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GENERALIZED FOURIER-FEYNMAN TRANSFORMS, CONVOLUTION PRODUCTS, AND FIRST VARIATIONS ON FUNCTION SPACE

SEUNG JUN CHANG, JAE GIL CHOI AND DAVID SKOUG

ABSTRACT. In this paper we examine the various relationships that exist among the first variation, the convolution product and the Fourier-Feynman transform for functionals of the form $F(x) = f(\langle\alpha_1, x\rangle, \dots, \langle\alpha_n, x\rangle)$ with x in a very general function space $C_{a,b}[0,T]$.

1. Introduction.

In [**11**], Kim, Ko, Park and Skoug, working in the setting of one-parameter Wiener space $C_0[0,T]$ established several interesting relationships involving the Fourier-Feynman transform, the convolution product, and the first variation of functionals F of the form

$$F(x) = f(\langle\alpha_1, x\rangle, \dots, \langle\alpha_n, x\rangle) \tag{1.1}$$

where $\langle\alpha, x\rangle$ denotes the Paley-Wiener-Zygmund stochastic integral $\int_0^T \alpha(t)\,dx(t)$.

In this paper, we also study functionals of the form (1.1) but with x in a very general function space $C_{a,b}[0,T]$ rather than in the Wiener space $C_0[0,T]$. The Wiener process used in [**11**] is free of drift and is stationary in time while the stochastic processes used in this paper are nonstationary in time and are subject to a drift $a(t)$. In turns out, as is pointed out in Remark 3.1 below, that including a drift term $a(t)$ makes establishing various relationships among transforms, convolution products, and first variations very difficult.

By choosing $a(t) = 0$ and $b(t) = t$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$, and so all of the results in [**11**] follow immediately from the results in this paper.

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In Section 3 of this paper we establish all seven of the distinct relationships involving exactly two of the three concepts of "transform," "convolution product" and "first variation" where each concept is used only once. In Section 4 we establish all 11 of the relationships involving all three of these concepts where each concept is used exactly once.

2. Definitions and preliminaries. In this section we briefly list the preliminaries from [**4**] and from [**6**] that we need to establish the various formulas in Sections 3 and 4 below; for more details see [**4, 6**].

Let $D = [0,T]$, and let $(\boldsymbol{\Omega}, \mathcal{B}, \mathcal{P})$ be a probability measure space. A real-valued stochastic process Y on $(\boldsymbol{\Omega}, \mathcal{B}, \mathcal{P})$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \dots, Y(t_n,\omega))$ is normally distributed with density function

$$
\begin{aligned}
K(\vec{t},\vec{\eta}) = {} & \bigg((2\pi)^n \prod_{j=1}^{n} (b(t_j) - b(t_{j-1}))\bigg)^{-1/2} \\
& \cdot \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\bigg\}
\end{aligned}
\tag{2.1}
$$

where $\vec{\eta} = (\eta_1, \dots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \dots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$ and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [**14**, page 18–20], Y induces a probability measure μ on the measurable space $(\mathbf{R}^D, \mathcal{B}^D)$ where $\mathbf{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbf{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbf{R}^D$ are measurable. The triple $(\mathbf{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [**14**, page 187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence, $(C_{a,b}[0,T],\mathcal{B}(C_{a,b}[0,T]),\mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable [**10**] provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho>0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N)=0$ for all $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., we write $F\approx G$.

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

(2.2)
$$L^2_{a,b}[0,T]=\left\{v:\int_0^T v^2(s)\,db(s)<\infty \text{ and } \int_0^T v^2(s)\,d|a|(s)<\infty\right\}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u,v\in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b}=\int_0^T u(t)\,v(t)\,d[b(t)+|a|(t)]. \tag{2.3}$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b}=\sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular, note that $\|u\|_{a,b}=0$ if and only if $u(t)=0$ almost everywhere on $[0,T]$. Furthermore $(L^2_{a,b}[0,T],\|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthogonal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j,\phi_k)_{a,b}=\begin{cases}0 & j\neq k\\ 1 & j=k,\end{cases}$$

and for each $v\in L^2_{a,b}[0,T]$, let

$$v_n(t)=\sum_{j=1}^{n}(v,\phi_j)_{a,b}\phi_j(t) \tag{2.4}$$

for $n = 1, 2, \ldots$. Then, for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T v_n(t)\, dx(t) \tag{2.5}$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0,T]$, the PWZ integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$.

We denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$E[F] = \int_{C_{a,b}[0,T]} F(x)\, d\mu(x) \tag{2.6}$$

whenever the integral exists.

We are now ready to state the definition of the generalized analytic Feynman integral.

Definition 2.1. Let $\mathbf{C}$ denote the complex numbers. Let $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0\}$ and $\widetilde{\mathbf{C}}_+ = \{\lambda \in \mathbf{C} : \lambda \neq 0 \text{ and } \operatorname{Re}\lambda \geq 0\}$. Let $F : C_{a,b}[0,T] \to \mathbf{C}$ be such that for each $\lambda > 0$, the function space integral

$$J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x)\, d\mu(x)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbf{C}_+$ we write

$$E^{\mathrm{an}_\lambda}[F] \equiv E_x^{\mathrm{an}_\lambda}[F(x)] = J^*(\lambda). \tag{2.7}$$

Let $q \neq 0$ be a real number, and let F be a functional such that $E^{\mathrm{an}_\lambda}[F]$ exists for all $\lambda \in \mathbf{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$E^{\mathrm{anf}_q}[F] \equiv E_x^{\mathrm{anf}_q}[F(x)] = \lim_{\lambda\to -iq} E^{\mathrm{an}_\lambda}[F] \tag{2.8}$$

where λ approaches $-iq$ through values in $\mathbf{C}_+$.

Next, see [**1, 4, 6**–**9**], we state the definition of the generalized analytic Fourier-Feynman transform (GFFT).

Definition 2.2. For $\lambda \in \mathbf{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$T_\lambda(F)(y) = E_x^{\mathrm{an}_\lambda}[F(y+x)]. \tag{2.9}$$

For $p \in (1,2]$, we define the L_p analytic GFFT, $T_q(p;F)$ of F, by the formula ($\lambda \in \mathbf{C}_+$)

$$T_q(p;F)(y) = \text{l.i.m.}_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.10}$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left| T_\lambda(F)(\rho y) - T_q(p;F)(\rho y) \right|^{p'} d\mu(y) = 0$$

where $1/p + 1/p' = 1$. We define the L_1 analytic GFFT, $T_q(1;F)$ of F, by the formula ($\lambda \in \mathbf{C}_+$)

$$T_q(1;F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.11}$$

if it exists.

We note that for $1 \le p \le 2$, $T_q(p;F)$ is only defined s-a.e. We also note that if $T_q(p;F)$ exists and if $F \approx G$, then $T_q(p;G)$ exists and $T_q(p;G) \approx T_q(p;F)$.

Definition 2.3. Let F and G be measurable functionals on $C_{a,b}[0,T]$. For $\lambda \in \widetilde{\mathbf{C}}_+$, we define their convolution product (if it exists) by

$$(F * G)_\lambda(y) \tag{2.12}$$
$$= \begin{cases} E_x^{\mathrm{an}_\lambda}[F((y+x)/\sqrt{2})G((y-x)/\sqrt{2})] & \lambda \in \mathbf{C}_+ \\ E_x^{\mathrm{anf}_q}[F((y+x)/\sqrt{2})G((y-x)/\sqrt{2})] & \lambda = -iq,\ q \in \mathbf{R},\ q \neq 0. \end{cases}$$

Remark 2.1. (i) When $\lambda = -iq$, we denote $(F * G)_\lambda$ by $(F * G)_q$.

(ii) Our definition of the convolution product is different than the definition given by Yeh in [**13**] and used by Yoo in [**15**]. In [**13**] and [**15**], Yeh and Yoo study relationships between their convolution product and Fourier-Wiener transform.

We next give the definition of the first variation of a functional F on $C_{a,b}[0,T]$.

Definition 2.4. Let F be a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional on $C_{a,b}[0,T]$, and let $w \in C_{a,b}[0,T]$. Then

$$\delta F(x|w) = \frac{\partial}{\partial h} F(x+hw)\Big|_{h=0} \tag{2.13}$$

(if it exists) is called the first variation of F.

Throughout this paper, when working with $\delta F(x|w)$, we will always require w to be an element of A where
(2.14)

$$A = \{w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)\,db(s) \text{ for some } z \in L^2_{a,b}[0,T]\}.$$

Let $\{\alpha_1, \dots, \alpha_n\}$ be an orthonormal set of functions from $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$. Then for $j \in \{1, \dots, n\}$, let

$$A_j = \int_0^T \alpha_j(t)\,da(t) \text{ and } B_j = \int_0^T \alpha_j^2(t)\,db(t), \tag{2.15}$$

and note that B_j is always positive, while A_j may be positive, negative or zero. Furthermore, for $\vec{u} \in \mathbf{R}^n$, we will write $f(\vec{u}) = f(u_1, \dots, u_n)$ and $f(\vec{u} + \langle \vec{\alpha}, y\rangle) = f(u_1 + \langle \alpha_1, y\rangle, \dots, u_n + \langle \alpha_n, y\rangle)$.

Next we state a very fundamental integration formula for the function space $C_{a,b}[0,T]$.

Theorem 2.1. *Let $\{\alpha_1, \dots, \alpha_n\}$ be an orthonormal set of functions from $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$. Let $f : \mathbf{R}^n \to \mathbf{C}$ be Lebesgue measurable, and let*

$$F(x) = f(\langle \alpha_1, x\rangle, \dots, \langle \alpha_n, x\rangle) = f(\langle \vec{\alpha}, x\rangle). \tag{2.16}$$

Then

(2.17)

$$E[F] \equiv \int_{C_{a,b}[0,T]} f(\langle\vec{\alpha}, x\rangle)\, d\mu(x) = \left(\prod_{j=1}^{n} 2\pi B_j\right)^{-1/2} \int_{\mathbf{R}^n} f(\vec{u}) \exp\left\{ -\sum_{j=1}^{n} \frac{(u_j - A_j)^2}{2B_j} \right\} d\vec{u}$$

in the sense that, if either side exists, both sides exist and equality holds.

We finish this section by describing the spaces $\mathcal{B}(p;m)$ of the functionals that we will be working with in this paper. Let n be a positive integer(fixed throughout this paper), and let $\{\alpha_1, \dots, \alpha_n\}$ be an orthonormal set of functions from $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$. Let m be a nonnegative integer(fixed throughout this paper). Then, for $1 \le p < \infty$, let $\mathcal{B}(p;m)$ be the space of all functionals of the form (2.16) for s-a.e. $x \in C_{a,b}[0,T]$ where all of the kth order partial derivatives $f_{j_1,\dots,j_k}(u_1,\dots,u_n) = f_{j_1,\dots,j_k}(\vec{u})$ of $f : \mathbf{R}^n \to \mathbf{R}$ are continuous and in $L^p(\mathbf{R}^n)$ for $k \in \{0, 1, \dots, m\}$ and each $j_i \in \{1, \dots, n\}$. Also let $\mathcal{B}(\infty;m)$ be the space of all functionals of the form (2.16) for s-a.e. $x \in C_{a,b}[0,T]$ where for $k = 0, 1, \dots, m$, all of the kth order partial derivatives $f_{j_1,\dots,j_k}(\vec{u})$ of f are in $C_0(\mathbf{R}^n)$, the space of bounded continuous functions on $\mathbf{R}^n$ that vanish at infinity. Note that $\mathcal{B}(p;m+1) \subset \mathcal{B}(p;m)$ for $m = 0, 1, \dots$.

We will concentrate on the space $\mathcal{B}(2;m)$ since the L^2-theory is more relevant in quantum mechanics and other applications than the L^p-theory for $1 \le p < 2$.

3. Relationships involving two concepts. In this section, we establish all of the various relationships involving exactly two of the three concepts of the generalized L_2 Fourier-Feynman transform, the convolution product and the first variation for functionals in $\mathcal{B}(2;m)$. These seven distinct relationships are given by equations (3.17)–(3.19), (3.28), (3.29), (3.33) and (3.37) below.

Remark 3.1. Let $F \in \mathcal{B}(2;m)$ be given by equation (2.16). In evaluating $E[F(\lambda^{-1/2}x)]$ for $\lambda > 0$, the expression

$$H(\lambda;\vec{u}) = \exp\bigg\{-\sum_{j=1}^{n}\frac{(\sqrt{\lambda}u_j - A_j)^2}{2B_j}\bigg\} \tag{3.1}$$

occurs with A_j and B_j given by equation (2.15) above. Clearly for $\lambda > 0$, $|H(\lambda;\vec{u})| \leq 1$ for all $\vec{u} \in \mathbf{R}^n$ since $B_j > 0$ for all $j = 1,\dots,n$. But for $\lambda \in \widetilde{\mathbf{C}}_+$, $|H(\lambda;\vec{u})|$ is not necessarily bounded by 1. Note that for each $\lambda \in \widetilde{\mathbf{C}}_+$, $\sqrt{\lambda} = c + di$ with $c \geq |d| \geq 0$. Hence, for each $\lambda \in \widetilde{\mathbf{C}}_+$

$$H(\lambda;\vec{u}) = \exp\bigg\{-\sum_{j=1}^{n}\frac{[(c^2 - d^2 + 2cdi)u_j^2 - 2(c+di)A_ju_j + A_j^2]}{2B_j}\bigg\}, \tag{3.2}$$

and so

$$|H(\lambda;\vec{u})| = \exp\bigg\{-\sum_{j=1}^{n}\frac{[(c^2 - d^2)u_j^2 - 2cA_ju_j + A_j^2]}{2B_j}\bigg\}. \tag{3.3}$$

Note that for $\lambda \in \mathbf{C}_+$, $\operatorname{Re}(\sqrt{\lambda}) = c > |d| = |\operatorname{Im}(\sqrt{\lambda})| \geq 0$, which implies that $c^2 - d^2 > 0$. Hence, for each $\lambda \in \mathbf{C}_+$, $H(\lambda;\vec{u})$, as a function of $\vec{u}$, is an element of $L^p(\mathbf{R}^n)$ for all $p \in [1,+\infty]$; in fact, $H(\lambda;\vec{u})$ also belongs to $C_0(\mathbf{R}^n)$, the space of bounded continuous functions on $\mathbf{R}^n$ that vanish at infinity. However, if $\lambda = -iq \in \widetilde{\mathbf{C}}_+ - \mathbf{C}_+$, then $\sqrt{\lambda} = \sqrt{-iq} = c + di$ with $c = \sqrt{|q|/2} = |d|$. Thus, for $\lambda = -iq$, $q \in \mathbf{R} - \{0\}$, $c^2 - d^2 = 0$, and so

$$|H(-iq;\vec{u})| = \exp\bigg\{\sum_{j=1}^{n}\frac{[\sqrt{2|q|}A_ju_j - A_j^2]}{2B_j}\bigg\}, \tag{3.4}$$

which is not necessarily in $L^p(\mathbf{R}^n)$ for any $p \in [1,+\infty]$. Thus, to obtain the existence of $T_q(2;F), T_q(2;G), (F*G)_q$, etc., we will need to put additional restrictions on F and G besides simply requiring them to be elements of $\mathcal{B}(2;m)$.

The inequality

$$|H(\lambda;\vec{u})| \leq \exp\bigg\{\sum_{j=1}^{n}\frac{cA_ju_j}{B_j}\bigg\} \leq \exp\bigg\{\bigg(\frac{1+|q|}{2}\bigg)^{1/2}\sum_{j=1}^{n}\frac{|A_ju_j|}{B_j}\bigg\}, \tag{3.5}$$

which holds for all $\lambda \in \widetilde{\mathbf{C}}_+$ with $c = \mathrm{Re}\,(\sqrt{\lambda}) < ((1+|q|)/2)^{1/2}$, is used several times below.

Remark 3.2. Note that in the setting of [**11**], $a(t) = 0$ and $b(t) = t$ implies that $A_j = 0$ and $B_j = 1$ for all $j = 1, \dots, n$; and thus $|H(\lambda; \vec{u})| \le 1$ for all $\lambda \in \widetilde{\mathbf{C}}_+$.

The following lemma, used in [**4**], follows easily from the definitions of $\delta F(y|w)$ and $\mathcal{B}(p;m)$.

Lemma 3.1. *Let $1 \le p \le +\infty$ be given, let m be a positive integer, let $F \in \mathcal{B}(p;m)$ be given by equation* (2.16), *and let w be an element of A. Then*

$$\delta F(y|w) = \sum_{j=1}^{n} \langle \alpha_j, w\rangle f_j(\langle \vec{\alpha}, y\rangle) \tag{3.6}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore, as a function of y, $\delta F(y|w) \in \mathcal{B}(p; m-1)$.

Let $F \in \mathcal{B}(2;m)$ be given by (2.16), let $G \in \mathcal{B}(2;m)$ be given by

$$G(y) = g(\langle \vec{\alpha}, y\rangle), \tag{3.7}$$

and for $\lambda \in \widetilde{\mathbf{C}}_+$, let

$$K_n(\lambda) = \left(\prod_{j=1}^{n} \frac{\lambda}{2\pi B_j} \right)^{1/2}, \tag{3.8}$$

and let

$$\psi(\lambda; \vec{\xi}) = K_n(\lambda) \int_{\mathbf{R}^n} f\left(\frac{\vec{\xi} + \vec{u}}{\sqrt{2}} \right) g\left(\frac{\vec{\xi} - \vec{u}}{\sqrt{2}} \right) H(\lambda; \vec{u})\, d\vec{u}. \tag{3.9}$$

Lemma 3.2. *Let $\lambda \in \widetilde{\mathbf{C}}_+$ be given. Let $G \in \mathcal{B}(2;m)$ be given by* (3.7). *Let $F \in \mathcal{B}(2;m)$ given by* (2.16) *be such that*

$$M(\lambda; \vec{\xi}) = \int_{\mathbf{R}^n} \left| f\left(\frac{\vec{\xi} + \vec{u}}{\sqrt{2}} \right) H(\lambda; \vec{u}) \right|^2 d\vec{u} \tag{3.10}$$

is an element of $L^1(\mathbf{R}^n)$. *Then* $\psi(\lambda;\cdot)\in L^2(\mathbf{R}^n)$ *where* $\psi(\lambda;\vec{\xi})$ *is given by equation* (3.9).

Proof. By Hölder's inequality,

$$\begin{aligned}(3.11)\quad &\int_{\mathbf{R}^n}|\psi(\lambda;\vec{\xi})|^2\,d\vec{\xi}\\ &\le |K_n(\lambda)|^2\int_{\mathbf{R}^n}\bigg[\int_{\mathbf{R}^n}\bigg|f\bigg(\frac{\vec{\xi}+\vec{u}}{\sqrt{2}}\bigg)g\bigg(\frac{\vec{\xi}-\vec{u}}{\sqrt{2}}\bigg)H(\lambda;\vec{u})\bigg|\,d\vec{u}\bigg]^2 d\vec{\xi}\\ &\le |K_n(\lambda)|^2\int_{\mathbf{R}^n}\bigg[\int_{\mathbf{R}^n}\bigg|f\bigg(\frac{\vec{\xi}+\vec{u}}{\sqrt{2}}\bigg)H(\lambda;\vec{u})\bigg|^2 d\vec{u}\int_{\mathbf{R}^n}\bigg|g\bigg(\frac{\vec{\xi}-\vec{u}}{\sqrt{2}}\bigg)\bigg|^2 d\vec{u}\bigg]d\vec{\xi}\\ &= |K_n(\lambda)|^2 2^{n/2}\|g\|_2^2\int_{\mathbf{R}^n}M(\lambda;\vec{\xi})\,d\vec{\xi}<\infty\end{aligned}$$

since

$$\int_{\mathbf{R}^n}\bigg|g\bigg(\frac{\vec{\xi}-\vec{u}}{\sqrt{2}}\bigg)\bigg|^2 d\vec{u}=2^{n/2}\int_{\mathbf{R}^n}|g(\vec{w})|^2\,d\vec{w}=2^{n/2}\|g\|_2^2. \qquad \square$$

In order to show that $(F*G)_q$ exists as an element of $\mathcal{B}(2;m)$, it will be helpful to first show that for each $\lambda\in\widetilde{\mathbf{C}}_+$, $\psi_{i_1,\dots,i_l}(\lambda;\vec{\xi})$ belongs to $L^2(\mathbf{R}^n)$, as a function of $\vec{\xi}$, for each $l\in\{0,\dots,m\}$ and each $i_j\in\{1,\dots,n\}$.

Lemma 3.3. *Let* λ *and* G *be as in Lemma* 3.2. *Let* $F\in\mathcal{B}(2;m)$ *given by* (2.16) *be such that*

$$(3.12)\qquad \int_{\mathbf{R}^n}\bigg|f_{j_1,\dots,j_k}\bigg(\frac{\vec{\xi}+\vec{u}}{\sqrt{2}}\bigg)H(\lambda;\vec{u})\bigg|^2 d\vec{u}$$

is an element of $L^1(\mathbf{R}^n)$ *for each* $k\in\{0,\dots,m\}$ *and each* $j_i\in\{1,\dots,n\}$. *Then* $\psi_{i_1,\dots,i_l}(\lambda;\vec{\xi})$ *is an element of* $L^2(\mathbf{R}^n)$ *for each* $l\in\{0,\dots,m\}$ *and each* $i_j\in\{1,\dots,n\}$ *and hence* $\psi(\lambda;\langle\vec{\alpha},y\rangle)$ *is an element of* $\mathcal{B}(2;m)$.

Proof. This lemma follows immediately from Lemma 3.2 because $\psi_{i_1,\dots,i_l}(\lambda;\vec{\xi})$ is the sum of 2^l terms, each of which can be shown to belong to $L^2(\mathbf{R}^n)$ using Hölder's inequality, condition (3.12), and the fact that all of the jth order partial derivatives of g belong to $L^2(\mathbf{R}^n)$ for each $j \in \{0, \dots, m\}$ because $G \in \mathcal{B}(2;m)$. Thus, $\psi(\lambda; \langle\vec{\alpha}, y\rangle) \in \mathcal{B}(2;m)$. □

Theorem 3.4. *Let $q \in \mathbf{R} - \{0\}$, let $G \in \mathcal{B}(2;m)$ be given by (3.7), and let $F \in \mathcal{B}(2;m)$ given by (2.16) be such that*

$$\int_{\mathbf{R}^n} \left| f_{j_1,\dots,j_k}\left(\frac{\vec{\xi}+\vec{u}}{\sqrt{2}}\right)\right|^2 \exp\left\{2\left(\frac{1+|q|}{2}\right)^{1/2} \sum_{j=1}^{n} \frac{|A_j u_j|}{B_j}\right\} d\vec{u}, \tag{3.13}$$

as a function of $\vec{\xi}$, is an element of $L^1(\mathbf{R}^n)$ for each $k \in \{0, \dots, m\}$ and each $j_i \in \{1, \dots, n\}$. Then $(F*G)_q$ exists as an element of $\mathcal{B}(2;m)$ and for s-a.e. $y \in C_{a,b}[0,T]$ is given by the formula

$$(F*G)_q(y) = \psi(-iq; \langle\vec{\alpha}, y\rangle) \tag{3.14}$$

with $\psi(\lambda; \vec{\xi})$ given by equation (3.9).

Proof. First note that inequality (3.5) together with condition (3.13) implies that condition (3.12) holds with $\lambda = -iq$. Hence by Lemma 3.3, $\psi(-iq; \langle\vec{\alpha}, y\rangle)$ belongs to $\mathcal{B}(2;m)$.

Next note that for all $\lambda \in \widetilde{\mathbf{C}}_+$ with $\mathrm{Re}\,(\sqrt{\lambda}) < ((1+|q|)/2)^{1/2}$,

$$\begin{aligned} &\left| K_n(\lambda) f\left(\frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\right) H(\lambda; \vec{u}) g\left(\frac{\langle\vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\right)\right| \\ &\qquad \le K_n(1+|q|) \left| f\left(\frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\right)\right| \\ &\qquad \cdot \exp\left\{\left(\frac{1+|q|}{2}\right)^{1/2} \sum_{j=1}^{n} \frac{|A_j u_j|}{B_j}\right\} \left| g\left(\frac{\langle\vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\right)\right| \end{aligned} \tag{3.15}$$

for s-a.e. $y \in C_{a,b}[0,T]$. But the right side of (3.15) is independent of λ, and as a function of $\vec{u}$, is an element of $L^1(\mathbf{R}^n)$ since the product of two L^2-functions is an L^1-function. Hence, by the dominated convergence theorem,

$$\begin{aligned}(F*G)_q(y) &= E_x^{\mathrm{anf}_q}\left[F\left(\frac{y+x}{\sqrt{2}}\right)G\left(\frac{y-x}{\sqrt{2}}\right)\right]\\ &= \lim_{\lambda\to -iq} E_x^{\mathrm{an}_\lambda}\left[F\left(\frac{y+x}{\sqrt{2}}\right)G\left(\frac{y-x}{\sqrt{2}}\right)\right]\\ &= \lim_{\lambda\to -iq} K_n(\lambda)\int_{\mathbf{R}^n} f\left(\frac{\langle\vec{\alpha},y\rangle+\vec{u}}{\sqrt{2}}\right)g\left(\frac{\langle\vec{\alpha},y\rangle-\vec{u}}{\sqrt{2}}\right)H(\lambda;\vec{u})\,d\vec{u}\\ &= K_n(-iq)\int_{\mathbf{R}^n} f\left(\frac{\langle\vec{\alpha},y\rangle+\vec{u}}{\sqrt{2}}\right)g\left(\frac{\langle\vec{\alpha},y\rangle-\vec{u}}{\sqrt{2}}\right)H(-iq;\vec{u})\,d\vec{u}\\ &= \psi(-iq;\langle\vec{\alpha},y\rangle)\end{aligned}$$

for s-a.e. $y\in C_{a,b}[0,T]$. Thus, $(F*G)_q$ is an element of $\mathcal{B}(2;m)$. □

Theorem 3.5. *Let q, G and F be as in Theorem 3.4. Then, for each $w\in A$, $\delta(F*G)_q(y|w)$ exists as an element of $\mathcal{B}(2;m-1)$, and for s-a.e. $y\in C_{a,b}[0,T]$ is given by the formula*

$$\begin{aligned}(3.16)\quad &\delta(F*G)_q(y|w)\\ &= \frac{K_n(-iq)}{\sqrt{2}}\int_{\mathbf{R}^n}\left[f\left(\frac{\langle\vec{\alpha},y\rangle+\vec{u}}{\sqrt{2}}\right)\sum_{l=1}^n\langle\alpha_l,w\rangle g_l\left(\frac{\langle\vec{\alpha},y\rangle-\vec{u}}{\sqrt{2}}\right)\right.\\ &\quad\left.+\,g\left(\frac{\langle\vec{\alpha},y\rangle-\vec{u}}{\sqrt{2}}\right)\sum_{l=1}^n\langle\alpha_l,w\rangle f_l\left(\frac{\langle\vec{\alpha},y\rangle+\vec{u}}{\sqrt{2}}\right)\right]H(-iq;\vec{u})\,d\vec{u}.\end{aligned}$$

Proof. By Theorem 3.4, $(F*G)_q$ exists as an element of $\mathcal{B}(2;m)$ and is given by equation (3.14). Hence, by Lemma 3.1, $\delta(F*G)_q(y|w)$ exists as an element of $\mathcal{B}(2;m-1)$ and is given by the formula

$$(3.17)\qquad \delta(F*G)_q(y|w)=\sum_{l=1}^n\langle\alpha_l,w\rangle\psi_l(-iq;\langle\vec{\alpha},y\rangle).$$

Formula (3.16) now follows directly from (3.17) and (3.9). □

Next we obtain formulas for the convolution product of the first variation of functionals. In Theorem 3.6, we take the convolution with respect to the first argument of the variations while in Theorem 3.7, we take the convolution with respect to the second argument of the variations.

Theorem 3.6. *Let q, G and F be as in Theorem 3.4. Then, for each $w \in A$, the convolution product $(\delta F(\cdot|w) * \delta G(\cdot|w))_q$ exists as an element of $\mathcal{B}(2; m-1)$, and for s-a.e. $y \in C_{a,b}[0,T]$ is given by the formula*

$$
\begin{aligned}
(3.18)\quad & (\delta F(\cdot|w) * \delta G(\cdot|w))_q(y) \\
&= K_n(-iq)\int_{\mathbf{R}^n}\left[\sum_{j=1}^{n}\langle\alpha_j, w\rangle f_j\left(\frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\right)\right] \\
&\qquad \cdot\left[\sum_{l=1}^{n}\langle\alpha_l, w\rangle g_l\left(\frac{\langle\vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\right)\right] H(-iq;\vec{u})\, d\vec{u}.
\end{aligned}
$$

Proof. The fact that for all $j, l \in \{1, \dots, n\}$,

$$
\eta(-iq;\vec{\xi}) \equiv \int_{\mathbf{R}^n} f_j\left(\frac{\vec{\xi}+\vec{u}}{\sqrt{2}}\right) g_l\left(\frac{\vec{\xi}-\vec{u}}{\sqrt{2}}\right) H(-iq;\vec{u})\, d\vec{u}
$$

belongs to $L^2(\mathbf{R}^n)$ follows from Hölder's inequality. Also $\eta(-iq;\langle\vec{\alpha}, y\rangle)$ is an element of $\mathcal{B}(2; m-1)$ since $\eta_{j_1,\dots,j_k}(-iq;\vec{\xi}) \in L^2(\mathbf{R}^n)$ for each $k \in \{0, \dots, m-1\}$ and each $j_i \in \{1, \dots, n\}$. Equation (3.18) then follows immediately using the definition of the convolution product, equation (3.6) and the formula $\delta G(y|w) = \sum_{l=1}^{n}\langle\alpha_l, w\rangle g_l(\langle\vec{\alpha}, y\rangle)$. □

Theorem 3.7. *Let $q \in \mathbf{R} - \{0\}$, $F \in \mathcal{B}(2; m)$ be given by (2.16), and let $G \in \mathcal{B}(2; m)$ be given by (3.7). Then, for each $w \in A$ and for s-a.e. $y \in C_{a,b}[0,T]$, $(\delta F(y|\cdot) * \delta G(y|\cdot))_q(w)$ exists and is given by the formula*

$$
\begin{aligned}
(3.19)\quad & (\delta F(y|\cdot) * \delta G(y|\cdot))_q(w) \\
&= \left[\delta F(y|w/\sqrt{2}) + \left(\frac{i}{2q}\right)^{1/2}\sum_{j=1}^{n} A_j f_j(\langle\vec{\alpha}, y\rangle)\right]
\end{aligned}
$$

$$\cdot \left[\delta G(y|w/\sqrt{2}) - \left(\frac{i}{2q}\right)^{1/2} \sum_{l=1}^{n} A_l g_l(\langle \vec{\alpha}, y\rangle)\right]$$
$$- \frac{i}{2q} \sum_{l=1}^{n} B_l f_l(\langle \vec{\alpha}, y\rangle) g_l(\langle \vec{\alpha}, y\rangle).$$

Furthermore, as a function of y, $(\delta F(y|\cdot) * \delta G(y|\cdot))_q(w)$ is an element of $\mathcal{B}(1; m-1)$.

Proof. Using equation (2.17), together with Definition 2.1, it follows that

$$E_x^{\mathrm{anf}_q}[\langle \alpha_j, x\rangle \langle \alpha_l, x\rangle] = \begin{cases} (i/q) A_j A_l & j \neq l \\ (i/q) B_l + (i/q) A_l^2 & j = l \end{cases} \tag{3.20}$$

and that

$$E_x^{\mathrm{anf}_q}[\langle \alpha_j, x\rangle] = \left(\frac{i}{q}\right)^{1/2} A_j. \tag{3.21}$$

Equation (3.19) now follows from the calculation below:

$$\begin{aligned}
&(\delta F(y|\cdot) * \delta G(y|\cdot))_q(w) \\
&= E_x^{\mathrm{anf}_q}\left[\delta F\left(y \middle| \frac{w+x}{\sqrt{2}}\right) \delta G\left(y \middle| \frac{w-x}{\sqrt{2}}\right)\right] \\
&= E_x^{\mathrm{anf}_q}\left[\left(\sum_{j=1}^{n} \frac{\langle \alpha_j, w+x\rangle}{\sqrt{2}} f_j(\langle \vec{\alpha}, y\rangle)\right)\left(\sum_{l=1}^{n} \frac{\langle \alpha_l, w-x\rangle}{\sqrt{2}} g_l(\langle \vec{\alpha}, y\rangle)\right)\right] \\
&= \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} f_j(\langle \vec{\alpha}, y\rangle) g_l(\langle \vec{\alpha}, y\rangle) E_x^{\mathrm{anf}_q}[\langle \alpha_j, w+x\rangle \langle \alpha_l, w-x\rangle] \\
&= \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} f_j(\langle \vec{\alpha}, y\rangle) g_l(\langle \vec{\alpha}, y\rangle) \\
&\qquad \cdot \big\{E_x^{\mathrm{anf}_q}[\langle \alpha_j, w\rangle \langle \alpha_l, w\rangle - \langle \alpha_j, w\rangle \langle \alpha_l, x\rangle + \langle \alpha_j, x\rangle \langle \alpha_l, w\rangle] \\
&\qquad\qquad - E_x^{\mathrm{anf}_q}[\langle \alpha_j, x\rangle \langle \alpha_l, x\rangle]\big\} \\
&= \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} f_j(\langle \vec{\alpha}, y\rangle) g_l(\langle \vec{\alpha}, y\rangle)
\end{aligned}$$

$$\cdot \left[\langle \alpha_j, w\rangle\langle \alpha_l, w\rangle - \left(\frac{i}{q}\right)^{1/2} A_l\langle \alpha_j, w\rangle + \left(\frac{i}{q}\right)^{1/2} A_j\langle \alpha_l, w\rangle\right]$$

$$- \frac{1}{2}\left[\frac{i}{q}\sum_{j\neq l} f_j(\langle\vec{\alpha}, y\rangle)g_l(\langle\vec{\alpha}, y\rangle)A_jA_l\right.$$

$$\left. + \frac{i}{q}\sum_{l=1}^{n} f_l(\langle\vec{\alpha}, y\rangle)g_l(\langle\vec{\alpha}, y\rangle)(B_l + A_l^2)\right]$$

$$= \sum_{j=1}^{n}\left[\langle \alpha_j, w/\sqrt{2}\rangle + \left(\frac{i}{2q}\right)^{1/2} A_j\right] f_j(\langle\vec{\alpha}, y\rangle)$$

$$\cdot \sum_{l=1}^{n}\left[\langle \alpha_l, w/\sqrt{2}\rangle - \left(\frac{i}{2q}\right)^{1/2} A_l\right] g_l(\langle\vec{\alpha}, y\rangle)$$

$$- \frac{i}{2q}\sum_{l=1}^{n} B_l f_l(\langle\vec{\alpha}, y\rangle)g_l(\langle\vec{\alpha}, y\rangle)$$

$$= \left[\delta F(y|w/\sqrt{2}) + \left(\frac{i}{2q}\right)^{1/2}\sum_{j=1}^{n} A_j f_j(\langle\vec{\alpha}, y\rangle)\right]$$

$$\cdot \left[\delta G(y|w/\sqrt{2}) - \left(\frac{i}{2q}\right)^{1/2}\sum_{l=1}^{n} A_l g_l(\langle\vec{\alpha}, y\rangle)\right]$$

$$- \frac{i}{2q}\sum_{l=1}^{n} B_l f_l(\langle\vec{\alpha}, y\rangle)g_l(\langle\vec{\alpha}, y\rangle)$$

using equations (3.20) and (3.21). □

Our next theorem is a statement of Theorem 5.1 in [**4**].

Theorem 3.8. *Let $q \in \mathbf{R} - \{0\}$ be given. Let $F \in \mathcal{B}(2; m)$ given by* (2.16) *be such that*

$$\int_{\mathbf{R}^n} |f_{j_1,\dots,j_k}(\vec{\xi} + \vec{u})| \exp\left\{\left(\frac{1+|q|}{2}\right)^{1/2}\sum_{j=1}^{n}\frac{|A_ju_j|}{B_j}\right\} d\vec{u} \tag{3.22}$$

is an element of $L^2(\mathbf{R}^n)$ for each $k \in \{0,\dots,m\}$ and each $j_i \in \{1,\dots,n\}$. Then $T_q(2;F)$ is an element of $\mathcal{B}(2;m)$ and is given by *the formula*

$$T_q(2;F)(y) = \phi(-iq;\langle\vec{\alpha},y\rangle) \tag{3.23}$$

with

$$\phi(\lambda;\vec{\xi}) = K_n(\lambda)\int_{\mathbf{R}^n} f(\vec{\xi}+\vec{u})H(\lambda;\vec{u})\,d\vec{u}, \quad \lambda\in\widetilde{\mathbf{C}}_+. \tag{3.24}$$

Remark 3.3. If $G\in\mathcal{B}(2;m)$ given by (3.7) is such that

$$\int_{\mathbf{R}^n} |g_{j_1,\dots,j_k}(\vec{\xi}+\vec{u})|\exp\left\{\left(\frac{1+|q|}{2}\right)^{1/2}\sum_{j=1}^{n}\frac{|A_ju_j|}{B_j}\right\}d\vec{u} \tag{3.25}$$

is an element of $L^2(\mathbf{R}^n)$ for each $k\in\{0,\dots,m\}$ and each $j_i\in\{1,\dots,n\}$, then $T_q(2;G)\in\mathcal{B}(2;m)$ and for s-a.e. $y\in C_{a,b}[0,T]$,

$$T_q(2;G)(y) = \theta(-iq;\langle\vec{\alpha},y\rangle) \tag{3.26}$$

with

$$\theta(\lambda;\vec{\xi}) = K_n(\lambda)\int_{\mathbf{R}^n} g(\vec{\xi}+\vec{u})H(\lambda;\vec{u})\,d\vec{u}, \quad \lambda\in\widetilde{\mathbf{C}}_+. \tag{3.27}$$

Remark 3.4. In the proof of Theorem 5.1 in [**4**] we showed that for ϕ given by (3.24),

$$\phi(\lambda;\vec{\xi}) \longrightarrow \phi(-iq;\vec{\xi}) \text{ almost everywhere on } \mathbf{R}^n,$$
$$\|\phi(\lambda;\cdot)\|_2 \longrightarrow \|\phi(-iq;\cdot)\|_2,$$

and hence that

$$\|\phi(\lambda;\cdot)-\phi(-iq;\cdot)\|_2 \longrightarrow 0$$

as $\lambda\to -iq$ through values in $\mathbf{C}_+$. Of course the same conclusions hold for $\theta(\lambda;\vec{\xi})$ given by (3.27).

In our next remark we note that the transform with respect to the first argument of the variation equals the variation of the transform.

Remark 3.5. For $w \in A$, $q \in \mathbf{R} - \{0\}$ and F as in Theorem 3.8 above, Corollary 5.2 in [**4**] implies that

$$\begin{aligned} \delta T_q(2;F)(y|w) =& T_q(2;\delta F(\cdot|w))(y) \\ =& \sum_{j=1}^{n} \langle \alpha_j, w\rangle \phi_j(-iq;\langle\vec{\alpha},y\rangle), \end{aligned} \tag{3.28}$$

which, as a function of y, is an element of $\mathcal{B}(2;m-1)$.

In our next theorem we take the transform with respect to the second argument of the variation.

Theorem 3.9. *Let $q \in \mathbf{R}-\{0\}$ be given, and let $F \in \mathcal{B}(2;m)$ be given by equation* (2.16)*. Then, for each $w \in A$ and s-a.e. $y \in C_{a,b}[0,T]$,*

$$T_q(2;\delta F(y|\cdot))(w) = \delta F(y|w) + \left(\frac{i}{q}\right)^{1/2} \sum_{j=1}^{n} A_j f_j(\langle\vec{\alpha},y\rangle) \tag{3.29}$$

which, as a function of y, is an element of $\mathcal{B}(2;m-1)$.

Proof. For all $\lambda \in \mathbf{C}_+$, using (3.6) we see that

$$\begin{aligned} T_\lambda(\delta F(y|\cdot))(w) &= E_x^{\mathrm{an}_\lambda}[\delta F(y|w+x)] \\ &= E_x^{\mathrm{an}_\lambda}\left[\sum_{j=1}^{n}\langle\alpha_j, w+x\rangle f_j(\langle\vec{\alpha},y\rangle)\right] \\ &= \sum_{j=1}^{n} f_j(\langle\vec{\alpha},y\rangle) E_x^{\mathrm{an}_\lambda}[\langle\alpha_j,w\rangle + \langle\alpha_j,x\rangle] \\ &= \sum_{j=1}^{n} f_j(\langle\vec{\alpha},y\rangle)\left[\langle\alpha_j,w\rangle + \frac{A_j}{\sqrt{\lambda}}\right]. \end{aligned}$$

Letting $\lambda \to -iq$ though value in $\mathbf{C}_+$ yields (3.29) as desired. □

Theorem 3.10. *Let q, F and G be as in Theorem* 3.4*. Furthermore, assume that*

$$\int_{\mathbf{R}^n} \left|\psi_{j_1,\dots,j_k}(-iq;\vec{\xi}+\vec{u})\right| \exp\left\{\left(\frac{1+|q|}{2}\right)^{1/2} \sum_{j=1}^{n} \frac{|A_j u_j|}{B_j}\right\} d\vec{u} \tag{3.30}$$

is an element of $L^2(\mathbf{R}^n)$ *for each* $k \in \{0, \dots, m\}$ *and each* $j_i \in \{1, \dots, n\}$ *with* ψ *given by equation* (3.9). *Then* $T_q(2; (F * G)_q)$ *is an element of* $\mathcal{B}(2; m)$ *and is given by the formula*

$$T_q(2; (F * G)_q)(y) = \Psi(-iq; \langle \vec{\alpha}, y \rangle) \tag{3.31}$$

with

$$\Psi(\lambda; \vec{\xi}) = K_n(\lambda) \int_{\mathbf{R}^n} \psi(\lambda; \vec{\xi} + \vec{u}) H(\lambda; \vec{u})\, d\vec{u}, \quad \lambda \in \widetilde{\mathbf{C}}_+. \tag{3.32}$$

Proof. By Theorem 3.4 we know that $(F * G)_q(y) = \psi(-iq; \langle \vec{\alpha}, y \rangle) \in \mathcal{B}(2; m)$. Then by Theorem 3.8 with F replaced with $(F * G)_q$, we know that

(3.33)

$$\begin{aligned} T_q(2; (F * G)_q)(y) &= T_q(2; \psi(-iq; \cdot))(y) \\ &= \Psi(-iq; \langle \vec{\alpha}, y \rangle) \\ &= K_n(-iq) \int_{\mathbf{R}^n} \psi(-iq; \langle \vec{\alpha}, y \rangle + \vec{u}) H(-iq; \vec{u})\, d\vec{u} \end{aligned}$$

is an element of $\mathcal{B}(2; m)$. □

Theorem 3.11. *Let* q *and* F *be as in Theorem* 3.8. *Let* $G \in \mathcal{B}(2; m)$ *be as in Remark* 3.3. *Furthermore, assume that* ϕ *given by equation* (3.24) *is such that*

(3.34)

$$\int_{\mathbf{R}^n} \left[\int_{\mathbf{R}^n} \left| f_{j_1, \dots, j_k}\left(\frac{\vec{\xi} + \vec{u}}{\sqrt{2}} + \vec{v} \right) \right| \exp\left\{ \left(\frac{1 + |q|}{2} \right)^{1/2} \sum_{j=1}^{n} \frac{|A_j v_j|}{B_j} \right\} d\vec{v} \right]^2 \cdot \exp\left\{ 2 \left(\frac{1 + |q|}{\sqrt{2}} \right)^{1/2} \sum_{j=1}^{n} \frac{|A_j u_j|}{B_j} \right\} d\vec{u}$$

is an element of $L^1(\mathbf{R}^n)$ *for each* $k \in \{0, \dots, m\}$ *and each* $j_i \in \{1, \dots, n\}$. *Then the convolution product* $(T_q(2; F) * T_q(2; G))_q$ *exists* *as an element of* $\mathcal{B}(2;m)$ *and for s-a.e.* $y \in C_{a,b}[0,T]$ *is given by the formula*

$$(T_q(2;F) * T_q(2;G))_q(y) = \Phi(-iq; \langle \vec{\alpha}, y\rangle) \tag{3.35}$$

where

$$\Phi(\lambda;\vec{\xi}) = K_n(\lambda)\int_{\mathbf{R}^n} \phi\bigg(\lambda; \frac{\vec{\xi}+\vec{u}}{\sqrt{2}}\bigg)\theta\bigg(\lambda; \frac{\vec{\xi}-\vec{u}}{\sqrt{2}}\bigg) H(\lambda;\vec{u})\,d\vec{u}, \quad \lambda \in \widetilde{\mathbf{C}}_+ \tag{3.36}$$

with $\theta(\lambda;\vec{\xi})$ *given by equation* (3.27). *That is to say, for s-a.e.* $y \in C_{a,b}[0,T]$,

$$\begin{aligned}(3.37)\quad &(T_q(2;F) * T_q(2;G))_q(y)\\ &= K_n(-iq)\int_{\mathbf{R}^n} \phi\bigg(-iq; \frac{\langle \vec{\alpha}, y\rangle+\vec{u}}{\sqrt{2}}\bigg)\theta\bigg(-iq; \frac{\langle \vec{\alpha}, y\rangle-\vec{u}}{\sqrt{2}}\bigg) H(-iq;\vec{u})\,d\vec{u}.\end{aligned}$$

Proof. By Theorem 3.8 and Remark 3.4 we know that $T_q(2;F)(y) = \phi(-iq;\langle \vec{\alpha}, y\rangle)$ and $T_q(2;G)(y) = \psi(-iq;\langle \vec{\alpha}, y\rangle)$ are elements of $\mathcal{B}(2;m)$.

Using inequalities (3.5) and (3.34) we see that

$$\int_{\mathbf{R}^n} \bigg|\phi_{j_1,\dots,j_k}\bigg(-iq; \frac{\vec{\xi}+\vec{u}}{\sqrt{2}}\bigg) H(-iq;\vec{u})\bigg|^2 d\vec{u} \tag{3.38}$$

is an element of $L^1(\mathbf{R}^n)$ for each $k \in \{0,\dots,m\}$ and each $j_i \in \{1,\dots,n\}$. Applying Lemma 3.3 with (3.12) replaced with (3.38), we also see that $\Phi_{i_1,\dots,i_l}(-iq;\langle \vec{\alpha}, y\rangle)$ belongs to $\mathcal{B}(2;m)$ for each $l \in \{0,\dots,m\}$ and each $i_j \in \{1,\dots,n\}$.

Note that for all $\lambda \in \widetilde{\mathbf{C}}_+$ with $\operatorname{Re}(\sqrt{\lambda}) < ((1+|q|)/2)^{1/2}$,

$$\begin{aligned}
(3.39)\quad &\left|K_n(\lambda)\phi\left(\lambda; \frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\right) H(\lambda;\vec{u})\theta\left(\lambda; \frac{\langle\vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\right)\right| \\
&\le K_n(1+|q|)\left|\phi\left(\lambda; \frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\right)\right| \\
&\quad\cdot \exp\left\{\left(\frac{1+|q|}{2}\right)^{1/2} \sum_{j=1}^n \frac{|A_j u_j|}{B_j}\right\}\left|\theta\left(\lambda; \frac{\langle\vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\right)\right| \\
&\le \big(K_n(1+|q|)\big)^3 \int_{\mathbf{R}^n} \left|f\left(\frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}} + \vec{v}\right)\right| \\
&\qquad\qquad \cdot \exp\left\{\left(\frac{1+|q|}{2}\right)^{1/2} \sum_{j=1}^n \frac{|A_j v_j|}{B_j}\right\} d\vec{v} \\
&\quad\cdot \exp\left\{2\left(\frac{1+|q|}{2}\right)^{1/2} \sum_{j=1}^n \frac{|A_j u_j|}{B_j}\right\} \\
&\quad\cdot \int_{\mathbf{R}^n} \left|g\left(\frac{\langle\vec{\alpha}, y - \vec{u}}{\sqrt{2}} + \vec{v}\right)\right| \exp\left\{\left(\frac{1+|q|}{2}\right)^{1/2} \sum_{j=1}^n \frac{|A_j v_j|}{B_j}\right\} d\vec{v}
\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$. But the last expression of (3.39) is independent of λ, and as a function of $\vec{u}$, is an element of $L^1(\mathbf{R}^n)$ since the product of two L^2-functions is an L^1-function. Hence, by the dominated convergence theorem,

$$\begin{aligned}
&(T_q(2;F) * T_q(2;G))_q(y) \\
&= \lim_{\lambda\to -iq} E_x^{\mathrm{an}_\lambda}\left[T_\lambda(F)\left(\frac{y+x}{\sqrt{2}}\right) T_\lambda(G)\left(\frac{y-x}{\sqrt{2}}\right)\right] \\
&= \lim_{\lambda\to -iq} K_n(\lambda) \int_{\mathbf{R}^n} \phi\left(\lambda; \frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\right)\theta\left(\lambda; \frac{\langle\vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\right) H(\lambda;\vec{u})\, d\vec{u} \\
&= \Phi(-iq; \langle\vec{\alpha}, y\rangle)
\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$. □

4. Relationships involving three concepts. In this section, we examine all of the various relationships involving the transform, the convolution product and the first variation where each operation is used exactly once. There are more than six possibilities since one can take the transform or the convolution with respect to either the first or the second argument of the variation. To exhaust all possibilities we need to take the variation of the expressions in (3.33) and (3.37), the transform of the expressions in (3.17), (3.18) and (3.19), and the convolution of the expressions in equations (3.28) and (3.29). It turns out that there are 11 distinct formulas, and these are given by equations (4.1) through (4.11) below.

In our first theorem we obtain a formula for the transform with respect to y of the expressions in equation (3.17). We obtain the same formula by taking the variation of the expressions in equation (3.33).

Theorem 4.1. *Let q, F, G and ψ be as in Theorem 3.10. Then for each $w \in A$,*

$$\begin{aligned} T_q(2;\delta(F*G)_q)(\cdot|w)(y) &= \delta T_q(2;(F*G)_q)(y|w) \\ &= K_n(-iq)\sum_{j=1}^{n}\langle\alpha_j,w\rangle\int_{\mathbf{R}^n}\psi_j(-iq;\langle\vec{\alpha},y\rangle+\vec{v})H(-iq;\vec{v})\,d\vec{v} \end{aligned} \tag{4.1}$$

which, as a function of y, is in an element of $\mathcal{B}(2;m-1)$.

Proof. The first equality in (4.1) follows from Remark 3.5 with F replaced with $(F*G)_q$. The second equality in (4.1) follows from equation (3.33) and Lemma 3.1. □

In our second theorem we obtain a formula for the transform with respect to w of the expressions in equation (3.17).

Theorem 4.2. *Let q, G, F and ψ be as in Theorem 3.4. Then, for each $w \in A$,*

$$\begin{aligned} T_q(2;\delta(F*G)_q(y|\cdot))(w) \\ = \delta(F*G)_q(y|w)+\left(\frac{i}{q}\right)^{1/2}\sum_{j=1}^{n}A_j\psi_j(\langle\vec{\alpha},y\rangle) \end{aligned} \tag{4.2}$$

which, as a function of y, is an element of $\mathcal{B}(2;m-1)$.

Proof. This result follows immediately by replacing F with $(F * G)_q$ in equation (3.29). □

Next we take the transform of the expressions in equation (3.18). Again there are two cases since we can take the transform either with respect to y (Theorem 4.3 below) or else with respect to w (Theorem 4.4 below).

Theorem 4.3. *Let q, F, G and ψ be as in Theorem* 3.10. *Then for each $w \in A$,*

$$(4.3)\quad \begin{aligned} &T_q(2;(\delta F(\cdot|w) * \delta G(\cdot|w))_q)(y) \\ &= K_n^2(-iq)\sum_{j=1}^n\sum_{l=1}^n \langle \alpha_j, w\rangle\langle \alpha_l, w\rangle \int_{\mathbf{R}^n} H(-iq;\vec{v}) \\ &\cdot \bigg(\int_{\mathbf{R}^n} f_j\bigg(\frac{\langle\vec{\alpha}, y\rangle + \vec{v} + \vec{u}}{\sqrt{2}}\bigg) g_l\bigg(\frac{\langle\vec{\alpha}, y\rangle + \vec{v} - \vec{u}}{\sqrt{2}}\bigg) H(-iq;\vec{u})\, d\vec{u}\bigg)\, d\vec{v} \end{aligned}$$

which, as a function of y, is an element of $\mathcal{B}(2; m-1)$.

Proof. A direct calculation using (3.18) shows that (4.3) holds for s-a.e. $y \in C_{a,b}[0,T]$. Now condition (3.30) implies that the righthand side of (4.3) belongs to $\mathcal{B}(2; m-1)$ because for each j and l in $\{1, \dots, n\}$,

$$\frac{1}{2}K_n(-iq)\int_{\mathbf{R}^n} f_j\bigg(\frac{\vec{\xi} + \vec{v} + \vec{u}}{\sqrt{2}}\bigg) g_l\bigg(\frac{\vec{\xi} + \vec{v} - \vec{u}}{\sqrt{2}}\bigg) H(-iq;\vec{u})\, d\vec{u}$$

is one of the four terms involved in the calculation of

$$\psi_{j,l}(-iq;\vec{\xi}+\vec{v}) = \frac{\partial}{\partial \xi_l}\bigg(\frac{\partial}{\partial \xi_j}\psi(-iq;\vec{\xi}+\vec{v})\bigg). \qquad \square$$

Theorem 4.4. *Let q, G and F be as in Theorem* 3.4, *and let $w \in A$. Then, taking the L_2 analytic transform with respect to the second argument of the variations of the expressions in equation* (3.18) *yields the formula*

$$
\begin{aligned}
(4.4)\quad & E_x^{\mathrm{anf}_q}[(\delta F(\cdot|w+x) * \delta G(\cdot|w+x))_q(y)] \\
&= K_n(-iq)\sum_{j=1}^{n}\sum_{l=1}^{n}\left[\langle\alpha_j, w\rangle + \left(\frac{i}{q}\right)^{1/2} A_j\right]\left[\langle\alpha_l, w + \left(\frac{i}{q}\right)^{1/2} A_l\right] \\
&\qquad \cdot \int_{\mathbf{R}^n} f_j\left(\frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\right) g_l\left(\frac{\langle\vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\right) H(-iq;\vec{u})\, d\vec{u} \\
&\quad + \frac{i}{q} K_n(-iq)\sum_{l=1}^{n} B_l \int_{\mathbf{R}^n} f_l\left(\frac{\langle\vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}}\right) g_l\left(\frac{\langle\vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}}\right) H(-iq;\vec{u})\, d\vec{u}
\end{aligned}
$$

which, as a function of y*, is an element of* $\mathcal{B}(2; m-1)$.

Proof. To obtain formula (4.4), we simply substitute the righthand side of equation (3.18) into the lefthand side of (4.4) and then evaluate this analytic Feynman integral using (3.20) and (3.21). □

Our next goal is to obtain formulas for the transform of the convolution product with respect to the second argument of the variations. Again, there are two cases since we can take the transform of the expressions in equation (3.19) either with respect to w (Theorem 4.5 below) or else with respect to y (Theorem 4.6 below).

Theorem 4.5. *Let* q*,* F *and* G *be as in Theorem* 3.7 *above. Then for each* $w \in A$ *and s-a.e.* $y \in C_{a,b}[0,T]$*,*

$$
\begin{aligned}
(4.5)\quad & T_q(2; (\delta F(y|\cdot) * \delta G(y|\cdot))_q)(w) \\
&= \left[\delta F(y|w/\sqrt{2}) + \left(\frac{2i}{q}\right)^{1/2}\sum_{j=1}^{n} A_j f_j(\langle\vec{\alpha}, y\rangle)\right]\delta G(y|w/\sqrt{2})
\end{aligned}
$$

which, as a function of y*, is an element of* $\mathcal{B}(2; m-1)$.

Proof. Using equation (3.19), we obtain that the lefthand side of (4.5) equals the analytic Feynman integral

$$E_x^{\mathrm{anf}_q}\bigg[\bigg(\delta F\bigg(y\bigg|\frac{w+x}{\sqrt{2}}\bigg)+\bigg(\frac{i}{2q}\bigg)^{1/2}\sum_{j=1}^{n}A_j f_j(\langle\vec{\alpha},y\rangle)\bigg)$$
$$\cdot\bigg(\delta G\bigg(y\bigg|\frac{w+x}{\sqrt{2}}\bigg)-\bigg(\frac{i}{2q}\bigg)^{1/2}\sum_{l=1}^{n}A_l g_l(\langle\vec{\alpha},y\rangle)\bigg)$$
$$-\frac{i}{2q}\sum_{l=1}^{n}B_l f_l(\langle\vec{\alpha},y\rangle)g_l(\langle\vec{\alpha},y\rangle)\bigg].$$

Then, using (3.20) and (3.21) to evaluate the above Feynman integral yields the righthand side of (4.5). □

We omit the proof of our next theorem because it is very similar to the last two proofs given above.

Theorem 4.6. *Let* q, F *and* G *be as in Theorem* 3.4, *and let* $w \in A$. *Then the transform of the expressions in equation* (3.19) *with respect to* y *is given by the formula*

$$(4.6)\quad E_x^{\mathrm{anf}_q}[(\delta F(y+x|\cdot)*\delta G(y+x|\cdot))_q(w)]$$
$$=K_n(-iq)\sum_{j=1}^{n}\sum_{l=1}^{n}\bigg[\bigg\langle\alpha_j,\frac{w}{\sqrt{2}}\bigg\rangle+\bigg(\frac{i}{2q}\bigg)^{1/2}A_j\bigg]$$
$$\cdot\bigg[\bigg\langle\alpha_l,\frac{w}{\sqrt{2}}\bigg\rangle+\bigg(\frac{i}{2q}\bigg)^{1/2}A_l\bigg]$$
$$\cdot\int_{\mathbf{R}^n}f_j(\langle\vec{\alpha},y\rangle+\vec{u})g_l(\langle\vec{\alpha},y\rangle+\vec{u})H(-iq;\vec{u})d\vec{u}$$
$$-\frac{i}{2q}K_n(-iq)\sum_{l=1}^{n}B_l\int_{\mathbf{R}^n}f_l(\langle\vec{\alpha},y\rangle+\vec{u})g_l(\langle\vec{\alpha},y\rangle+\vec{u})H(-iq;\vec{u})\,d\vec{u}$$

which, as a function of y, *is an element of* $\mathcal{B}(2;m-1)$.

In our next theorem we obtain a formula for the variation of the expressions in equation (3.37).

Theorem 4.7. *Let q, F, G, ϕ and θ be as in Theorem 3.11. Then for each $w \in A$,*

$$(4.7) \quad \delta(T_q(2;F) * T_q(2;G))_q(y|w)$$
$$= K_n(-iq) \int_{\mathbf{R}^n} \Big[\phi\Big(-iq; \frac{\langle \vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}} \Big) \sum_{l=1}^{n} \Big\langle \alpha_l, \frac{w}{\sqrt{2}} \Big\rangle \theta_l \Big(-iq; \frac{\langle \vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}} \Big)$$
$$+ \theta\Big(-iq; \frac{\langle \vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}} \Big) \sum_{j=1}^{n} \Big\langle \alpha_j, \frac{w}{\sqrt{2}} \Big\rangle \phi_j \Big(-iq; \frac{\langle \vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}} \Big) \Big] H(-iq; \vec{u})\, d\vec{u}$$

which, as a function of y, is an element of $\mathcal{B}(2; m-1)$.

Proof. By Theorem 3.11, $(T_q(2;F) * T_q(2;G))_q(y)$ is an element of $\mathcal{B}(2;m)$ and so by Lemma 3.1, $\delta(T_q(2;F) * T_q(2;G))_q(y|w)$ belongs to $\mathcal{B}(2;m-1)$. Using (3.37) and (3.6) we obtain equation (4.7). □

Next we obtain formulas for the convolution product of the expressions in equation (3.28). Again there are two cases since we can take the convolution product with respect to the first argument or the second argument of the variation.

Theorem 4.8. *Let q, F, G, ϕ, θ and w be as in Theorem 3.11. Then*

$$(4.8) \quad (\delta T_q(2;F)(\cdot|w) * \delta T_q(2;G)(\cdot|w))_q(y)$$
$$= K_n(-iq) \sum_{j=1}^{n} \sum_{l=1}^{n} \langle \alpha_j, w\rangle \langle \alpha_l, w\rangle$$
$$\cdot \int_{\mathbf{R}^n} \phi_j \Big(-iq; \frac{\langle \vec{\alpha}, y\rangle + \vec{u}}{\sqrt{2}} \Big) \theta_l \Big(-iq; \frac{\langle \vec{\alpha}, y\rangle - \vec{u}}{\sqrt{2}} \Big) H(-iq; \vec{u})\, d\vec{u},$$

which, as a function of y, is an element of $\mathcal{B}(2; m-1)$.

Proof. By Remark 3.5 we know that $\delta T_q(2;F)(\cdot|w)$ and $\delta T_q(2;G)(\cdot|w)$ are elements of $\mathcal{B}(2;m-1)$. Hence, by Theorem 3.4 their convolution product is an element of $\mathcal{B}(2;m-1)$. Equation (4.8) now follows by a direct calculation. □

Theorem 4.9. *Let $q \in \mathbf{R} - \{0\}$, let $F \in \mathcal{B}(2;m)$ be as in Theorem 3.8, and let $G \in \mathcal{B}(2;m)$ be as in Remark 3.3. Then, for* *each* $w \in A$,

$$(4.9)\quad (\delta T_q(2;F)(y|\cdot) * \delta T_q(2;G)(y|\cdot))_q(w)$$
$$= \left[\delta T_q(2;F)(y|w/\sqrt{2}) + \left(\frac{i}{2q}\right)^{1/2} \sum_{j=1}^{n} A_j \phi_j(-iq;\langle\vec{\alpha},y\rangle)\right]$$
$$\cdot \left[\delta T_q(2;G)(y|w/\sqrt{2}) - \left(\frac{i}{2q}\right)^{1/2} \sum_{l=1}^{n} A_l \theta_l(-iq;\langle\vec{\alpha},y\rangle)\right]$$
$$- \frac{i}{2q} \sum_{l=1}^{n} B_l \phi_l(-iq;\langle\vec{\alpha},y\rangle)\theta_l(-iq;\langle\vec{\alpha},y\rangle)$$

which, as a function of y, *is an element of* $\mathcal{B}(1;m-1)$ *where* ϕ *and* θ *are given by equations* (3.24) *and* (3.27), *respectively.*

Proof. Equation (4.9) follows directly by replacing F, G, f and g in equation (3.19) with $T_q(2;F)$, $T_q(2;G)$, ϕ and θ, respectively. □

We finish this section by taking the convolution product of the expressions in equation (3.29), first with respect to w and then with respect to y.

Theorem 4.10. *Let* $q \in \mathbf{R} - \{0\}$, *let* $F \in \mathcal{B}(2;m)$ *be given by* (2.16), *and let* $G \in \mathcal{B}(2;m)$ *be given by* (3.7). *Then, for each* $w \in A$,

$$(4.10)\quad (T_q(2;\delta F(y|\cdot)) * T_q(2;\delta G(y|\cdot)))_q(w)$$
$$= \left(\delta F\ (y|w/\sqrt{2}) + \left[\left(\frac{i}{q}\right)^{1/2} + \left(\frac{i}{2q}\right)^{1/2}\right] \sum_{j=1}^{n} A_j f_j(\langle\vec{\alpha},y\rangle)\right)$$
$$\cdot \left(\delta G(y|w/\sqrt{2}) + \left[\left(\frac{i}{q}\right)^{1/2} - \left(\frac{i}{2q}\right)^{1/2}\right] \sum_{l=1}^{n} A_l g_l(\langle\vec{\alpha},y\rangle)\right)$$
$$- \left(\frac{i}{2q}\right)^{1/2} \sum_{l=1}^{n} B_l f_l(\langle\vec{\alpha},y\rangle) g_l(\langle\vec{\alpha},y\rangle)$$

which, as a function of y, *is an element of* $\mathcal{B}(1;m-1)$.

Proof. The conclusions of this theorem follow immediately by using equations (3.29) and (3.19). □

Theorem 4.11. *Let q, G and F be as in Theorem* 3.4. *Then for all $w \in A$, the convolution product with respect to y of the expressions in equation* (3.29) *is given by the formula*

$$K_n(-iq)\int_{\mathbf{R}^n}\bigg(\sum_{j=1}^{n}\bigg[\langle\alpha_j,w\rangle+\bigg(\frac{i}{q}\bigg)^{1/2}A_j\bigg]f_j\bigg(\frac{\langle\vec{\alpha},y\rangle+\vec{u}}{\sqrt{2}}\bigg)\bigg) \cdot\bigg(\sum_{l=1}^{n}\bigg[\langle\alpha_l,w\rangle+\bigg(\frac{i}{q}\bigg)^{1/2}A_l\bigg]g_l\bigg(\frac{\langle\vec{\alpha},y\rangle-\vec{u}}{\sqrt{2}}\bigg)\bigg)H(-iq;\vec{u})\,d\vec{u} \tag{4.11}$$

which, as a function of y, is an element of $\mathcal{B}(2;m-1)$.

Proof. Formula (4.11) follows by a direct calculation using the right-hand side of equation (3.29) and then equations (3.6) and (2.12). □

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Department of Mathematics, Dankook University, Cheonan 330-714, Korea
Email address: sejchang@dankook.ac.kr

Department of Mathematics, Dankook University, Cheonan 330-714, Korea
Email address: jgchoi@dankook.ac.kr

Department of Mathematics, University of Nebraska, Lincoln, Nebraska, 68588-0130
Email address: dskoug@math.unl.edu