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GRADE STABLE LIE ALGEBRAS I

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ABSTRACT. We define a grade stable Lie algebra. The Lie automorphism group $\mathrm{Aut}_{\mathrm{Lie}}(W^+(2))$ of the Witt type Lie algebra $W^+(2)$ is unknown [**4, 8, 11**]. It is not easy to find an auto-invariant set of $W^+(2)$ [**1, 2**]. The automorphism group $\mathrm{Aut}_{\mathrm{Lie}}(S^+(2))$ of the special type Lie subalgebra $S^+(2)$ of the Lie algebra $W^+(2) = W(0,0,2)$ is found in the paper [**10**]. Since the Witt type Lie algebra $W(1,0,2)$ containing $W^+(2)$ is grade stable, we can find the Lie automorphism group $\mathrm{Aut}_{\mathrm{Lie}}(W(1,0,2))$ of the Lie algebra $W(1,0,2)$.

1. Introduction. The Lie algebra automorphism groups of some self-centralizing Lie algebras are found in the papers [**6, 7, 8**]. Since the Witt type algebra $W(1,0,2)$ is not self-centralizing, it is an interesting problem to find its Lie algebra automorphism group $\mathrm{Aut}_{\mathrm{Lie}}(W(1,0,2))$ [**4, 11**]. The Lie algebra $W(1,0,2)$ is $\mathbf{Z}$-graded, and we define a graded Lie algebra to be grade stable if all Lie automorphisms preserve the grade. We first show that the degree zero component is preserved by any Lie automorphism (Lemma 2); this is an extension of the results of [**1, 2**]. Then we show that any Lie automorphism θ of $W(1,0,2)$ satisfies

$$\theta(\partial_1) = \partial_1 \text{ and } \theta(\partial_2) = c_1\partial_2, \text{ or } \theta(\partial_1) = -\partial_1 \text{ and } \theta(\partial_1) = c_2\partial_2$$

for nonzero scalars c_1, c_2 (Lemma 3). Finally we determine the Lie automorphism group of $W(1,0,2)$ (Theorem 1); it follows that $W(1,0,2)$ is grade stable.

2. Preliminaries. Let $\mathbf{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout the paper, $\mathbf{N}$ and $\mathbf{Z}$ will denote the nonnegative integers and the integers, respectively. Let $\mathbf{F}^\bullet$

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be the multiplicative group of nonzero elements of $\mathbf{F}$. The Witt type algebra $W(n,m,s)$ has the standard basis

$$(1)\quad B_{W(n,m,s)} = \{e^{a_1x_1}\cdots e^{a_nx_n}x_1^{i_1}\cdots x_m^{i_m}x_{m+1}^{i_{m+1}}\cdots x_{m+s}^{i_{m+s}}\partial_u \mid a_1,\dots,a_n,i_1,\dots,i_m \in \mathbf{Z}, i_{m+1},\dots i_{m+s}\in \mathbf{N},\ 1\le u\le m+s,\ n\le m\}$$

and the usual multiplication, i.e., Lie bracket, [**4, 8, 10, 11**]. Let L be a Lie algebra over $\mathbf{F}$ with a basis $S=\{s_u \mid u\in I\}$. The Lie algebra L is degreeing if for any $s\in S$ we define the Lie degree $\deg_{\text{Lie}}(s)\in\mathbf{Z}$ of s. Thus, for any l of L, we may define $\deg_{\text{Lie}}(l)$ as the highest Lie degree of the nonzero basis terms of l. An element l of L is degree stable if for any $l_1\in L$, $\deg_{\text{Lie}}([Cl,l_1])\le\deg_{\text{Lie}}(l_1)$ holds. For a degreeing Lie algebra L, the degree stabilizer $\text{St}_{\text{Lie}}(L)$ of the Lie algebra L is the vector subspace of L spanned by all elements which are degree stable. For any $\theta\in\text{Aut}_{\text{Lie}}(L)$, we have the following diagram:

$$(2)\quad \begin{array}{ccccc} L & \longrightarrow & \theta(L) & = & L \\ \uparrow & & \uparrow & & \\ \text{St}_{\text{Lie}}(L) & \longrightarrow & \theta(\text{St}_{\text{Lie}}(L)) & & \end{array}$$

where $\text{Aut}_{\text{Lie}}(L)$ is the automorphism group of the Lie algebra L and $\text{St}_{\text{Lie}}(L)\rightharpoonup L$, respectively, $\theta(\text{St}_{\text{Lie}}(L))\rightharpoonup\theta(L)$ is an embedding as vector spaces. It is an interesting to note that the equality

$$(3)\quad \text{St}_{\text{Lie}}(L)=\theta(\text{St}_{\text{Lie}}(L)),$$

sometimes holds and sometimes does not hold for any $\theta\in\text{Aut}_{\text{Lie}}(L)$. A Lie algebra L is degree-stabilizing if $\text{St}_{\text{Lie}}(L)$ is auto-invariant, i.e., the equality (3) holds [**1, 2, 10**]. For each basis element $e^{a_1x_1}\cdots e^{a_nx_n}x_1^{i_1}\cdots x_{m+s}^{i_{m+s}}\partial_u$ of $W(n,m,s)$, we define the Lie degree $\deg_{\text{Lie}}(e^{a_1x_1}\cdots e^{a_nx_n}x_1^{i_1}\cdots x_{m+s}^{i_{m+s}}\partial_u)=|i_1|+\cdots+|i_m|+i_{m+1}+i_{m+s}$ where $|i_u|$ is the absolute value of i_u, $1\le u\le m$ [**11**]. For any element l of $W(n,m,s)$, we can define the Lie degree $\deg_{\text{Lie}}(l)$ as the highest degree of nonzero basis terms of l. The Lie algebra $W(n,m,s)$ is $\mathbf{Z}^n$-graded as follows:

$$(4)\quad W(n,m,s)=\bigoplus_{(a_1,\dots,a_n)\in\mathbf{Z}^n} W_{(a_1,\dots,a_n)},$$

where the $(a_1, \dots, a_n)$-homogeneous component $W_{(a_1,\dots,a_n)}$ is the vector space spanned by

$$\{e^{a_1x_1}\cdots e^{a_nx_n}x_1^{i_1}\cdots x_{n+m}^{i_{n+m}}\partial_u | i_1, \dots, i_n \in \mathbf{Z}, \\ i_{n+1}, \dots, i_{n+m} \in \mathbf{N},\ 1 \le u \le n+m\}.$$

An A-graded Lie algebra $L = \oplus_{a\in A} L_a$ is graded stable, if for any Lie algebra automorphism θ of L, $\theta(L_a) \subset L_a$ where A is an additive group [**1, 2**]. For any l in a Lie algebra L, l_1 is ad-diagonal with respect to l, if $[l, l_1] = \text{cl}$ holds where $c \in \mathbf{F}$. An element l of a Lie algebra L is ad-diagonal with respect to a subset B of L, if $[l, l_1] = c_1 l_1$ holds for any $l_1 \in B$ where c_1 is a scalar which depends on l and l_1. For a given basis B of a Lie algebra L, the toral $\text{tor}_L(B)$ of B is n, if there are n ad-diagonal elements $\{l_1, \dots l_n\}$ with respect to B such that the set $\{l_1, \dots, l_n\}$ is the linearly independent maximal set. For a Lie algebra L, $\text{Tor}\,(L)$ is defined as follows:

$$\text{Tor}\,(L) = \max\{\text{tor}_L(B) \mid B \text{ is a basis of } L\}.$$

A Lie algebra L is n-toral, if $\text{Tor}\,(L)$ is equal to n. The Lie algebras $W(0,1,0)$ and $W(0,0,1)$ are 1-toral and self-centralizing [**6**]. If L is self-centralizing, then $\text{Tor}\,(L) \le 1$. Note that $W(0,n,0) = W(n)$ and $W(0,0,n) = W^+(n)$. The Lie subalgebra $S(0,0,2)$ of $W(0,0,2)$ spanned by $\{[x^a\partial_2, y^i\partial_1] | a, i \in \mathbf{N}\}$ is simple such that $\text{Tor}\,(S(0,0,2))$ is one [**9**].

3. Automorphism group Aut$\,(W(1,0,2))$ of $W(1,0,2)$. Note that the Lie algebra $W(1,2,0)$ is spanned by the standard basis $\{e^{ax}x^iy^j\partial_u \mid a, i, j \in \mathbf{Z}, 1 \le u \le 2\}$. The Lie subalgebra $W(1,0,2)$ of $W(1,2,0)$ is spanned by $\{e^{a_1x}x^iy^j\partial_u \mid a_1 \in \mathbf{Z}, i, j \in \mathbf{N}, 1 \le u \le 2\}$. The simple Lie algebra $W(1,2,0)$ has simple Lie subalgebras $W(1,1,1)$, $W(1,0,2)$, $W(1,0,1)$, $W(0,2,0)$, $W(0,1,1)$, $W(0,0,2)$ and $W(0,0,1)$ [**3, 5, 8**].

Proposition 1. *There is a Lie monomorphism θ from the Lie algebra $W(1,0,2)$ to itself such that θ is not surjective.*

Proof. For any element $e^{ax}x^iy^p\partial_u$ of $W(1,0,2)$, $1 \le u \le 2$, and $k \in \mathbf{Z}^\bullet$, we define $\mathbf{F}$-map θ from $W(1,0,2)$ to itself as follows:

$$\theta(e^{ax}x^iy^p\partial_u) = k^{-1+a+i+p}e^{akx}x^iy^p\partial_u.$$

Then θ can be linearly extended to a Lie endomorphism of $W(1,0,2)$. The Lie algebra $W(1,0,2)$ is simple. It's easy to prove that if k is 1 or -1, then θ can be an automorphism of $W(0,0,1)$ and if k is not equal to 1 or -1, then θ is a monomorphism of $W(0,0,1)$ such that θ is not surjective. This completes the proof of the proposition. □

Lemma 1. *The stabilizer* $\mathrm{St}_{\mathrm{Lie}}(W(1,0,2))$ *of the Lie algebra* $W(1,0,2)$ *is spanned by* $x\partial_2$, $y\partial_2$, ∂_1 *and* ∂_2 *[**10**].*

Proof. The proof of the lemma is straightforward by the order of the algebra $W(1,0,2)$, so it is omitted. □

Lemma 2. *For any nonzero Lie automorphism* θ *of* $W(1,0,2)$, $\theta(x\partial_1) = x\partial_1 + c\partial_1$ *and* $\theta(y\partial_2) = y\partial_2 + c'\partial_2$ *hold, and* $\theta(W_0)$ *is a subset of* W_0 *where* W_0 *is the* 0*-homogeneous component of* $W(1,0,2)$ *for* $c, c' \in \mathbf{F}$.

Proof. Let θ be any automorphism θ of $W(1,0,2)$. For any element $\sum_{a,i,j} c_{a,i,j,1}e^{ax}x^i y^j\partial_1 + \sum_{b,h,k} c_{b,h,k,2}e^{bx}x^h y^k\partial_2$ of $W(1,0,2)$, we have that

$$(5)\quad \theta\Big(\Big[y\partial_2, \sum_{a,i,j} c_{a,i,j,1}e^{ax}x^i y^j\partial_1 + \sum_{b,h,k} c_{b,h,k,2}e^{bx}x^h y^k\partial_2\Big]\Big)$$
$$= \theta\Big(\sum_{a,i,j} ic_{a,i,j,1}e^{ax}x^i y^j\partial_1 + \sum_{b,h,k} (k-1)c_{b,h,k,2}e^{bx}x^h y^k\partial_2\Big).$$

This implies that

$$\deg\Big(\theta\Big(\sum_{a,i,j} ic_{a,i,j,1}e^{ax}x^i y^j\partial_1 + \sum_{b,h,k} (k-1)c_{b,h,k,2}e^{bx}x^h y^k\partial_2\Big)\Big)$$
$$\leq \deg\Big(\theta\Big(\sum_{a,i,j} c_{a,i,j,1}e^{ax}x^i y^j\partial_1 + \sum_{b,h,k} c_{b,h,k,2}e^{bx}x^h y^k\partial_2\Big)\Big).$$

Thus, $\theta(y\partial_2)$ is in $\mathrm{St}_{\mathrm{Lie}}(W(1,0,2))$. By Lemma 1, we have that $\theta(y\partial_2) = c_1x\partial_2 + c_2y\partial_2 + c_3\partial_1 + c_4\partial_2$ where $c_1, \dots, c_4 \in \mathbf{F}$. First let us assume that c_1 is not equal to zero. We have two cases: $c_2 \neq 0$ and $c_2 = 0$.

Let us assume that c_2 is not equal to zero. Since $\theta(y^p\partial_2)$, $p \neq 1$, centralizes $\theta(y\partial_2)$ and c_1 is not equal to zero, we have that

$$\theta(y^p\partial_2) = c_{p,b_p,s_p,0,1}e^{b_px}x^{s_p}\partial_1 + c_{p,b_p,s_p,0,2}e^{b_px}x^{s_p}\partial_2 + \#_1 \tag{6}$$

where either $c_{p,b_p,s_p,0,1}e^{b_px}x^{s_p}\partial_1$ or $c_{p,b_p,s_p,0,2}e^{b_px}x^{s_p}\partial_2$ is the maximal term of the element $\theta(y^p\partial_2)$ and $\#_1$ is the sum of the remaining terms of the element $\theta(y^p\partial_2)$ with appropriate coefficients. Note that the Lie algebra $W(0,0,1)$ spanned by $S = \{\theta(y^p\partial_2) \mid p \in \mathbf{N}\}$ is simple and self-centralizing. Since $\theta(y\partial_2)$ is an ad-diagonal element with respect to S and c_1 is not zero, we have that $\theta(y^p\partial_2)$, $p \neq 1$, does not have a term with y^r, $r \geq 1$. Since $x^q\partial_1$, $q \in \mathbf{N}$, centralizes $y\partial_2$, $\theta(x^q\partial_1)$ can be written as follows:

$$\theta(x^q\partial_1) = c_{q,d_q,t_q,0,1}e^{d_qx}x^{t_q}\partial_1 + c_{q,d_q,t_q,0,2}e^{d_qx}x^{t_q}\partial_2 + \#_2 \tag{7}$$

as $\theta(y^p\partial_2)$. Since c_1 is not zero, if $\theta(y^p\partial_2)$, $p \in \mathbf{N}$, is in the 0-homogeneous component W_0, then the proof of lemma is obvious, i.e., we can derive a contradiction easily. Thus, without loss of generality, we can assume that b_p is not zero in (6). For any p_1 and q_1, $\theta(x^{q_1}\partial_1)$ and $\theta(y^{p_1}\partial_2)$ have terms in the same nonzero homogeneous components. Otherwise, they cannot centralize each other. By (6) and (7), there are also two positive integers p_1 and q_1 such that $\theta(x^{q_1}\partial_1)$ and $\theta(y^{p_1}\partial_2)$ have terms which are not in W_0. There is a $c \in \mathbf{F}$ so that $[\theta(x^{q_1}\partial_1), \theta(x^{q_1}\partial_1) - c\theta(y^{p_1}\partial_2)]$ is not zero. This contradiction shows that c_2 is equal to zero. We can put $\theta(y\partial_2) = c_1x\partial_2 + c_3\partial_1 + c_4\partial_2$. This implies that $\theta(y\partial_2)$ cannot be an ad-diagonal element with respect to $\theta(y^2\partial_2)$. This contradiction shows that c_1 is zero. Second, let us assume that c_1 is equal to zero. We have that $\theta(y\partial_2) = c_2y\partial_2 + c_3\partial_1 + c_4\partial_2$. Since $x^k\partial_1$, $k \in \mathbf{N}$, centralizes $y\partial_2$ and $W(1,0,2)$ is $\mathbf{Z}$-graded, we have that the maximal term of $\theta(x^k\partial_1)$ is either $e^{ax}x^hy^p\partial_1$ or $e^{ax}x^hy^p\partial_2$ with an appropriate coefficient. This implies that $\theta(x\partial_1)$ can be written as follows:

$$\theta(x\partial_1) = \#_3 + c_{1,0,0,0,1}\partial_1,$$

where $\#_3$ is the sum of the nonzero terms of $\theta(x\partial_1)$ such that the terms are not in W_0. Since the Lie algebra spanned by $\{\theta(x^k\partial_1) \mid k \in \mathbf{N}\}$ is isomorphic to $W(0,0,1)$, we have that $\theta(x\partial_1) = c_{1,0,0,0,1}\partial_1$ for $c_{1,0,0,0,1} \in \mathbf{F}^\bullet$. Since $x\partial_1$ is an ad-diagonal element with respect to ∂_1, we can prove that $\theta(\partial_1) = c_{0,r,0,0,1}e^{rx}\partial_1$ with an appropriate coefficient. By $\theta([\partial_1, x^2\partial_1]) = 2\theta(x\partial_1)$, we can prove that $\theta(x^2\partial_1) = c_{2,-r,0,0,1}e^{-rx}\partial_1$ with an appropriate coefficient. Similarly we can also prove that $\theta(x^3\partial_1) = c_{3,-2r,0,0,1}e^{-2rx}\partial_1$ with an appropriate coefficient. Since θ is surjective, there is an element $l \in W(1,0,1)$ such that $\theta(l) = e^{2rx}\partial_1$. We have that $\theta([l, x^3\partial_1]) = c''\partial_1$ with an appropriate nonzero coefficient c''. This implies that

$$[l, x^3\partial_1] = c'''x\partial_1 \tag{8}$$

with an appropriate nonzero coefficient. There is no element l of $W(1,0,2)$ such that (8) holds. This contradiction shows that c_3 is zero. Since $y\partial_2$ is an ad-diagonal element with respect to ∂_2, we have that $\theta(\partial_2) = c_5\partial_2$ for $c_5 \in \mathbf{F}^\bullet$. Since ∂_2 and $y\partial_2$ centralizes ∂_1, we are able to prove that

$$\theta(\partial_1) = c_{0,\gamma,u,0,1}e^{\gamma x}x^u\partial_1 + \cdots + c_{0,0,v,0,1}x^v\partial_1 + \cdots + c_{0,0,0,0,1}\partial_1$$

with appropriate coefficients. Similarly, for $x^k\partial_1$, $k \neq 1$, we are also able to prove that

$$\theta(x^k\partial_1) = c_{k,\mu,w,0,1}e^{\mu x}x^w\partial_1 + \cdots + c_{k,0,\sigma,0,1}x^\sigma\partial_1 + \cdots + c_{k,0,0,0,1}\partial_1$$

with appropriate coefficients. Since the Lie algebra spanned by $\{\theta(x^k\partial_1) \mid k \in \mathbf{N}\}$ is isomorphic to $W(0,0,1)$, we can prove that $\theta(x\partial_1) = x\partial_1 + c_6\partial_1$ and $\theta(\partial_1) = c_7\partial_1$ where $c_6 \in \mathbf{F}$ and $c_7 \in \mathbf{F}^\bullet$. Thus, by induction on i_1 and j_1 of $x^{i_1}y^{j_1}\partial_t$, $1 \le t \le 2$, we can also prove routinely that $\theta(W_0)$ is a subset of W_0 where W_0 is the 0-homogeneous component of $W(1,0,2)$. Therefore, we have proved the lemma. □

Lemma 3. *For any $\theta \in \mathrm{Aut}_{\mathrm{Lie}}(W(1,0,2))$, either $\theta(\partial_1) = \partial_1$ and $\theta(\partial_2) = c_1\partial_2$ hold, or $\theta(\partial_1) = -\partial_1$ and $\theta(\partial_2) = c_2\partial_2$ hold where $c_1, c_2 \in \mathbf{F}^\bullet$.*

Proof. Let θ be an automorphism of the Lie algebra $W(1,0,2)$. By Lemma 2, we have that $\theta(x\partial_1) = x\partial_1 + c_1\partial_1$ and $\theta(y\partial_2) = y\partial_2 + c_2\partial_2$ hold for $c_1, c_2 \in \mathbf{F}$. By Lemma 2, we also have that $\theta(\partial_1) = c_3\partial_1$ and $\theta(\partial_2) = c_4\partial_2$ for $c_3, c_4 \in \mathbf{F}^\bullet$. Since ∂_1 is an ad-diagonal element with respect to $e^x\partial_1$ and $y\partial_2$ centralizes $e^x\partial_1$, we can prove that $\theta(e^x\partial_1) = c_5e^{rx}\partial_1$ such that $c_5r = 1$. Since θ is surjective, r is either 1 or -1. Otherwise, by Proposition 1, θ can be a monomorphism which is not surjective. Therefore, we have proven the lemma. $\square$

Notes. For any basis element $e^{rx}x^py^i\partial_u$, $1 \leq u \leq 2$, of the Lie algebra $W(1,0,2)$ and $c_1, \ldots, c_4 \in \mathbf{F}^\bullet$, let us define $\mathbf{F}$-maps $\theta^+_{a_1,d_1,c_1,c_2}$ and $\theta^-_{a_2,d_2,c_3,c_4}$ from $W(1,0,2)$ to itself respectively as follows:

(9)
$$\theta^+_{a_1,d_1,c_1,c_2}(e^{rx}x^iy^p\partial_u) = a_1^r d_1^{1-p+\delta_{1,u}} e^{rx}(x+c_1)^i(y+c_2)^p\partial_u$$
(10)
$$\theta^-_{a_2,d_2,c_3,c_4}(e^{rx}x^iy^p\partial_u) = (-1)^{1-i}{a_2}^r d_2^{1-p+\delta_{1,u}} e^{-rx}(x+c_3)^i(y+c_4)^p\partial_u.$$

Then $\mathbf{F}$-maps $\theta^+_{a_1,d_1,c_1,c_2}$ and $\theta^-_{a_2,d_2,c_3,c_4}$ can be linearly extended to Lie automorphisms of the algebra $W(1,0,2)$ where $\delta_{1,u}$ is the Kronecker delta. $\square$

Lemma 4. *For any $\theta \in \mathrm{Aut}_{\mathrm{Lie}}(W(1,0,2))$, the automorphism θ is either $\theta^+_{a_1,d_1,c_1,c_2}$ or $\theta^-_{a_2,d_2,c_3,c_4}$ with appropriate scalars as shown in the Notes.*

Proof. Let θ be an automorphism of the Lie algebra $W(1,0,2)$. By Lemma 3, either $\theta(\partial_1) = \partial_1$ and $\theta(\partial_2) = d_1\partial_2$ hold or $\theta(\partial_1) = -\partial_1$ and $\theta(\partial_2) = d_2\partial_2$ hold where $d_1, d_2 \in \mathbf{F}^\bullet$. Let us prove the lemma by using following two cases: Case I. $\theta(\partial_1) = \partial_1$ and $\theta(\partial_2) = d_1\partial_2$ hold, and Case II. $\theta(\partial_1) = -\partial_1$ and $\theta(\partial_2) = d_2\partial_2$ hold. $\square$

Case I. Let us assume that $\theta(\partial_1) = \partial_1$ and $\theta(\partial_2) = d_1\partial_2$ hold. This implies that $\theta(x\partial_1) = (x+c_1)\partial_1$ for $c_1 \in \mathbf{F}$. Thus, since ∂_1 centralizes $y^p\partial_2$, $p \in \mathbf{N}$, and $\theta(\partial_2) = d_1\partial_2$, we are able to prove that $\theta(y\partial_2) = (y+c_2)\partial_2$ where $c_2 \in \mathbf{F}$. By induction on p of $y^p\partial_2$, $p \in \mathbf{N}$, we are also able to prove that

$$\theta(y^p\partial_2) = d_1^{1-p}(y+c_2)^p\partial_2. \tag{11}$$

Similarly, we can prove inductively that

$$\theta(x^i\partial_1) = (x+c_1)^i\partial_1 \tag{12}$$

for $i \in \mathbf{N}$. By $[\partial_1, \theta(x\partial_2)] = d_1\partial_2$ and ∂_2 centralizing $x\partial_2$, we have that $\theta(x\partial_2) = d_1 x\partial_2 + C(1)\partial_1 + C(2)\partial_2$ with appropriate scalars. By $\theta([y\partial_2, x\partial_2]) = -\theta(x\partial_2)$, we have that $C(1) = 0$, i.e., $\theta(x\partial_2) = d_1 x\partial_2 + C(2)\partial_2$. By $[\partial_1, \theta(xy\partial_2)] = y\partial_2 + c_2\partial_2$, we have that $\theta(xy\partial_2) = xy\partial_2 + c_2 x\partial_2 + \sum_i C(i,1)y^i\partial_1 + \sum_j C(j,2)y^j\partial_2$ with appropriate scalars. By $[\partial_2, \theta(xy\partial_2)] = \theta(x\partial_2)$, we have that $\theta(xy\partial_2) = xy\partial_2 + c_2 x\partial_2 + (C(2)y/d_1)\partial_2 + C(0,1)\partial_1 + C(0,2)\partial_2$ with appropriate scalars. Since $y\partial_2$ centralizes $xy\partial_2$, we have that $C(0,2) = (c_2 C(2)/d_1)$, i.e., $\theta(xy\partial_2) = (x + (C(2)/d_1))(y + c_2)\partial_2 + C(0,1)\partial_1$. Since $x\partial_1$ is an ad-diagonal element with respect to $xy\partial_2$, we have that $c_1 = (C(2)/d_1)$ and $C(0,1) = 0$, i.e., $\theta(xy\partial_2) = (x + c_1)(y + c_2)\partial_2$. Thus, by induction on i and p of $x^i y^p \partial_2$, we can prove that

$$\theta(x^i y^p \partial_2) = d_1^{1-p}(x + c_1)^i (y + c_2)^p \partial_2. \tag{13}$$

By $[d_2\partial_2, \theta(y\partial_1)] = \partial_1$ and ∂_1 centralizing $y\partial_1$, we have that $\theta(y\partial_1) = d_1^{-1} y\partial_1 + C(3)\partial_1 + C(4)\partial_2$ with appropriate scalars. Similarly to $\theta(x\partial_2)$ we can prove that

$$\theta(x^i y^p \partial_1) = d_1^{-p}(x + c_1)^i (y + c_2)^p \partial_1. \tag{14}$$

Since the Lie subalgebra $W(1,0,0)$ of the Lie algebra $W(1,0,2)$ spanned by $\{e^{ax}\partial_1 \mid a \in \mathbf{Z}\}$ is self-centralizing, the element ∂_1 is an ad-diagonal element with respect to $e^x\partial_1$, and the element ∂_2 centralizes the element $e^x\partial_1$, we can prove that $\theta(e^x\partial_1) = a_1 e^x\partial_1 + a_2 e^x\partial_2$ where $a_1, a_2 \in \mathbf{F}$. By $[\theta(e^{-x}\partial_1), a_1 e^x\partial_1 + a_2 e^x\partial_2] = 2\partial_1$, we can also prove that $\theta(e^{-x}\partial_1) = a_1^{-1} e^{-x}\partial_1$. Thus, by induction on r of the element $e^{rx}\partial_u$, $1 \le u \le 2$, we can prove that $\theta(e^{rx}\partial_u) = a_1^r e^{rx}\partial_u$. This implies that, by (13), we can prove that

$$\theta(e^{rx} x^i y^p \partial_u) = a_1^r d_1^{1-p+\delta_{1,u}} e^{rx}(x + c_1)^i (y + c_2)^p \partial_u, \tag{15}$$

where $\delta_{1,u}$ is the Kronecker delta. This implies that θ is the automorphism $\theta^+_{a_1,d_1,c_1,c_2}$ of the Lie algebra $W(1,0,2)$ in the Notes.

Case II. Let us assume that $\theta(\partial_1) = -\partial_1$ and $\theta(\partial_2) = d_2\partial_2$ hold. Similarly to Case I, we can prove that

$$\theta(e^{rx} x^i y^p \partial_u) = (-1)^{1-i} {a_2}^r d_2^{1-p+\delta_{1,u}} e^{-rx}(x + c_3)^i (y + c_4)^p \partial_u, \tag{16}$$

where $\delta_{1,u}$ is the Kronecker delta. This implies that θ is the automorphism $\theta^{-}_{a_2,d_2,c_3,c_4}$ of the Lie algebra $W(1,0,2)$ in the Notes.

Thus, we have proven the lemma by the Notes, Case I and Case II. □

Theorem 1. *The automorphism group* $\mathrm{Aut}_{\mathrm{Lie}}(W(1,0,2))$ *of the Lie algebra* $W(1,0,2)$, *is generated by* $\theta^{+}_{a_1,d_1,c_1,c_2}$ *and* $\theta^{-}_{a_2,d_2,c_3,c_4}$ *with appropriate scalars as shown in the Notes.*

Proof. The proof of the theorem is obvious by the Notes and Lemma 4. Let us omit the details of the proof. □

Note that the Lie algebra $W(1,0,1)$ spanned by the standard basis $B = \{e^{ax}x^i\partial \mid a \in \mathbf{Z}, i \in \mathbf{N}\}$ is self-centralizing [**7**].

Corollary 1. *The Lie automorphism group of the Lie algebra* $W(1,0,1)$ *is a subgroup of the Lie automorphism group of the Lie algebra* $W(1,0,2)$ *and* $\mathrm{Aut}_{\mathrm{Lie}}(W(1,0,1))$ *is generated by* $\theta^{+}_{a_1,1,c_1,0}$ *and* $\theta^{-}_{a_2,1,c_3,0}$ *with appropriate scalars as shown in the Notes.*

Proof. The proof of the corollary is straightforward by Theorem 1 and Theorem 4.3 in [**7**]. □

Corollary 2. *The Lie algebra* $W(1,0,2)$ *is graded stable with respect to its standard basis.*

Proof. The proof of the corollary is straightforward by Theorem 1. □

Corollary 3. *The Lie subalgebra* L *of* $W(1,0,2)$ *spanned by* $\{x^py^i\partial_u \mid p+i \leq 1, 1 \leq u \leq 2\}$ *is auto-invariant* [**1, 2**].

Proof. The proof of the corollary is obvious by Theorem 1. Let us omit the details of the proof. □

Proposition 2. *For the Lie algebra* $W(1,0,2)$, $\mathrm{Tor}(W(1,0,2))$ *is one, i.e.,* $W(1,0,2)$ *is one-toral* [**1, 2**].

Proof. Because of the standard basis of $W(1,0,2)$, $\mathrm{Tor}\,(W(1,0,2)) \geq 1$. Let us assume that there is a basis B of $W(1,0,2)$ such that $0 \neq \mathrm{tor}_{W(1,0,2)}(B)$. Let us show that $\mathrm{tor}_{W(1,0,2)}(B) \leq 1$. Let l be an ad-diagonal element with respect to B. Since $W(1,0,2)$ is $\mathbf{Z}$-graded in (4) and simple, l is in the 0-homogeneous component W_0. By taking a sufficiently large integer a and $l_1 \in B$ such that l_1 has a term in the a-homogeneous component W_a, we have that $[l, l_1] \neq cl_1$, for $c \in \mathbf{F}$. This contradiction shows that l has no terms with ∂_1. This implies that l can be written as follows:

$$l = \sum_p c_p y^p \partial_2$$

with appropriate scalars. Since l is ad-diagonal with respect to a basis of the simple Lie subalgebra $W(0,0,1)$ of $W(1,0,2)$ spanned by $\{y^i\partial_2 \mid i \in \mathbf{N}\}$, we can prove that $l = y\partial_2 + c_0\partial_2$. Let l_2 be an ad-diagonal with respect to B such that $l \neq l_2$. Since l centralizes l_2 and $W(0,0,1)$ is self-centralizing, l is a scalar multiple of l_2. This implies that $\mathrm{Tor}\,(W(1,0,2))$ is one. Thus, we have proven the proposition. □

Theorem 2. *Let L_1 be a simple Lie algebra which is also a self-centralizing Lie algebra. If L_2 is not a self-centralizing Lie algebra, then there is no nonzero Lie algebra homomorphism from L_2 to L_1. There is no Lie isomorphism from L_1 to L_2.*

Proof. The proof of the theorem is easy, and so is omitted. □

If a Lie algebra L is self-centralizing, then $\mathrm{Tor}\,(L) \leq 1$. There is a non self-centralizing Lie algebra L such that $\mathrm{Tor}\,(L) = 1$ [**10**].

Corollary 4. *There is no nonzero Lie algebra homomorphism from the Lie algebra $W(1,0,2)$ to the Lie algebra $W(0,1,0)$, respectively $W(0,0,1)$. There is no Lie isomorphism from the Lie algebra $W(1,0,2)$ to the Lie algebra $W(0,1,0)$, respectively $W(0,0,1)$.*

Proof. The proof of the corollary is straightforward by Theorem 2. □

Theorem 3. *Let L_1 be a Lie algebra such that L_1 contains a maximal Lie subalgebra L_{n_1} with* $\mathrm{Tor}\,(L_{n_1}) = n_1$ *and L_2 be a Lie algebra such that L_2 contains a maximal Lie subalgebra L_{n_2} with* $\mathrm{Tor}\,(L_{n_2}) = n_2$. *If $n_1 > n_2$, then there is no nonzero Lie algebra homomorphism from L_{n_1} to L_{n_2}. If $n_1 > n_2$, then there is no Lie isomorphism from L_{n_1} to L_{n_2}.*

Proof. Since $s > t$, if there is a nonzero Lie algebra homomorphism from L_s to L_t, then we derive a contradiction easily. This completes the proof of the theorem. □

Corollary 5. *Let L_s be s-toral, and let L_t be t-toral Lie algebras. If $s > t$, then there is no nonzero Lie algebra homomorphism from L_s to L_t. If $s > t$, then there is no Lie isomorphism from L_s to L_t.*

Proof. The proof of the corollary is straightforward by Theorem 3. □

Note that the converse of Theorems 2 and 3 are generally untrue.

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