10}^{t_2,2}),\\
&(D_2^{\hat{d},2}),(D_2^{z,2}),(D_{10}^{3}),(\mathbf{Z}_{10}^{t_1,3}),(D_2^{\hat{d},3}),(D_{10}^{4}).
\end{aligned}
\tag{4.30}
$$

On the other hand,

$$
\deg_{\mathcal{A}}^{1}=\deg{}_{\mathcal{A}}^{0}\star\left(\sum_{j=0}^{s}\sum_{k=1}^{\infty}\widetilde{m}_j^{k}(A)\sum_{l=1}^{k}\deg_{\mathcal{V}_{j,l}}+\sum_{j=0}^{s}\widetilde{m}_j(l_0^2)\sum_{l=1}^{l_0-1}\deg_{\mathcal{V}_{j,l}}\right)
$$

$$\begin{aligned}&\overset{(l_0=2)}{=} \prod_{i=0}^{1} \deg_{\mathcal{V}_i} \star \left(1\cdot \deg_{\mathcal{V}_{1,1}} + 1\cdot \deg_{\mathcal{V}_{0,1}}\right)\\ &= \Theta_1\left[\texttt{showdegree[D10]}\,(1,1,0,0,0,0,0,0,1,1,0,0,0,0,0,0)\right]\\ &= -(D_{10}) - (\mathbf{Z}_{10}^{t_1}) - (D_2^d) - (D_2^{\hat{d}}) + (\widetilde{D}_1^z) + 2(\widetilde{D}_1) + (D_1^z)\\ &\quad + 2(D_1) + (\mathbf{Z}_2^-) - 3(\mathbf{Z}_1).\end{aligned}$$

Since $Z_0 \simeq \mathcal{V}_{0,2}$, we have that $\mathcal{J}(Z_0) = \{(D_{10}\times S^1), (D_{10}^2)\}$. By (N2), except for possibly (D_{10}), $(\mathbf{Z}_{10}^{t_1})$, $(D_2^{\hat{d}})$ and (D_{10}^2), every orbit type listed in (4.30) will appear in the value of $\deg_\infty - \deg_0$, namely:

$$\begin{gathered}(\mathbf{Z}_{10}^{t_2}), (D_2^z), (\mathbf{Z}_{10}^{t_1,2}), (\mathbf{Z}_{10}^{t_2,2}), (D_2^{\hat{d},2}),\\ (D_2^{z,2}), (D_{10}^3), (\mathbf{Z}_{10}^{t_1,3}), (D_2^{\hat{d},3}), (D_{10}^4).\end{gathered}$$

Conclusion. Under assumptions (H0)–(H2), (H3$_l$) (with $l_0 = 2$) and (H4$_0$), by Theorem 4.1, there exist altogether at least 15 nonstationary solutions of (4.10). To be more specific, there are: 2 nonstationary solutions with least symmetries $(\mathbf{Z}_{10}^{t_2,2})$, 5 nonstationary solutions with least symmetries $(D_2^{z,2})$, 2 nonstationary solutions with least symmetries $(\mathbf{Z}_{10}^{t_1,3})$, 5 nonstationary solutions with least symmetries $(D_2^{\hat{d},3})$ and 1 nonstationary solution with least symmetry (D_{10}^4).

TABLE 7. Eigenvalues of A and B, $\Gamma = D_{12}$.

	c	d	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_7
A	-1	2.5	4	3.3	1.5	-1	-3.5	-5.3	-6
B	9	3.7	16.4	15.4	12.7	9	5.3	2.6	1.6

Dihedral symmetry D_{12}. Let $\Gamma = D_{12}$ and $V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_4 \oplus \mathcal{V}_5 \oplus \mathcal{V}_7$. Consider the potential $\varphi : V \to \mathbf{R}$ satisfying (H0)–(H2) with the matrices A and B being of the type

$$C = \begin{bmatrix} c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c \end{bmatrix}.$$

It can easily be obtained that $\sigma(C) = \{\mu_0 = c + 2d, \mu_1 = c + \sqrt{3}d, \mu_2 = c + d, \mu_3 = c, \mu_4 = c - d, \mu_5 = c - \sqrt{3}d, \mu_7 = c - 2d\}$, where each μ_i has its eigenspace $E(\mu_i) \simeq \mathcal{V}_i$. Take $c = -2$ and $d = 2\sqrt{3}$ for A and $c = 3$, $d = \sqrt{3}$ for B, and list eigenvalues of A and B in Table 7. Notice that the assumptions (H3$_l$) (for $l_0 = 2$) and (H4$_l$) (for $l_\infty = 3$) are satisfied in this case. The dominating orbit types in W are (D_{12}), (D_{12}^d), $(\mathbf{Z}_{12}^{t_1})$, $(\mathbf{Z}_{12}^{t_2})$, $(\mathbf{Z}_{12}^{t_3})$, $(\mathbf{Z}_{12}^{t_4})$, $(\mathbf{Z}_{12}^{t_5})$, $(\widetilde{D}_6^d)$, (D_4^z), $(D_4^{\hat{d}})$.

Using Table 7, we compute the numbers

$$\widetilde{m}_0(l_0^2) = 1, \quad \widetilde{m}_1^1(A) = 1, \quad \widetilde{m}_1^2(A) = 1, \quad \widetilde{m}_0^4(B) = 1, \quad \widetilde{m}_1^3(B) = 1,$$
$$\widetilde{m}_2^3(B) = 1, \quad \widetilde{m}_3(l_\infty^2) = 1, \quad \widetilde{m}_4^2(B) = 1, \quad \widetilde{m}_5^1(B) = 1, \quad \widetilde{m}_7^1(B) = 1.$$

Compute the value of $\deg_{\mathcal{B}}^1$

$$\begin{aligned} \deg_{\mathcal{B}}^1 &= \deg_{\mathcal{A}}^0 \star \left(\sum_{j=0}^{s} \sum_{k=1}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} + \sum_{j=0}^{s} \widetilde{m}_j(l_\infty^2) \sum_{l=1}^{l_\infty - 1} \deg_{\mathcal{V}_{j,l}} \right) \\ &\overset{l_\infty=3}{=} \prod_{i\in\{0,\dots,5,7\}} \deg_{\mathcal{V}_i} \star \Bigg(1 \cdot \left(\sum_{l=1}^{4} \deg_{\mathcal{V}_{0,l}} \right) \\ &\quad + 1 \cdot \left(\sum_{l=1}^{3} \deg_{\mathcal{V}_{1,l}} \right) + 1 \cdot \left(\sum_{l=1}^{3} \deg_{\mathcal{V}_{2,l}} \right) + 1 \cdot \left(\sum_{l=1}^{2} \deg_{\mathcal{V}_{3,l}} \right) \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + 1\cdot\left(\sum_{l=1}^{2}\deg_{\mathcal{V}_{4,l}}\right) + 1\cdot\deg_{\mathcal{V}_{5,1}} + 1\cdot\deg_{\mathcal{V}_{7,1}}\Bigg)\\
&= \prod_{i\in\{0,\dots,5,7\}}\deg_{\mathcal{V}_i}\star\big(\deg_{\mathcal{V}_{0,1}}+\deg_{\mathcal{V}_{1,1}}+\deg_{\mathcal{V}_{2,1}}\\
&\qquad\qquad +\deg_{\mathcal{V}_{3,1}}+\deg_{\mathcal{V}_{4,1}}+\deg_{\mathcal{V}_{5,1}}+\deg_{\mathcal{V}_{7,1}}\big)\\
&\quad + \prod_{i\in\{0,\dots,5,7\}}\deg_{\mathcal{V}_i}\star\big(\deg_{\mathcal{V}_{0,2}}+\deg_{\mathcal{V}_{1,2}}\\
&\qquad\qquad +\deg_{\mathcal{V}_{2,2}}+\deg_{\mathcal{V}_{3,2}}+\deg_{\mathcal{V}_{4,2}}\big)\\
&\quad + \prod_{i\in\{0,\dots,5,7\}}\deg_{\mathcal{V}_i}\star\big(\deg_{\mathcal{V}_{0,3}}+\deg_{\mathcal{V}_{1,3}}+\deg_{\mathcal{V}_{2,3}}\big)\\
&\quad + \prod_{i\in\{0,\dots,5,7\}}\deg_{\mathcal{V}_i}\star\deg_{\mathcal{V}_{0,4}}\\
&= \Theta_1\left[\texttt{showdegree[D12]}\,(1,1,1,1,1,1,0,1,0,1,1,1,1,1,1,0,1,0)\right]\\
&\quad + \Theta_2\left[\texttt{showdegree[D12]}\,(1,1,1,1,1,1,0,1,0,1,1,1,1,1,0,0,0,0)\right]\\
&\quad + \Theta_3\left[\texttt{showdegree[D12]}\,(1,1,1,1,1,1,0,1,0,1,1,1,0,0,0,0,0,0)\right]\\
&\quad + \Theta_4\left[\texttt{showdegree[D12]}\,(1,1,1,1,1,1,0,1,0,1,0,0,0,0,0,0,0,0)\right]\\
&= -(D_{12}) - (D_{12}^d) - (\mathbf{Z}_{12}^{t_1}) - (\mathbf{Z}_{12}^{t_2}) - (\mathbf{Z}_{12}^{t_3}) - (\mathbf{Z}_{12}^{t_4}) - (\mathbf{Z}_{12}^{t_5})\\
&\quad - (\widetilde{D}_6^d) + (D_6^d) + 2(D_6) + (\mathbf{Z}_6^d) + 2(\mathbf{Z}_6^{t_1}) + 2(\mathbf{Z}_6^{t_2}) + 3(D_4)\\
&\quad + 3(D_4^d) + (D_4^z) + (D_4^{\hat{d}}) + 2(\widetilde{D}_3) - 3(D_3) - (D_3^z)\\
&\quad + 2(\widetilde{D}_3^z) - 2(\mathbf{Z}_4^d) - 2(\mathbf{Z}_4) - 2(\widetilde{D}_2^d) - 2(\widetilde{D}_2) + 2(D_2^d) - 2(D_2)\\
&\quad - 3(\widetilde{D}_2^z) + 4(\widetilde{D}_1) - 4(D_1) + 4(\widetilde{D}_1^z) - 4(D_1^z) + 4(\mathbf{Z}_2) - (D_{12}^2)\\
&\quad - (\mathbf{Z}_{12}^{t_1,2}) - (\mathbf{Z}_{12}^{t_2,2}) - (\mathbf{Z}_{12}^{t_3,2}) - (\mathbf{Z}_{12}^{t_4,2}) - (\widetilde{D}_6^{d,2}) + (D_6^{d,2})\\
&\quad + (D_6^2) + (D_4^{d,2}) + 3(D_4^2) + (D_4^{\hat{d},2}) + (D_4^{z,2}) + (\mathbf{Z}_6^{d,2}) + (\mathbf{Z}_6^{t_1,2})\\
&\quad + 2(\mathbf{Z}_6^{t_2,2}) + 2(\widetilde{D}_3^2) - 2(D_3^2) + (\widetilde{D}_3^{z,2}) - (D_3^{z,2}) - (\mathbf{Z}_4^{d,2}) - 2(\mathbf{Z}_4^2)\\
&\quad - (\widetilde{D}_2^{d,2}) - 3(\widetilde{D}_2^2) + (D_2^{d,2}) - (D_2^2) - 2(\widetilde{D}_2^{z,2}) + 3(\widetilde{D}_1^2) - 3(D_1^2)\\
&\quad + 3(\widetilde{D}_1^{z,2}) - 3(D_1^{z,2}) + 3(\mathbf{Z}_2^2) - (D_{12}^3) + (D_6^3) - (\mathbf{Z}_{12}^{t_1,3})\\
&\quad - (\mathbf{Z}_{12}^{t_2,3}) + (D_4^{d,3}) + (D_4^{\hat{d},3}) + 2(D_4^3) + (\widetilde{D}_3^3) - (D_3^3) + (\mathbf{Z}_6^{t_1,3})\\
&\quad + (\mathbf{Z}_6^{t_2,3}) - (\widetilde{D}_2^{z,3}) - (\widetilde{D}_2^{d,3}) + (D_2^{d,3}) - 2(\widetilde{D}_2^3) - (D_2^3) - (\mathbf{Z}_4^{d,3})\\
&\quad - (\mathbf{Z}_4^3) + 2(\widetilde{D}_1^{z,3}) - 2(D_1^{z,3}) + 2(\widetilde{D}_1^3) - 2(D_1^3) + 2(\mathbf{Z}_2^3) - (D_{12}^4)
\end{aligned}
$$

$$+ (D_6^4) + 2(D_4^4) + (\widetilde{D}_3^4) - (D_3^4) - (\widetilde{D}_2^4) - (D_2^4) - (\mathbf{Z}_4^2) + (\mathbf{Z}_2^4).$$

Since $\mathbf{Z}_\infty = \mathcal{V}_{3,3} \simeq \mathcal{U}_3$, we have the set of all orbit types is $\mathcal{J}(\mathcal{U}_3) = \{(\mathbf{Z}_{12}^{t_3}), (D_6^d), (\widetilde{D}_6^d), (\mathbf{Z}_6^d)\}$. By (Y2′) and Proposition 4.2 (i), there exist the following nontrivial terms in $\deg_\infty$:

$$\begin{aligned}&(D_{12}), (D_{12}^d), (\mathbf{Z}_{12}^{t_1}), (\mathbf{Z}_{12}^{t_2}), (\mathbf{Z}_{12}^{t_3}), (\mathbf{Z}_{12}^{t_4}), (\mathbf{Z}_{12}^{t_5}), (\widetilde{D}_6^d), (D_4^z),\\ &(D_4^{\hat{d}}), (D_{12}^2), (\mathbf{Z}_{12}^{t_1,2}), (\mathbf{Z}_{12}^{t_2,2}), (\mathbf{Z}_{12}^{t_3,2}), (\mathbf{Z}_{12}^{t_4,2}), (\widetilde{D}_6^{d,2}),\\ &(D_4^{\hat{d},2}), (D_4^{z,2}), (D_{12}^3), (\mathbf{Z}_{12}^{t_1,3}), (\mathbf{Z}_{12}^{t_2,3}), (D_4^{\hat{d},3}), (D_{12}^4).\end{aligned} \tag{4.31}$$

On the other hand,

$$\begin{aligned}\deg_{\mathcal{A}}^1 &= \deg_{\mathcal{A}}^0 \star \Big(\sum_{j=0}^{s} \sum_{k=1}^{\infty} \widetilde{m}_j^k(A) \sum_{l=1}^{k} \deg_{\mathcal{V}_{j,l}} + \sum_{j=0}^{s} \widetilde{m}_j(l_0^2) \sum_{l=1}^{l_0-1} \deg_{\mathcal{V}_{j,l}} \Big)\\ &\overset{l_0=2}{=} \prod_{i=0}^{2} \deg_{\mathcal{V}_i} \star \big(1 \cdot \deg_{\mathcal{V}_{0,1}} + 1 \cdot \deg_{\mathcal{V}_{1,1}} + 1 \cdot \deg_{\mathcal{V}_{2,1}}\big)\\ &= \Theta_1 \left[\texttt{showdegree[D12]}\ (1,1,1,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0)\right]\\ &= -(D_{12}) - (\mathbf{Z}_{12}^{t_1}) - (\mathbf{Z}_{12}^{t_2}) - (D_4^d) - (D_4^{\hat{d}}) + (\widetilde{D}_2^z) + (D_2^z) + (\widetilde{D}_2^d)\\ &\quad + (D_2^d) + 2(\widetilde{D}_2) + 2(D_2) + (\mathbf{Z}_4^d) - 2(\widetilde{D}_1^z) - 2(D_1^z) - 3(\widetilde{D}_1)\\ &\quad - 3(D_1) - (\mathbf{Z}_2^-) - 3(\mathbf{Z}_2) + 5(\mathbf{Z}_1).\end{aligned}$$

Since $Z_0 = \mathcal{V}_{0,2}$, we have the set of all orbit types is $\mathcal{J}(Z_0) = \{(D_{12} \times S^1), (D_{12})\}$. By (N2) and Proposition 4.2 (ii), except for possibly (D_{12}), $(\mathbf{Z}_{12}^{t_1})$, $(\mathbf{Z}_{12}^{t_2})$ and $(D_4^{\hat{d}})$, every orbit type listed in (4.31) will appear in $\deg_\infty - \deg_0$, namely,

$$\begin{aligned}&(D_{12}^d), (\mathbf{Z}_{12}^{t_3}), (\mathbf{Z}_{12}^{t_4}), (\mathbf{Z}_{12}^{t_5}), (\widetilde{D}_6^d), (D_4^z), (D_{12}^2), (\mathbf{Z}_{12}^{t_1,2}),\\ &(\mathbf{Z}_{12}^{t_2,2}), (\mathbf{Z}_{12}^{t_3,2}), (\mathbf{Z}_{12}^{t_4,2}), (\widetilde{D}_6^{d,2}), (D_4^{\hat{d},2}), (D_4^{z,2}),\\ &(D_{12}^3), (\mathbf{Z}_{12}^{t_1,3}), (\mathbf{Z}_{12}^{t_2,3}), (D_4^{\hat{d},3}), (D_{12}^4).\end{aligned}$$

Conclusion. Under assumptions (H0)–(H2), ($H3_l$) (with $l_0 = 2$) and ($H4_l$) (with $l_\infty = 3$), by Theorem 4.1, there exist altogether at least 20 nonstationary solutions of (4.10). To be more specific, there are: 1 nonstationary solution with least symmetry (D_{12}^4), 1 nonstationary solution with least symmetry (D_{12}^d), 2 nonstationary solutions with least symmetries $(\mathbf{Z}_{12}^{t_1,3})$, 2 nonstationary solutions with least symmetries $(\mathbf{Z}_{10}^{t_2,3})$, 2 nonstationary solutions with least symmetries $(\mathbf{Z}_{10}^{t_3,2})$, 2 nonstationary solutions with least symmetries $(\mathbf{Z}_{10}^{t_4,2})$, 2 nonstationary solutions with least symmetries $(\mathbf{Z}_{10}^{t_5})$, 2 nonstationary solutions with least symmetries $(\widetilde{D}_6^{d,2})$, 3 nonstationary solutions with least symmetries $(D_4^{\hat{d},3})$ and 3 nonstationary solutions with least symmetries $(D_4^{z,2})$.

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ENDNOTES

1. The equivariant degree Maple© Library package is available at `http://krawcewicz.net/degree` or `http://www.math.ualberta.ca/~wkrawcew/degree`.

2. For the conventions used in this article, especially for complete lists of real (or complex) irreducible Γ-representations (for example, with Γ being the dihedral groups D_N as used for the computational examples), we refer to [**2**].

3. The eigenvalues are evaluated only up to 10^{-1}, which is sufficient for determining the numbers $\widetilde{m}_j^k$ for the computations of degree.

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