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THE REFLEXIVE AND ANTI-REFLEXIVE SOLUTIONS OF A LINEAR MATRIX EQUATION AND SYSTEMS OF MATRIX EQUATIONS

MEHDI DEHGHAN AND MASOUD HAJARIAN

ABSTRACT. An $n \times n$ complex matrix P is said to be a generalized reflection matrix if $P^* = P$ and $P^2 = I$ (where P^* is the conjugate transpose of P). An $n \times n$ complex matrix A is said to be a reflexive (anti-reflexive) matrix with respect to the generalized reflection matrix P if $A = PAP$ $(A = -PAP)$. The reflexive and anti-reflexive matrices have wide applications in information theory, linear estimate theory and numerical analysis. In this paper, we will consider the matrix equations

$$\text{(I)} \qquad A_1XB_1 = D_1,$$

$$\text{(II)} \qquad A_1X = C_1, \qquad XB_2 = C_2,$$

and

$$\text{(III)} \quad A_1X = C_1,\ XB_2 = C_2,\ A_3X = C_3,\ XB_4 = C_4,$$

over reflexive and anti-reflexive matrices. We first introduce several decompositions of $A_1, B_1, C_1, B_2, C_2, A_3, C_3, B_4$ and C_4, then by applying these decompositions, the necessary and sufficient conditions for the solvability of matrix equations (I), (II) and (III) over reflexive or anti-reflexive matrices are proposed. Also some general expressions of the solutions for solvable cases are obtained.

1. Introduction. Throughout the paper, the notation $\mathbf{C}^{m\times n}$ represents the vector space of all $m \times n$ matrices over the complex field. The conjugate transpose of a matrix $A \in \mathbf{C}^{m\times n}$ is denoted as A^*. The unit matrix is denoted by I. We define a conditional inverse of $A \in \mathbf{C}^{m\times n}$, denoted by A^-, to be any matrix $B \in \mathbf{C}^{n\times m}$

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satisfying $ABA = A$. We denote a reflexive inverse of a matrix A by A^+ which satisfies simultaneously $AA^+A = A$ and $A^+AA^+ = A^+$. For two matrices $A = (a_{ij})_{m\times n}$ and B, their Kronecker product is defined as:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

We use $\mathrm{vec}\,(A)$ to represent the $mn \times 1$ vector formed by the vertical concatenation of the respective columns of matrix A. That is, if $A = (\, a_1 \; a_2 \; \cdots \; a_n \,)$, where a_i, $i = 1, 2, \ldots, n$, are column vectors with dimension m, then

$$\mathrm{vec}\,(A) = (\, a_1^T \quad a_2^T \quad \cdots \quad a_n^T \,)^T.$$

Given the $mn \times 1$ vector w, we use $\mathrm{Invec}_{m,n}(w)$ to denote the $m \times n$ matrix $W \in \mathbf{C}^{m\times n}$ such that $\mathrm{vec}\,(W) = w$. In addition, given a matrix A, define $L_A = I - A^+A$ and $R_A = I - AA^+$ where A^+ is any arbitrary but fixed reflexive inverse of the matrix A.

The reflexive and anti-reflexive matrices with respect to a generalized reflection matrix P have applications in system and control theory, in engineering, in scientific computations and various other fields [**1, 2, 3, 44**] which can be defined as follows:

Definition 1.1. A matrix $P \in \mathbf{C}^{n\times n}$ is said to be a generalized reflection matrix if P satisfies that $P^* = P$, $P^2 = I$. Let $P \in \mathbf{C}^{n\times n}$ be a given generalized reflection matrix. A matrix $A \in \mathbf{C}^{n\times n}$ is said to be an $n \times n$ reflexive (anti-reflexive) matrix with respect to P if A satisfies $A = PAP$ ($A = -PAP$). We denote the set of all $n \times n$ reflexive (anti-reflexive) matrices by $\mathbf{C}_r^{n\times n}(P)$ ($\mathbf{C}_a^{n\times n}(P)$).

We know that solving linear matrix equations is a topic of very active research in computational mathematics and has been widely applied in various areas, such as principal component analysis, biology, electricity, solid mechanics, automatics control theory, vibration theory, and so on. A large number of papers have presented several methods for solving matrix equations [**4, 8, 9, 10, 19, 30, 32, 33, 42, 43**]. Dai [**7**] and Chu [**5**] studied the linear matrix equation

$$AXB = C,$$

with a symmetric condition on the solution. In [**25**] Li and Wu gave symmetric and skew-antisymmetric solutions to certain matrix equations

$$A_1X = C_1, \qquad XB_3 = C_3,$$

over the real quaternion algebra H. Mitra [**26, 27**] provided conditions for the existence of a solution and a representation of a general common solution to the pair of individually consistent simultaneous linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ where A_i, B_i and C_i are known matrices defined over the complex field. In [**24**], Jiang and Wei studied the matrix equations

$$X - AXB = C, \qquad X - A\overline{X}B = C,$$

by the method of characteristic polynomial, and derived explicit solutions. In [**16**], Deng and Hu established necessary and sufficient conditions for the existence of and expressions for the general solutions of the linear matrix equation

$$AXA^T + BYB^T = C,$$

with the unknown X and Y. In [**45**] by applying the canonical correlation decomposition (CCD) of matrix pairs, we obtain expressions of the least-squares solutions of the matrix equation

$$AXB + CYD = E,$$

and sufficient and necessary conditions for the existence and uniqueness of the solutions. Conditions for the existence of a common solution to the pair of linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ have also been studied by Shinozki and Sibuya [**34**] and von der Woude [**35**]. Chu [**5**] derived a numerical algorithm for the common solution to equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$. Also, von Rosen [**36**] studied common solutions to the matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ for the special case of $C_1 = C_2 = 0$ [**29**].

Ding and Chen presented the hierarchical gradient iterative (HGI) algorithms for general matrix equations [**17, 22**] and hierarchical least-squares-iterative (HLSI) algorithms for generalized coupled Sylvester matrix equation and general coupled matrix equations [**18, 19**]. The HGI algorithms [**17, 22**] and HLSI algorithms [**19, 20, 22**] for solving general (coupled) matrix equations are innovational and computationally efficient numerical ones and were proposed based on the hierarchical identification principle [**18, 19**] which regards the unknown matrix as the system parameter matrix to be identified. Recently Dehghan and Hajarian [**12**] proposed an efficient iterative method for solving the second-order Sylvester matrix equation

$$EVF^2 - AVF - CV = BW.$$

In [**8, 11, 13–15**], some iterative algorithms were proposed to solve the generalized coupled Sylvester matrix equations and the Sylvester matrix equation over reflexive and anti-reflexive matrices. In [**39, 40, 41**], the authors investigated symmetric, persymmetric and centrosymmetric solutions to several systems of matrix equations.

In this article, we give the reflexive and anti-reflexive solutions of three matrix equations

$$A_1XB_1 = D_1, \tag{1.1}$$

$$A_1X = C_1, \qquad XB_2 = C_2, \tag{1.2}$$

and

$$A_1X = C_1, \qquad XB_2 = C_2, \qquad A_3X = C_3, \qquad XB_4 = C_4, \tag{1.3}$$

where $A_1 \in \mathbf{C}^{m\times n}$, $B_1 \in \mathbf{C}^{n\times l}$, $D_1 \in \mathbf{C}^{m\times l}$, $C_1 \in \mathbf{C}^{m\times n}$, $B_2 \in \mathbf{C}^{n\times s}$, $C_2 \in \mathbf{C}^{n\times s}$, $A_3 \in \mathbf{C}^{k\times n}$, $C_3 \in \mathbf{C}^{k\times n}$, $B_4 \in \mathbf{C}^{n\times t}$ and $C_4 \in \mathbf{C}^{n\times t}$.

This paper is organized as follows. In Section 2, we first review some properties of the generalized reflection matrix P and subsets $\mathbf{C}_r^{n\times n}(P)$ and $\mathbf{C}_a^{n\times n}(P)$ of $\mathbf{C}^{n\times n}$; then we introduce decompositions of $A_1, B_1, D_1, C_1, B_2, C_2, A_3, C_3, B_4$ and C_4. By using these decompositions, the necessary and sufficient conditions for the existence and the expressions of the general solutions to the matrix equations (1.1)–(1.3) are proposed.

2. Main results. In this section we first review some properties of the generalized reflection matrix P and subsets $\mathbf{C}_r^{n\times n}(P)$ and $\mathbf{C}_a^{n\times n}(P)$ of $\mathbf{C}^{n\times n}$. Then we give necessary and sufficient conditions for the existence and the expression for the reflexive and anti-reflexive solutions of (1.1)–(1.3). Now we state the following preliminary results and lemmas. Their proofs can be found in [**6, 31**].

Let $P \in \mathbf{C}^{n\times n}$ be a generalized reflection matrix. We can express the matrix P by the following form [**6**]:

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*. \tag{2.1}$$

where $U = (U_1, U_2)$ is a unitary matrix and $U_1 \in \mathbf{C}^{n\times r}$, $U_2 \in \mathbf{C}^{n\times(n-r)}$.

Lemma 2.1. *The matrix $A \in \mathbf{C}_r^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} U^*, \tag{2.2}$$

where $A_1 \in \mathbf{C}^{r\times r}$, $A_4 \in \mathbf{C}^{(n-r)\times(n-r)}$ and U is as in (2.1).

Lemma 2.2. *The matrix $A \in \mathbf{C}_a^{n\times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix} U^*, \tag{2.3}$$

where $A_2 \in \mathbf{C}^{r\times(n-r)}$, $A_3 \in \mathbf{C}^{(n-r)\times r}$ and U is as in (2.1).

Now, for the matrices $A_1 \in \mathbf{C}^{m\times n}$, $B_1 \in \mathbf{C}^{n\times l}$, $C_1 \in \mathbf{C}^{m\times n}$, $B_2 \in \mathbf{C}^{n\times s}$, $C_2 \in \mathbf{C}^{n\times s}$, $A_3 \in \mathbf{C}^{k\times n}$, $C_3 \in \mathbf{C}^{k\times n}$, $B_4 \in \mathbf{C}^{n\times t}$ and $C_4 \in \mathbf{C}^{n\times t}$, we introduce the following decompositions

$$A_1 U = (A_{11}, A_{12}) \text{ where } A_{11} \in \mathbf{C}^{m\times r} \text{ and } A_{12} \in \mathbf{C}^{m\times(n-r)}, \tag{2.4}$$

$$U^*B_1 = (B_{11}^T, B_{12}^T)^T \text{ where } B_{11} \in \mathbf{C}^{r\times l} \text{ and } B_{12} \in \mathbf{C}^{(n-r)\times l}, \tag{2.5}$$

$$C_1U = (C_{11}, C_{12}) \text{ where } C_{11} \in \mathbf{C}^{m\times r} \text{ and } C_{12} \in \mathbf{C}^{m\times (n-r)}, \tag{2.6}$$

$$U^*B_2 = (B_{21}^T, B_{22}^T)^T \text{ where } B_{21} \in \mathbf{C}^{r\times s} \text{ and } B_{22} \in \mathbf{C}^{(n-r)\times s}, \tag{2.7}$$

$$U^*C_2 = (C_{21}^T, C_{22}^T)^T \text{ where } C_{21} \in \mathbf{C}^{r\times s} \text{ and } C_{22} \in \mathbf{C}^{(n-r)\times s}, \tag{2.8}$$

$$A_3U = (A_{31}, A_{32}) \text{ where } A_{31} \in \mathbf{C}^{k\times r} \text{ and } A_{32} \in \mathbf{C}^{k\times (n-r)}, \tag{2.9}$$

$$C_3U = (C_{31}, C_{32}) \text{ where } C_{31} \in \mathbf{C}^{k\times r} \text{ and } C_{32} \in \mathbf{C}^{k\times (n-r)}, \tag{2.10}$$

$$U^*B_4 = (B_{41}^T, B_{42}^T)^T \text{ where } B_{41} \in \mathbf{C}^{r\times t} \text{ and } B_{42} \in \mathbf{C}^{(n-r)\times t}, \tag{2.11}$$

$$U^*C_4 = (C_{41}^T, C_{42}^T)^T \text{ where } C_{41} \in \mathbf{C}^{r\times t} \text{ and } C_{42} \in \mathbf{C}^{(n-r)\times t}. \tag{2.12}$$

In the rest of this paper, we will suppose without loss of generality, that the matrices $A_1, B_1, C_1, B_2, C_2, A_3, C_3, B_4$ and C_4 have the above decompositions.

• **Solution to the matrix equation (1.1).** First we consider the matrix equation $A_1XB_1 = D_1$.

Theorem 2.1. *Let* $A_1 \in \mathbf{C}^{m\times n}$, $B_1 \in \mathbf{C}^{n\times l}$, $D_1 \in \mathbf{C}^{m\times l}$. *For the linear matrix equation* $A_1XB_1 = D_1$, *the following statements are equivalent*:

(1): *The linear matrix equation* $A_1XB_1 = D_1$ *has a solution* $X \in \mathbf{C}_r^{n\times n}(P)$.

(2): $R_M R_{A_{11}} D_1 = 0$, $R_{A_{11}} D_1 L_{B_{12}} = 0$, $D_1 L_{B_{11}} L_N = 0$, $R_{A_{12}} D_1 L_{B_{11}} = 0$.

(3): $MM^+ R_{A_{11}} D_1 B_{12}^+ B_{12} = R_{A_{11}} D_1$, $A_{12}A_{12}^+ D_1 L_{B_{11}} N^+ N = D_1 L_{B_{11}}$.

(4): $R_H R_{A_{12}} D_1 = 0$, $R_{A_{12}} D_1 L_{B_{11}} = 0$, $R_{A_{11}} D_1 L_{B_{12}} = 0$, $D_1 L_{B_{12}} L_Q = 0$.

(5): $HH^+ R_{A_{12}} D_1 B_{11} B_{11}^+ = R_{A_{12}} D_1$, $A_{11}A_{11}^+ D_1 L_{B_{12}} Q^+ Q = D_1 L_{B_{12}}$

where $M = R_{A_{11}} A_{12}$, $N = B_{12} L_{B_{11}}$, $S = A_{12} L_M$, $T = R_{B_{12}} N$, $F = NL_T$, $G = R_S A_{12}$, $H = R_{A_{12}} A_{11}$, $Q = B_{11} L_{B_{12}}$.

In that case, the reflexive solution of the matrix equation $A_1XB_1 = D_1$ *can be expressed as the following*

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* \in \mathbf{C}_r^{n\times n}(P), \tag{2.13}$$

where

$$X_1 = A_{11}^+(D_1 - A_{12}X_4B_{12})B_{11}^+ + L_{A_{11}}J + ZR_{B_{11}}, \tag{2.14}$$

$$\begin{aligned} X_4 = {} & M^+R_{A_{11}}D_1B_{12}^+ + L_M(V - S^+SVNN^+) - L_MS^+A_{12}L_GWTN^+ \\ & + (W - G^+GWTT^+)R_{B_{12}}, \end{aligned} \tag{2.15}$$

where J, V, W, Z *are arbitrary matrices with appropriate sizes, or*

$$\begin{aligned} X_1 = {} & H^+R_{A_{12}}D_1B_{11}^+ \\ & + L_H(V_1 - S_1^+S_1V_1QQ^+) \\ & - L_HS_1^+A_{11}L_{G_1}W_1T_1Q^+ \\ & + (W_1 - G_1^+G_1W_1T_1T_1^+)R_{B_{11}}, \end{aligned} \tag{2.16}$$

$$X_4 = A_{12}^+(D_1 - A_{12}X_1B_{12})B_{12}^+ + L_{A_{12}}J_1 + Z_1R_{B_{12}}, \tag{2.17}$$

where $S_1 = A_{11}L_H$, $T_1 = R_{B_{11}}Q$, $G_1 = R_{S_1}A_{11}$ *and* J_1, V_1, W_1, Z_1 *are arbitrary matrices with appropriate sizes.*

Proof. We first give a general matrix equation equivalent to the matrix equation $A_1XB_1 = D_1$. By substituting (2.4) and (2.5) in the matrix equation $A_1XB_1 = D_1$, we can obtain

$$\begin{aligned} & A_1XB_1 = D_1 \\ & \Longleftrightarrow (A_{11}\ A_{12})U^*U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^*U\begin{pmatrix} B_{11} \\ B_{12} \end{pmatrix} = D_1 \\ & \Longleftrightarrow A_{11}X_1B_{11} + A_{12}X_4B_{12} = D_1. \end{aligned} \tag{2.18}$$

Hence the matrix equation (2.18) is equivalent to the matrix equation (1.1). From the results in [**37**], we have

(1): The linear matrix equation $A_{11}X_1B_{11} + A_{12}X_4B_{12} = D_1$ has solutions $X_1 \in \mathbf{C}^{r\times r}$ and $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$.

(2): $R_M R_{A_{11}} D_1 = 0$, $R_{A_{11}} D_1 L_{B_{12}} = 0$, $D_1 L_{B_{11}} L_N = 0$, $R_{A_{12}} D_1 L_{B_{11}} = 0$.

(3): $MM^+R_{A_{11}}D_1B_{12}^+B_{12} = R_{A_{11}}D_1$, $A_{12}A_{12}^+D_1L_{B_{11}}N^+N = D_1L_{B_{11}}$.

(4): $R_H R_{A_{12}} D_1 = 0$, $R_{A_{12}} D_1 L_{B_{11}} = 0$, $R_{A_{11}} D_1 L_{B_{12}} = 0$, $D_1 L_{B_{12}} L_Q = 0$.

(5): $HH^+R_{A_{12}}D_1B_{11}B_{11}^+ = R_{A_{12}}D_1$, $A_{11}A_{11}^+D_1L_{B_{12}}Q^+Q = D_1L_{B_{12}}$.

In that case, the solution of the matrix equation $A_{11}X_1B_{11}+A_{12}X_4B_{12} = D_1$ can be expressed by (2.14)–(2.15) or (2.16)–(2.17); therefore, the reflexive solution of the matrix equation $A_1XB_1 = D_1$ can be expressed by

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* \in \mathbf{C}_r^{n\times n}(P), \tag{2.19}$$

where X_1 and X_4 are obtained from (2.14)–(2.15) or (2.16)–(2.17).

The proof is completed. □

Similarly to the proof of Theorem (2.1), we can prove the following theorem.

Theorem 2.2. *Let $A_1 \in \mathbf{C}^{m\times n}$, $B_1 \in \mathbf{C}^{n\times l}$, $D_1 \in \mathbf{C}^{m\times l}$. For the linear matrix equation $A_1XB_1 = D_1$, the following statements are equivalent*:

(1): *The linear matrix equation $A_1XB_1 = D_1$ has a solution $X \in \mathbf{C}_a^{n\times n}(P)$.*

(2): $R_M R_{A_{11}} D_1 = 0$, $R_{A_{11}} D_1 L_{B_{11}} = 0$, $D_1 L_{B_{12}} L_N = 0$, $R_{A_{12}} D_1 L_{B_{12}} = 0$.

(3): $MM^+ R_{A_{11}} D_1 B_{11}^+ B_{11} = R_{A_{11}} D_1$, $A_{12} A_{12}^+ D_1 L_{B_{12}} N^+ N = D_1 L_{B_{12}}$.

(4): $R_H R_{A_{12}} D_1 = 0$, $R_{A_{12}} D_1 L_{B_{12}} = 0$, $R_{A_{11}} D_1 L_{B_{11}} = 0$, $D_1 L_{B_{11}} L_Q = 0$.

(5): $HH^+ R_{A_{12}} D_1 B_{12} B_{12}^+ = R_{A_{12}} D_1$, $A_{11} A_{11}^+ D_1 L_{B_{11}} Q^+ Q = D_1 L_{B_{11}}$.

where $M = R_{A_{11}} A_{12}$, $N = B_{11} L_{B_{12}}$, $S = A_{12} L_M$, $T = R_{B_{11}} N$, $F = NL_T$, $G = R_S A_{12}$, $H = R_{A_{12}} A_{11}$, $Q = B_{12} L_{B_{11}}$.

In that case, the anti-reflexive solution of the matrix equation $A_1XB_1 = D_1$ can be expressed as the following

$$X = U \begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix} U^* \in \mathbf{C}_a^{n\times n}(P), \tag{2.20}$$

where

$$X_2 = A_{11}^+ (D_1 - A_{12} X_4 B_{11}) B_{12}^+ + L_{A_{11}} J + Z R_{B_{12}}, \tag{2.21}$$

$$\begin{aligned} X_3 = {} & M^+ R_{A_{11}} D_1 B_{11}^+ + L_M (V - S^+ S V N N^+) \\ & - L_M S^+ A_{12} L_G W T N^+ + (W - G^+ G W T T^+) R_{B_{11}}, \end{aligned} \tag{2.22}$$

where J, V, W, Z are arbitrary matrices with appropriate sizes, or

$$\begin{aligned} X_2 = {} & H^+ R_{A_{12}} D_1 B_{12}^+ + L_H (V_1 - S_1^+ S_1 V_1 Q Q^+) \\ & - L_H S_1^+ A_{11} L_{G_1} W_1 T_1 Q^+ \\ & + (W_1 - G_1^+ G_1 W_1 T_1 T_1^+) R_{B_{12}}, \end{aligned} \tag{2.23}$$

$$X_3 = A_{12}^+ (D_1 - A_{12} X_1 B_{11}) B_{11}^+ + L_{A_{12}} J_1 + Z_1 R_{B_{11}}, \tag{2.24}$$

where $S_1 = A_{11} L_H$, $T_1 = R_{B_{12}} Q$, $G_1 = R_{S_1} A_{11}$ and J_1, V_1, W_1, Z_1 are arbitrary matrices with appropriate sizes.

• **Solution to matrix equations (1.2).** Now we consider the matrix equations $A_1X = C_1$, $XB_2 = C_2$ over reflexive and anti-reflexive matrices. In the following theorems, we give several necessary and sufficient conditions for the solvability of these matrix equations over reflexive or anti-reflexive matrices, and several general expressions of the solutions of the matrix equations.

Theorem 2.3. *Suppose that $A_1 \in \mathbf{C}^{m\times n}$, $C_1 \in \mathbf{C}^{m\times n}$, $B_2 \in \mathbf{C}^{n\times s}$, $C_2 \in \mathbf{C}^{n\times s}$. The matrix equations $A_1X = C_1$, $XB_2 = C_2$ have a solution $X \in \mathbf{C}_r^{n\times n}(P)$, if and only if*

$$A_{11}A_{11}^{+}C_{11} = C_{11}, \qquad C_{21}B_{21}^{+}B_{21} = C_{21}, \tag{2.25}$$

$$A_{12}A_{12}^{+}C_{12} = C_{12}, \qquad C_{22}B_{22}^{+}B_{22} = C_{22}, \tag{2.26}$$

$$W_1(C_{21}B_{21}^{+} - A_{11}^{+}C_{11})B_{21} = 0, \tag{2.27}$$

$$W_2(C_{22}B_{22}^{+} - A_{12}^{+}C_{12})B_{22} = 0, \tag{2.28}$$

where $S_1 = L_{A_{11}}$, $S_2 = L_{A_{12}}$, $W_1 = R_{S_1}$ and $W_2 = R_{S_2}$.

In that case, the reflexive solution of the matrix equations $A_1X = C_1$, $XB_2 = C_2$ can be expressed by the following form:

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* \in \mathbf{C}_r^{n\times n}(P), \tag{2.29}$$

where

$$\begin{aligned} X_1 = {} & A_{11}^{+}C_{11} + L_{A_{11}}S_1^{+}L_{W_1}(C_{21}B_{21}^{+} - A_{11}^{+}C_{11})B_{21}B_{21}^{+} \\ & + L_{A_{11}}(Y - S_1^{+}S_1YB_{21}B_{21}^{+}) \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} X_4 = {} & A_{12}^{+}C_{12} + L_{A_{12}}S_2^{+}L_{W_2}(C_{22}B_{22}^{+} - A_{12}^{+}C_{12})B_{22}B_{22}^{+} \\ & + L_{A_{12}}(Z - S_2^{+}S_2ZB_{22}B_{22}^{+}) \end{aligned} \tag{2.31}$$

and Y, Z are any matrices with appropriate dimensions.

Proof. We first prove the necessity. Assume that the matrix equations $A_1X = C_1$, $XB_2 = C_2$ have a solution $X \in \mathbf{C}_r^{n\times n}(P)$. From Lemma 2.1, $X \in \mathbf{C}_r^{n\times n}(P)$ can be expressed as

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^*, \tag{2.32}$$

where $X_1 \in \mathbf{C}^{r\times r}$, $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$. By applying (2.4)–(2.8), we can write
(2.33)

$$\begin{cases} A_1X = C_1, \\ XB_2 = C_2, \end{cases} \iff \begin{cases} (A_{11}\ A_{12})U^*U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* = (C_{11}\ C_{12})U^*, \\ U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^*U \begin{pmatrix} B_{21} \\ B_{22} \end{pmatrix} = U \begin{pmatrix} C_{21} \\ C_{22} \end{pmatrix}, \end{cases}$$
$$\iff \begin{cases} A_{11}X_1 = C_{11},\ X_1B_{21} = C_{21}, \\ A_{12}X_4 = C_{12},\ X_4B_{22} = C_{22}. \end{cases}$$

Therefore, the matrix equations (2.33) are equivalent to the system of matrix equations (1.2). From [**37**], we have $A_{11}A_{11}^+C_{11} = C_{11}$, $C_{21}B_{21}^+B_{21} = C_{21}$, $A_{12}A_{12}^+C_{12} = C_{12}$, $C_{22}B_{22}^+B_{22} = C_{22}$, $W_1(C_{21}B_{21}^+ - A_{11}^+C_{11})B_{21} = 0$, $W_2(C_{22}B_{22}^+ - A_{12}^+C_{12})B_{22} = 0$, and

$$\begin{aligned} X_1 = {} & A_{11}^+C_{11} + L_{A_{11}}S_1^+L_{W_1}(C_{21}B_{21}^+ - A_{11}^+C_{11})B_{21}B_{21}^+ \\ & + L_{A_{11}}(Y - S_1^+S_1YB_{21}B_{21}^+), \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} X_4 = {} & A_{12}^+C_{12} + L_{A_{12}}S_2^+L_{W_2}(C_{22}B_{22}^+ - A_{12}^+C_{12})B_{22}B_{22}^+ \\ & + L_{A_{12}}(Z - S_2^+S_2ZB_{22}B_{22}^+), \end{aligned} \tag{2.35}$$

where Y and Z are any matrices with appropriate dimensions. Now we substitute (2.34) and (2.35) in (2.32). The proof of the necessity is completed.

Now we prove the sufficiency. Suppose that (2.25)–(2.28) hold. It is well-known [**37**] that there exist $X_1 \in \mathbf{C}^{r\times r}$ and $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$ such that

$$\begin{cases} A_{11}X_1 = C_{11},\ X_1B_{21} = C_{21}, \\ A_{12}X_4 = C_{12},\ X_4B_{22} = C_{22}. \end{cases} \tag{2.36}$$

From previous results, (2.36) is equivalent to

$$A_1 U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* = C_1,$$

$$U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* B_2 = C_2.$$

Hence,

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* \in \mathbf{C}_r^{n\times n}(P)$$

($X_1 \in \mathbf{C}^{r\times r}$, and $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$) is the solution of the matrix equations (1.2). The proof of the sufficiency is completed. □

Similarly to the proof of Theorem 2.3, we can prove the following theorem.

Theorem 2.4. *Suppose that* $A_1 \in \mathbf{C}^{m\times n}$, $C_1 \in \mathbf{C}^{m\times n}$, $B_2 \in \mathbf{C}^{n\times s}$, $C_2 \in \mathbf{C}^{n\times s}$. *The matrix equations* $A_1X = C_1$, $XB_2 = C_2$ *have a solution* $X \in \mathbf{C}_a^{n\times n}(P)$, *if and only if*

$$A_{11}A_{11}^+C_{12} = C_{12}, \qquad C_{21}B_{22}^+B_{22} = C_{21}, \tag{2.37}$$

$$A_{12}A_{12}^+C_{11} = C_{11}, \quad C_{22}B_{21}^+B_{21} = C_{22}, \tag{2.38}$$

$$W_1(C_{21}B_{22}^+ - A_{11}^+C_{12})B_{22} = 0, \tag{2.39}$$

$$W_2(C_{22}B_{21}^+ - A_{12}^+C_{11})B_{21} = 0, \tag{2.40}$$

where $S_1 = L_{A_{11}}$, $S_2 = L_{A_{12}}$, $W_1 = R_{S_1}$ *and* $W_2 = R_{S_2}$.

In that case, the anti-reflexive solution of the matrix equations $A_1X = C_1$, $XB_2 = C_2$ *can be expressed by the following form*:

$$X = U \begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix} U^* \in \mathbf{C}_a^{n\times n}(P), \tag{2.41}$$

where

$$\begin{aligned} X_2 = {} & A_{11}^+C_{12} + L_{A_{11}}S_1^+L_{W_1}(C_{21}B_{22}^+ - A_{11}^+C_{12})B_{22}B_{22}^+ \\ & + L_{A_{11}}(Y - S_1^+S_1YB_{22}B_{22}^+), \end{aligned} \tag{2.42}$$

and

$$X_3 = A_{12}^{+}C_{11} + L_{A_{12}}S_2^{+}L_{W_2}(C_{22}B_{21}^{+} - A_{12}^{+}C_{11})B_{21}B_{21}^{+} + L_{A_{12}}(Z - S_2^{+}S_2 Z B_{21}B_{21}^{+}), \tag{2.43}$$

and Y, Z *are any matrices with appropriate dimensions.*

In the following theorems, we introduce some other conditions for the solvability of matrix equations $A_1X = C_1$, $XB_2 = C_2$ over reflexive and anti-reflexive matrices and represent some other forms of the general expression of the solution. We first define some matrices which are used in next theorems. The following definitions can be found in [**29**].

Suppose that $K_1 \in \mathbf{C}^{n\times q}$, $K_2 \in \mathbf{C}^{p\times q}$, $H_1 \in \mathbf{C}^{r\times s}$, $H_2 \in \mathbf{C}^{r\times t}$, $R_1 \in \mathbf{C}^{n\times s}$, and $R_2 \in \mathbf{C}^{p\times t}$. Let now$(K)$ be the number of rows in the matrix K and let $\mathrm{ncol}(K)$ be the number of columns in the matrix K. For $i, j = 1, 2$, $i \neq j$, define the following matrices. Let $\widetilde{K}_i = \begin{pmatrix} K_i \\ 0 \end{pmatrix}$ if $\mathrm{nrow}(K_j) > \mathrm{nrow}(K_i)$ where, $\mathrm{nrow}(\widetilde{K}_i) = \max(n, p)$, and let $\widetilde{K}_i = K_i$, otherwise. Let $\widetilde{H}_i = (H_i\ 0)$ if $\mathrm{nrow}(H_j) > \mathrm{ncol}(H_i)$, where $\mathrm{nrow}(\widetilde{H}_i) = \max(s, t)$, and let $\widetilde{H}_i = H_i$, otherwise. Also, let $\widetilde{R}_i = \begin{pmatrix} R_i \\ 0 \end{pmatrix}$, where $\mathrm{nrow}(\widetilde{R}_i) = \max(n, p)$ if $\mathrm{nrow}(R_i) \geq \mathrm{ncol}(R_j)$, and $\mathrm{nrow}(R_i) < \mathrm{nrow}(R_j)$. Let $\widetilde{R}_i = (R_i\ 0)$, where $\mathrm{nrow}(\widetilde{R}_i) = \max(s, t)$ if $\mathrm{nrow}(R_i) \geq \mathrm{nrow}(R_j)$ and $\mathrm{nrow}(R_i) < \mathrm{ncol}(R_j)$. Let $\widetilde{R}_i = \begin{pmatrix} \mathrm{nrow} & R_i & 0 \\ & 0 & 0 \end{pmatrix}$, where $\mathrm{nrow}(\widetilde{R}_i) = \max(n, p)$ and $\mathrm{ncol}(\widetilde{R}_i) = \max(s, t)$ if $\mathrm{nrow}(R_i) < \mathrm{nrow}(R_j)$ and $\mathrm{ncol}(R_i) < \mathrm{ncol}(R_j)$, and let $\widetilde{R}_i = R_i$ if $\mathrm{ncol}(R_i) \geq \mathrm{ncol}(R_j)$, and $\mathrm{nrow}(R_i) \geq \mathrm{nrow}(R_j)$, where $\mathrm{nrow}(\widetilde{R}_i) = \max(n, p)$ and $\mathrm{ncol}(\widetilde{R}_i) = \max(s, t)$.

In the following theorem, we use the matrices introduced above. We now present new necessary and sufficient conditions for the linear matrix equations (1.2) to have a common solution. We also give a new representation for the solution to the matrix equations, provided a solution exists.

Theorem 2.5. *Suppose that* $A_1 \in \mathbf{C}^{m\times n}$, $C_1 \in \mathbf{C}^{m\times n}$, $B_2 \in \mathbf{C}^{n\times s}$, $C_2 \in \mathbf{C}^{n\times s}$. *Also we assume* $d_1 = \max(m, r)$, $e_1 = \max(s, r)$, $d_2 = \max(m, n - r)$ *and* $e_2 = \max(s, n - r)$. *The matrix equations*

$A_1X = C_1$, $XB_2 = C_2$ have a solution $X \in \mathbf{C}_r^{n\times n}(P)$, if and only if

$$\widetilde{A}_{11}\widetilde{A}_{11}^{-}\widetilde{C}_{11}\widetilde{I}^{-}\widetilde{I} = \widetilde{C}_{11}, \qquad \widetilde{A}_{12}\widetilde{A}_{12}^{-}\widetilde{C}_{12}\widetilde{I}^{-}\widetilde{I} = \widetilde{C}_{12},$$
$$\mathrm{Invec}_{d_1,e_1}(\widetilde{M_1}\widetilde{M_1}^{-}\mathrm{vec}\,\widetilde{N}_1) = \widetilde{N}_1, \qquad \mathrm{Invec}_{d_2,e_2}(\widetilde{M_2}\widetilde{M_2}^{-}\mathrm{vec}\,\widetilde{N}_2) = \widetilde{N}_2,$$

where $\widetilde{M_1} = \widetilde{B}_{21}^* \otimes \widetilde{I} + \widetilde{E}_1^* \otimes \widetilde{D}_1$, $\widetilde{M_2} = \widetilde{B}_{22}^* \otimes \widetilde{I} + \widetilde{E}_2^* \otimes \widetilde{D}_2$, $\widetilde{D}_1 = -\widetilde{I}\widetilde{A}_{11}^{-}\widetilde{A}_{11}$, $\widetilde{D}_2 = -\widetilde{I}\widetilde{A}_{12}^{-}\widetilde{A}_{12}$, $\widetilde{E}_1 = \widetilde{I}\widetilde{I}^{-}\widetilde{B}_{21}$, $\widetilde{E}_2 = \widetilde{I}\widetilde{I}^{-}\widetilde{B}_{22}$, $\widetilde{N}_1 = \widetilde{C}_{21} - \widetilde{I}\widetilde{A}_{11}^{-}\widetilde{C}_{11}\widetilde{I}^{-}\widetilde{B}_{21}$ and $\widetilde{N}_2 = \widetilde{C}_{22} - \widetilde{I}\widetilde{A}_{12}^{-}\widetilde{C}_{12}\widetilde{I}^{-}\widetilde{B}_{22}$.

A representation of the reflexive solution to these matrix equations is

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* \in \mathbf{C}_r^{n\times n}(P), \tag{2.44}$$

where

(2.45)
$$\begin{aligned} X_1 = \widetilde{A}_{11}^{-}\widetilde{C}_{11}\widetilde{I}^{-} &+ \mathrm{Invec}_{r,r}\{\widetilde{M_1}^{-}\mathrm{vec}\widetilde{N}_1 + (I - \widetilde{M_1}^{-}\widetilde{M_1})\mathrm{vec}\,V_1\}, \\ &- \widetilde{A}_{11}^{-}\widetilde{A}_{11}\{\mathrm{Invec}_{r,r}(\widetilde{M_1}^{-}\mathrm{vec}\,N_1 + (I - \widetilde{M_1}^{-}\widetilde{M_1})\mathrm{vec}\,V_1)\}\widetilde{I}\widetilde{I}^{-}, \end{aligned}$$

and

(2.46)
$$\begin{aligned} X_4 = \widetilde{A}_{12}^{-}\widetilde{C}_{12}\widetilde{I}^{-} &+ \mathrm{Invec}_{n-r,n-r}\{\widetilde{M_2}^{-}\mathrm{vec}\,\widetilde{N}_2 + (I - \widetilde{M_2}^{-}\widetilde{M_2})\mathrm{vec}\,V_2\} \\ &- \widetilde{A}_{12}^{-}\widetilde{A}_{12}\{\mathrm{Invec}_{n-r,n-r}(\widetilde{M_2}^{-}\,vec\,N_2 + (I - \widetilde{M_2}^{-}\widetilde{M_2})\mathrm{vec}\,V_2)\}\widetilde{I}\widetilde{I}^{-}, \end{aligned}$$

and $V_1 \in \mathbf{C}^{r\times r}$, $V_2 \in \mathbf{C}^{n-r\times n-r}$ are arbitrary matrices.

Proof. The necessity. Let the matrix equations $A_1X = C_1$, $XB_2 = C_2$ has a solution $X \in \mathbf{C}_r^{n\times n}(P)$ which can be expressed by

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^*, \tag{2.47}$$

where $X_1 \in \mathbf{C}^{r\times r}$, $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$. We know the matrix equations (1.2) are equivalent to the matrix equations (2.33). It follows from [**29**] that $\widetilde{A}_{11}\widetilde{A}_{11}^{-}\widetilde{C}_{11}\widetilde{I}^{-}\widetilde{I} = \widetilde{C}_{11}$, $\widetilde{A}_{12}\widetilde{A}_{12}^{-}\widetilde{C}_{12}\widetilde{I}^{-}\widetilde{I} = \widetilde{C}_{12}$, $\mathrm{Invec}_{d_1,e_1}(\widetilde{M_1}\widetilde{M_1}^{-}\mathrm{vec}\,\widetilde{N}_1) = \widetilde{N}_1$, and $\mathrm{Invec}_{d_2,e_2}(\widetilde{M_2}\widetilde{M_2}^{-}\mathrm{vec}\,\widetilde{N}_2) = \widetilde{N}_2$ and

(2.48)
$$\begin{aligned} X_1 = \widetilde{A}_{11}^{-}\widetilde{C}_{11}\widetilde{I}^{-} &+ \mathrm{Invec}_{r,r}\{\widetilde{M_1}^{-}\mathrm{vec}\,\widetilde{N}_1 + (I - \widetilde{M_1}^{-}\widetilde{M_1})\mathrm{vec}\,V_1\} \\ &- \widetilde{A}_{11}^{-}\widetilde{A}_{11}\{\mathrm{Invec}_{r,r}(\widetilde{M_1}^{-}\mathrm{vec}\,N_1 + (I - \widetilde{M_1}^{-}\widetilde{M_1})\mathrm{vec}\,V_1)\}\widetilde{I}\widetilde{I}^{-}, \end{aligned}$$

and

(2.49)
$$X_4 = \widetilde{A}_{12}^{-}\widetilde{C}_{12}\widetilde{I}^{-} + \mathrm{Invec}_{n-r,n-r}\{\widetilde{M_2}^{-}\mathrm{vec}\,\widetilde{N}_2 + (I - \widetilde{M_2}^{-}\widetilde{M_2})\mathrm{vec}\,V_2\}$$
$$- \widetilde{A}_{12}^{-}\widetilde{A}_{12}\{\mathrm{Invec}_{n-r,n-r}(\widetilde{M_2}^{-}\mathrm{vec}\,N_2 + (I - \widetilde{M_2}^{-}\widetilde{M_2})\mathrm{vec}\,V_2)\}\widetilde{I}\widetilde{I}^{-},$$

where $V_1 \in \mathbf{C}^{r\times r}$, $V_2 \in \mathbf{C}^{n-r\times n-r}$ are arbitrary matrices. By substituting (2.48) and (2.49) in (2.47), we can finish easily the proof of necessity.

The sufficiency. Assume that $\widetilde{A}_{11}\widetilde{A}_{11}^{-}\widetilde{C}_{11}\widetilde{I}^{-}\widetilde{I} = \widetilde{C}_{11}$, $\widetilde{A}_{12}\widetilde{A}_{12}^{-}\widetilde{C}_{12}\widetilde{I}^{-}\widetilde{I} = \widetilde{C}_{12}$, $\mathrm{Invec}_{d_1,e_1}(\widetilde{M_1}\widetilde{M_1}^{-}\mathrm{vec}\,\widetilde{N}_1) = \widetilde{N}_1$, and $\mathrm{Invec}_{d_2,e_2}(\widetilde{M_2}\widetilde{M_2}^{-}\mathrm{vec}\,\widetilde{N}_2) = \widetilde{N}_2$. From results in [**29**], we obtain that there exist $X_1 \in \mathbf{C}^{r\times r}$, $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$ such that

$$\begin{cases} A_{11}X_1 = C_{11}, & X_1B_{21} = C_{21}, \\ A_{12}X_4 = C_{12}, & X_4B_{22} = C_{22}. \end{cases} \tag{2.50}$$

The above equation is equivalent to

$$A_1U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* = C_1,$$
$$U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^*B_2 = C_2.$$

Hence

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* \in \mathbf{C}_r^{n\times n}(P)$$
$$(X_1 \in \mathbf{C}^{r\times r},\ \text{and } X_4 \in \mathbf{C}^{(n-r)\times(n-r)})$$

is the solution of the matrix equations (1.2). The proof of the sufficiency is completed. □

Similarly, from Lemma 2.1 and Equation (2.33), we deduce the following results:

Theorem 2.6. *Suppose that* $A_1 \in \mathbf{C}^{m\times n}$, $C_1 \in \mathbf{C}^{m\times n}$, $B_2 \in \mathbf{C}^{n\times s}$, $C_2 \in \mathbf{C}^{n\times s}$. *Also we assume* $d_1 = \max(m, r)$, $e_1 = \max(s, n-r)$, $d_2 = \max(m, n-r)$ *and* $e_2 = \max(s, r)$. *The matrix equations*

$A_1X = C_1$, $XB_2 = C_2$ have the solution $X \in \mathbf{C}_a^{n\times n}(P)$, if and only if

$$\widetilde{A}_{11}\widetilde{A}_{11}^{-}\widetilde{C}_{12}\widetilde{I}^{-}\widetilde{I} = \widetilde{C}_{12}, \qquad \widetilde{A}_{12}\widetilde{A}_{12}^{-}\widetilde{C}_{11}\widetilde{I}^{-}\widetilde{I} = \widetilde{C}_{11},$$
$$\mathrm{Invec}_{d_1,e_1}(\widetilde{M_1}\widetilde{M_1}^{-}\mathrm{vec}\,\widetilde{N}_1) = \widetilde{N}_1, \qquad \mathrm{Invec}_{d_2,e_2}(\widetilde{M_2}\widetilde{M_2}^{-}\mathrm{vec}\,\widetilde{N}_2) = \widetilde{N}_2$$

where $\widetilde{M_1} = \widetilde{B}_{22}^{*} \otimes \widetilde{I} + \widetilde{E}_1^{*} \otimes \widetilde{D}_1$, $\widetilde{M_2} = \widetilde{B}_{21}^{*} \otimes \widetilde{I} + \widetilde{E}_2^{*} \otimes \widetilde{D}_2$, $\widetilde{D}_1 = -\widetilde{I}\widetilde{A}_{11}^{-}\widetilde{A}_{11}$, $\widetilde{D}_2 = -\widetilde{I}\widetilde{A}_{12}^{-}\widetilde{A}_{12}$, $\widetilde{E}_1 = \widetilde{I}\widetilde{I}^{-}\widetilde{B}_{22}$, $\widetilde{E}_2 = \widetilde{I}\widetilde{I}^{-}\widetilde{B}_{21}$, $\widetilde{N}_1 = \widetilde{C}_{21} - \widetilde{I}\widetilde{A}_{11}^{-}\widetilde{C}_{12}\widetilde{I}^{-}\widetilde{B}_{22}$ and $\widetilde{N}_2 = \widetilde{C}_{22} - \widetilde{I}\widetilde{A}_{12}^{-}\widetilde{C}_{11}\widetilde{I}^{-}\widetilde{B}_{21}$.

A representation of the anti-reflexive solution to these matrix equations is

$$X = U\begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix}U^{*} \in \mathbf{C}_a^{n\times n}(P), \tag{2.51}$$

where

$$\begin{aligned} X_2 = {} & \widetilde{A}_{11}^{-}\widetilde{C}_{12}\widetilde{I}^{-} + \mathrm{Invec}_{r,n-r}\{\widetilde{M_1}^{-}\mathrm{vec}\,\widetilde{N}_1 + (I - \widetilde{M_1}^{-}\widetilde{M_1})\mathrm{vec}\,V_1\} \\ & - \widetilde{A}_{11}^{-}\widetilde{A}_{11}\{\mathrm{Invec}_{r,n-r}(\widetilde{M_1}^{-}\mathrm{vec}\,N_1 + (I - \widetilde{M_1}^{-}\widetilde{M_1})\mathrm{vec}\,V_1)\}\widetilde{I}\widetilde{I}^{-}, \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} X_3 = {} & \widetilde{A}_{12}^{-}\widetilde{C}_{11}\widetilde{I}^{-} + \mathrm{Invec}_{n-r,r}\{\widetilde{M_2}^{-}\mathrm{vec}\,\widetilde{N}_2 + (I - \widetilde{M_2}^{-}\widetilde{M_2})\mathrm{vec}\,V_2\} \\ & - \widetilde{A}_{12}^{-}\widetilde{A}_{12}\{\mathrm{Invec}_{n-r,r}(\widetilde{M_2}^{-}\mathrm{vec}\,N_2 + (I - \widetilde{M_2}^{-}\widetilde{M_2})\mathrm{vec}\,V_2)\}\widetilde{I}\widetilde{I}^{-}, \end{aligned} \tag{2.53}$$

and $V_1 \in \mathbf{C}^{r\times n-r}$, $V_2 \in \mathbf{C}^{n-r\times r}$ are arbitrary matrices.

• **Reflexive solution to matrix equations (1.3).** Now we consider the linear matrix equations (1.3). We give necessary and sufficient conditions for the solvability of these matrix equations and the expressions for the reflexive and anti-reflexive solutions with respect to a generalized reflection matrix P solutions of (1.3).

Theorem 2.7. *Assume that $A_1 \in \mathbf{C}^{m\times n}$, $C_1 \in \mathbf{C}^{m\times n}$, $B_2 \in \mathbf{C}^{n\times s}$, $C_2 \in \mathbf{C}^{n\times s}$, $A_3 \in \mathbf{C}^{k\times n}$, $C_3 \in \mathbf{C}^{k\times n}$, $B_4 \in \mathbf{C}^{n\times t}$, $C_4 \in \mathbf{C}^{n\times t}$, and $K_1 = A_{31}L_{A_{11}}$, $N_1 = R_{B_{21}}B_{41}$, $K_2 = A_{32}L_{A_{12}}$, $N_2 = R_{B_{22}}B_{42}$, $E_1 = C_{31} - A_{31}A_{11}^{+}C_{11} - K_1C_{21}B_{21}^{+}$, $E = C_{41} - A_{11}^{+}C_{11}B_{41} - L_{A_{11}}C_{21}B_{21}^{+}B_{41} - L_{A_{11}}K_1^{+}E_1N_1$, and $F_1 = C_{32} - A_{32}A_{12}^{+}C_{12} - K_2C_{22}B_{22}^{+}$, $F = C_{42} -$*

$A_{12}^{+}C_{12}B_{42} - L_{A_{12}}C_{22}B_{22}^{+}B_{42} - L_{A_{12}}K_2^{+}F_1N_2$. Then the matrix equations $A_1X = C_1$, $XB_2 = C_2$, $A_3X = C_3$, $XB_4 = C_4$ have a solution $X \in \mathbf{C}_r^{n\times n}(P)$, if and only if

$$K_1K_1^{+}E_1R_{B_{21}} = E_1,\ EL_{N_1} = 0,\ R_{L_{A_{11}}L_{K_1}}E = 0,\ A_{11}C_{21} = C_{11}B_{21}, \tag{2.54}$$

$$K_2K_2^{+}F_1R_{B_{22}} = F_1,\ FL_{N_2} = 0,\ R_{L_{A_{12}}L_{K_2}}F = 0,\ A_{12}C_{22} = C_{12}B_{22}, \tag{2.55}$$

$$\begin{aligned} A_{11}A_{11}^{+}C_{11} = C_{11},\ A_{31}A_{31}^{+}C_{31} = C_{31},\\ C_{21}B_{21}^{+}B_{21} = C_{21},\ C_{41}B_{41}^{+}B_{41} = C_{41}, \end{aligned} \tag{2.56}$$

$$\begin{aligned} A_{12}A_{12}^{+}C_{12} = C_{12},\ A_{32}A_{32}^{+}C_{32} = C_{32},\\ C_{22}B_{22}^{+}B_{22} = C_{22},\ C_{42}B_{42}^{+}B_{42} = C_{42}. \end{aligned} \tag{2.57}$$

In that case, the reflexive solution of the matrix equations $A_1X = C_1$, $XB_2 = C_2$, $A_3X = C_3$, $XB_4 = C_4$ can be expressed as follows:

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* \in \mathbf{C}_r^{n\times n}(P), \tag{2.58}$$

where

$$\begin{aligned} X_1 = A_{11}^{+}C_{11} + L_{A_{11}}C_{21}B_{21}^{+} + L_{A_{11}}K_1^{+}E_1R_{B_{21}}\\ + EN_1^{+}R_{B_{21}} + L_{A_{11}}L_{K_1}Z_1R_{N_1}R_{B_{21}}, \end{aligned} \tag{2.59}$$

and

$$\begin{aligned} X_4 = A_{12}^{+}C_{12} + L_{A_{12}}C_{22}B_{22}^{+} + L_{A_{12}}K_2^{+}F_1R_{B_{22}}\\ + FN_2^{+}R_{B_{22}} + L_{A_{12}}L_{K_2}Z_2R_{N_2}R_{B_{22}}, \end{aligned} \tag{2.60}$$

where Z_1 and Z_2 are arbitrary matrices with compatible dimension.

Proof. We show that matrix X in (2.58)–(2.60) is a solution of (1.3) under the assumptions (2.54)–(2.57). Suppose that (2.54)–(2.57) hold. From [**38**], there exist $X_1 \in \mathbf{C}^{r\times r}$, $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$ of the forms (2.59) and (2.60) such that

$$\begin{cases} A_{11}X_1 = C_{11},\ X_1B_{21} = C_{21},\ A_{31}X_1 = C_{31},\ X_1B_{41} = C_{41},\\ A_{12}X_4 = C_{12},\ X_4B_{22} = C_{22},\ A_{32}X_4 = C_{32},\ X_4B_{42} = C_{42}, \end{cases}$$

which are equivalent to

$$\begin{aligned}
\begin{pmatrix} A_{11} & A_{12} \end{pmatrix}\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} &= \begin{pmatrix} C_{11} & C_{12} \end{pmatrix}, \\
\begin{pmatrix} A_{31} & A_{32} \end{pmatrix}\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} &= \begin{pmatrix} C_{31} & C_{32} \end{pmatrix}, \\
\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}\begin{pmatrix} B_{21} \\ B_{22} \end{pmatrix} &= \begin{pmatrix} C_{21} \\ C_{22} \end{pmatrix}, \\
\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}\begin{pmatrix} B_{41} \\ B_{42} \end{pmatrix} &= \begin{pmatrix} C_{41} \\ C_{42} \end{pmatrix}
\end{aligned}$$

These, in turn, are equivalent to

$$\begin{aligned}
A_1 U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* &= C_1, \\
A_3 U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* &= C_3, \\
U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* B_2 &= C_2, \\
U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* B_4 &= C_4,
\end{aligned}$$

This implies that

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^* \in \mathbf{C}_r^{n\times n}(P)$$

($X_1 \in \mathbf{C}^{r\times r}$, and $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$) is the solution of (1.3).

Now we prove that any solution of (1.3) can be expressed in the form of (2.58)–(2.60); then (2.54)–(2.57) hold. Assume (1.3) has a solution $X \in \mathbf{C}_r^{n\times n}(P)$, where X can be expressed as

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} U^*, \tag{2.61}$$

where $X_1 \in \mathbf{C}^{r\times r}$, and $X_4 \in \mathbf{C}^{(n-r)\times(n-r)}$. By using (2.4)–(2.12), we can obtain

(2.62)
$$\begin{cases} A_1X = C_1, \\ XB_2 = C_2, \\ A_3X = C_3, \\ XB_4 = C_4, cr \end{cases}$$
$$\Longleftrightarrow \begin{cases} (A_{11}\ A_{12})U^*U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* = (C_{11}\ C_{12})U^*, \\ U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^*U\begin{pmatrix} B_{21} \\ B_{22} \end{pmatrix} = U\begin{pmatrix} C_{21} \\ C_{22} \end{pmatrix}, \\ (A_{31}\ A_{32})U^*U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^* = (C_{31}\ C_{32})U^*, \\ U\begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix}U^*U\begin{pmatrix} B_{41} \\ B_{42} \end{pmatrix} = U\begin{pmatrix} C_{41} \\ C_{42} \end{pmatrix}, \end{cases}$$
$$\Longleftrightarrow \begin{cases} A_{11}X_1 = C_{11},\ X_1B_{21} = C_{21},\ A_{31}X_1 = C_{31}, \\ \qquad X_1B_{41} = C_{41}, \\ A_{12}X_4 = C_{12},\ X_4B_{22} = C_{22},\ A_{32}X_4 = C_{32}, \\ \qquad X_4B_{42} = C_{42}. \end{cases}$$

By using (2.62) and the results in [**37**], we can see that (2.54)–(2.57) hold and

$$X_1 = A_{11}^+C_{11} + L_{A_{11}}C_{21}B_{21}^+ + L_{A_{11}}K_1^+E_1R_{B_{21}} + EN_1^+R_{B_{21}} + L_{A_{11}}L_{K_1}Z_1R_{N_1}R_{B_{21}}, \tag{2.63}$$

and

$$X_4 = A_{12}^+C_{12} + L_{A_{12}}C_{22}B_{22}^+ + L_{A_{12}}K_2^+F_1R_{B_{22}} + FN_2^+R_{B_{22}} + L_{A_{12}}L_{K_2}Z_2R_{N_2}R_{B_{22}}. \tag{2.64}$$

By substituting (2.63) and (2.64) in (2.58), we obtain that the matrix equations (1.3) have the solution by form (2.58)–(2.60). The proof is complete. □

• **Anti-reflexive solution to matrix equations (1.3).** Similarly to the proof of Theorem 2.7, we can prove the following theorem.

Theorem 2.8. *Assume that* $A_1 \in \mathbf{C}^{m\times n}$, $C_1 \in \mathbf{C}^{m\times n}$, $B_2 \in \mathbf{C}^{n\times s}$, $C_2 \in \mathbf{C}^{n\times s}$, $A_3 \in \mathbf{C}^{k\times n}$, $C_3 \in \mathbf{C}^{k\times n}$, $B_4 \in \mathbf{C}^{n\times t}$, $C_4 \in \mathbf{C}^{n\times t}$, *and* $K_1 = A_{31}L_{A_{11}}$, $N_1 = R_{B_{22}}B_{42}$, $K_2 = A_{32}L_{A_{12}}$, $N_2 = R_{B_{21}}B_{41}$, $E_1 = C_{32}-A_{31}A_{11}^{+}C_{12}-K_1C_{21}B_{22}^{+}$, $E = C_{41}-A_{11}^{+}C_{12}B_{42}-L_{A_{11}}C_{21}B_{22}^{+}B_{42}-L_{A_{11}}K_1^{+}E_1N_1$, *and* $F_1 = C_{31} - A_{32}A_{12}^{+}C_{11} - K_2C_{22}B_{21}^{+}$, $F = C_{42} - A_{12}^{+}C_{11}B_{41} - L_{A_{12}}C_{22}B_{21}^{+}B_{41} - L_{A_{12}}K_2^{+}F_1N_2$. *Then the system of matrix equations* $A_1X = C_1$, $XB_2 = C_2$, $A_3X = C_3$, $XB_4 = C_4$ *has a solution* $X \in \mathbf{C}_a^{n\times n}(P)$, *if and only if*

$$K_1K_1^{+}E_1R_{B_{22}} = E_1,\ EL_{N_1} = 0,\ R_{L_{A_{11}}L_{K_1}}E = 0,\quad A_{11}C_{21} = C_{12}B_{22}, \tag{2.65}$$

$$K_2K_2^{+}F_1R_{B_{21}} = F_1,\ FL_{N_2} = 0,\ R_{L_{A_{12}}L_{K_2}}F = 0,\quad A_{12}C_{22} = C_{11}B_{21}, \tag{2.66}$$

$$A_{11}A_{11}^{+}C_{12} = C_{12},\ A_{31}A_{31}^{+}C_{32} = C_{32},\ C_{21}B_{22}^{+}B_{22} = C_{21},\quad C_{41}B_{42}^{+}B_{42} = C_{41}, \tag{2.67}$$

$$A_{12}A_{12}^{+}C_{11} = C_{11},\ A_{32}A_{32}^{+}C_{31} = C_{31},\ C_{22}B_{21}^{+}B_{21} = C_{22},\quad C_{42}B_{41}^{+}B_{41} = C_{42}. \tag{2.68}$$

In that case, the anti-reflexive solution of the matrix equations $A_1X = C_1$, $XB_2 = C_2$, $A_3X = C_3$, $XB_4 = C_4$ *can be expressed as follows*:

$$X = U\begin{pmatrix} 0 & X_2 \\ X_3 & 0 \end{pmatrix}U^* \in \mathbf{C}_a^{n\times n}(P) \tag{2.69}$$

where

$$X_2 = A_{11}^{+}C_{12} + L_{A_{11}}C_{21}B_{22}^{+} + L_{A_{11}}K_1^{+}E_1R_{B_{22}} + EN_1^{+}R_{B_{22}} + L_{A_{11}}L_{K_1}Z_1R_{N_1}R_{B_{22}}, \tag{2.70}$$

and

$$X_3 = A_{12}^{+}C_{11} + L_{A_{12}}C_{22}B_{21}^{+} + L_{A_{12}}K_2^{+}F_1R_{B_{21}} + FN_2^{+}R_{B_{21}} + L_{A_{12}}L_{K_2}Z_2R_{N_2}R_{B_{21}}, \tag{2.71}$$

where Z_1 *and* Z_2 *are arbitrary matrices with compatible dimensions.*

Notice that we can use Theorems 2.7 and 2.8 for solving the matrix equations $A_1X = C_1$, $XB_2 = C_2$.

3. Conclusion. It is known that matrix equations have nice applications in various branches of control and system theory; also reflexive and anti-reflexive matrices have wide applications in many fields. In this paper, we have considered the reflexive and anti-reflexive (with respect to a generalized reflection P) solutions of the matrix equation $A_1XB_1 = D_1$, the system of matrix equations $A_1X = C_1, XB_2 = C_2$ and the system of matrix equations $A_1X = C_1$, $XB_2 = C_2$, $A_3X = C_3$ and $XB_4 = C_4$. We have derived necessary and sufficient conditions for the existence and the expression of reflexive and anti-reflexive solutions to these matrix equations. The solvability conditions and explicit formulae for the solutions were given.

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Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Avenue, Tehran 15914, Iran
Email address: mdehghan@aut.ac.ir, mdehghan.aut@gmail.com

Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Avenue, Tehran 15914, Iran
Email address: mhajarian@aut.ac.ir, masoudhajarian@gmail.com