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CONVEXITY OF THE INTEGRAL ARITHMETIC MEAN OF A CONVEX FUNCTION

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ABSTRACT. In this paper it is proved that the integral arithmetic mean of a continuous function f is a convex function if and only if f is a convex function.

1. Introduction. For the convenience of the readers, we recall the main definitions as follows.

Definition 1. Let $D \subset \mathbf{R}^n$ be a convex set (if $n = 1$, then D is an interval). A function $f : D \to \mathbf{R}^n$ is called a convex function on D if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in D$.

Definition 2. Let I be an interval with nonempty interior. A function $F : I^n \to \mathbf{R}$ is called a Schur-convex function on I^n if

$$F(x_1, x_2, \dots, x_n) \leq F(y_1, y_2, \dots, y_n)$$

for any two n-tuples $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ in I^n, such that $x \prec y$ holds, i.e.,

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \dots, n-1$$

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and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component in x. F is called a strictly Schur-convex function on I if a strict inequality holds in Definition 2 whenever $x \prec y$ and x is not a permutation of y.

Definition 3. A set $D \subseteq \mathbf{R}^n$ is called a symmetric set if $xP \in D$ for any permutation P and all $x \in D$.

Definition 4. Let $D \subseteq \mathbf{R}^n$ be a symmetric set. A function $f : D \to \mathbf{R}$ is called a symmetric function if $f(xP) = f(x)$ for any permutation P and all $x \in D$.

The theory of convex functions and Schur-convex functions is an important research field in modern analysis and geometry. It can be used extensively in global Riemannian geometry [**6, 7**], operator inequalities [**1**], nonlinear PDEs of elliptic type [**10**], combinatorial optimization [**8**], isoperimetric problem for polytopes [**15**], linear regression [**13**], graphs and matrices [**2**], improperly posed problems [**14**], inequalities and extremum problems [**3**], nilpotent groups [**5**], global surface theory [**12**], and other related fields.

One of the focus problems in convex functions or Schur-convex functions theory is how to distinguish convex function or Schur-convex.

The following two criteria for convexity and Schur-convexity of functions were established in [**11**].

Theorem A. (1) *Let $I \subset \mathbf{R}$ be an open interval. If $f : I \to \mathbf{R}$ is a twice differentiable function, then f is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$.*

(2) *Let $D \subset \mathbf{R}^n$ $(n \geq 2, n \in \mathbf{N})$ be a convex set. If $f : D \to \mathbf{R}$ has continuous second partial derivatives, then f is convex if and only if the matrix*

$$L(x) = \begin{pmatrix} f''_{11} & f''_{12} & \cdots & f''_{1n} \\ f''_{21} & f''_{22} & \cdots & f''_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f''_{n1} & f''_{n2} & \cdots & f''_{nn} \end{pmatrix}$$

is positive semi-definite for all $x \in D$, *where*

$$f''_{ij} = \frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_i \partial x_j}$$

for $x = (x_1, x_2, \dots, x_n)$.

Theorem B. *Let* $D \subseteq \mathbf{R}^n (n \geq 2)$ *be a symmetric convex set. If* $f : D \to \mathbf{R}$ *is a symmetric convex function on* D, *then* f *is a Schur-convex function.*

The following Theorem C was obtained by Elezović and Pečarić [**4**] in 2000.

Theorem C. *Let* $I \subset \mathbf{R}$ *be an open interval and* $f : I \to \mathbf{R}$ *a continuous function. If*

$$(1) \qquad F(x,y) = \begin{cases} (1/y - x) \int_x^y f(t)\, dt & x, y \in I, x \neq y, \\ f(x) & x = y \in I, \end{cases}$$

then F *is Schur-convex on* I^2 *if and only if* f *is a convex function on* I.

The main purpose of this paper is to improve Theorem C to the following result.

Theorem. *In Theorem* C, *the condition that* F *be Schur-convex can be replaced by the condition* F *be convex.*

2. Proof of theorem. First we shall introduce and establish the following three lemmas, which will be used in the proof of our main result.

Lemma 1 [**9**]. *Let* $f : I \to \mathbf{R}$ *be convex on an open interval* I. *For any subinterval* $[a,b]$ *of* I, *there exists a sequence* $\{f_n\}$ *of convex infinitely differentiable functions* f_n *which converges uniformly to* f *on* $[a,b]$.

The following Lemma 2 can be derived directly from the definition of convex function.

Lemma 2. *Let $D \subset \mathbf{R}^n$ be a convex set and $f_n : D \to \mathbf{R}$ $(n = 1, 2, \dots)$ a sequence of continuous convex functions. If $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in D$, then f is convex on D.*

Lemma 3. *Let $I \subset \mathbf{R}$ be an open interval and $f : I \to \mathbf{R}$ a convex function with continuous second derivatives on I. Then the function F defined by (1) is convex on I^2.*

Proof. For any $x, y \in I$, the argument will be divided into the following three cases.

Case 1. $y > x$. Then simple computations yield

$$F''_{11} = (y-x)^{-3}\left[2\int_x^y f(t)dt - (y-x)^2 f'(x) - 2(y-x)f(x)\right],$$
$$F''_{12} = F''_{21} = (y-x)^{-2}[f(y)+f(x)] - 2(y-x)^{-3}\int_x^y f(t)\,dt$$

and

$$F''_{22} = (y-x)^{-1}f'(y) - 2(y-x)^{-2}f(y) + 2(y-x)^{-3}\int_x^y f(t)\,dt.$$

Let $g(t) = (t-x)^2 f'(x) + 2(t-x)f(x) - 2\int_x^t f(u)\,du$, $t \in (x, y)$. Then

$$g'(t) = 2(t-x)\left[f'(x) - \frac{f(t)-f(x)}{t-x}\right],$$

making use of the Lagrange mean value theorem we know that there exist $\xi_1(t) \in (x, t)$ and $\xi_2(t) \in (x, \xi_1(t))$ such that

$$\begin{aligned} g'(t) &= 2(t-x)[f'(x) - f'(\xi_1(t))] \\ &= -2(t-x)[\xi_1(t) - x]f''(\xi_2(t)). \end{aligned} \tag{2}$$

Equation (2) and Theorem A (1) imply that $g(t)$ is decreasing in $[x, y]$; hence $g(y) \le g(x) = 0$ and this leads to

$$F''_{11} \ge 0. \tag{3}$$

Let $L(x,y) = \begin{pmatrix} F''_{11} & F''_{12} \\ F''_{21} & F''_{22} \end{pmatrix}$. Then from (3) we clearly see that $L(x,y)$ is a positive semi-definite matrix if and only if

$$F''_{11}F''_{22} - F''_{12}F''_{21} \geq 0.$$

This is equivalent to

$$\begin{aligned}(4)\quad & [f(y)-f(x)]^2 + (y-x)^2 f'(y)f'(x) - 2(y-x) \\ & \times [f(y)f'(x) - f'(y)f(x)] - 2[f'(y)-f'(x)]\int_x^y f(t)\,dt \leq 0.\end{aligned}$$

Next, let

$$(5)\quad \begin{aligned} h(t) = {} & [f(t)-f(x)]^2 + (t-x)^2 f'(t)f'(x) - 2(t-x) \\ & \times [f(t)f'(x) - f'(t)f(x)] \\ & - 2[f'(t)-f'(x)]\int_x^t f(u)\,du, \quad t \in [x,y], \end{aligned}$$

then simple computations yield

$$(6)\qquad h(x) = 0$$

and

$$(7)\qquad h'(t) = f''(t)\left[(t-x)^2 f'(x) + 2(t-x)f(x) - 2\int_x^t f(u)\,du\right].$$

Now, taking

$$(8)\quad H(t) = (t-x)^2 f'(x) + 2(t-x)f(x) - 2\int_x^t f(u)\,du, \quad t \in [x,y],$$

then simple computations yield

$$(9)\qquad H(x) = 0,$$

$$H'(t) = 2(t-x)f'(x) + 2f(x) - 2f(t),$$

$$(10)\qquad H'(x) = 0$$

and

$$H''(t) = 2[f'(x) - f'(t)].$$

Theorem A(1) and the convexity of f on I imply that $H''(t) \leq 0$ for all $t \in [x, y]$. Therefore, (4) follows from the convexity of f and Theorem A (1) together with (5)–(10); hence, $L(x, y)$ is a positive semi-definite matrix in this case.

Case 2. $y < x$. The argument is trivial in this case follows from the symmetry of F and Case 1.

Case 3. $y = x$. From the definition of $F(x, y)$ we get

$$F_1'(x, x) = F_2'(x, x) = \frac{1}{2} f'(x),$$

$$F_{21}''(x, x) = F_{12}''(x, x) = \frac{1}{6} f''(x), \tag{11}$$

$$F_{11}''(x, x) = F_{22}''(x, x) = \frac{1}{3} f''(x). \tag{12}$$

Equations (11) and (12) imply that $L(x, x)$ is a positive semi-definite matrix.

Lemma 3 follows from the above three cases and Theorem A (2).

Proof of theorem. Necessity. If $F(x, y)$ is a convex function on I^2, then the symmetry of F on symmetric convex set I^2 and Theorem B together with Theorem C imply that f is a convex function on I.

Sufficiency. For any (x_1, y_1) and $(x_2, y_2) \in I^2$, there exist $a, b \in I$ such that (x_1, y_1) and $(x_2, y_2) \in [a, b]^2$. Since f is a continuous convex function on I, there exist convex functions sequences $\{f_n\}$, $n = 1, 2, \ldots$, with continuous second derivatives and converge to f uniformly on $[a, b]$. Let

$$F_n(x, y) = \begin{cases} (1/y - x) \int_x^y f_n(t)\, dt & x, y \in [a, b],\ x \neq y, \\ f_n(x) & x = y \in [a, b]. \end{cases}$$

Then Lemma 3 implies that F_n is a convex function on $[a,b]^2$ and by Lemma 2 we know that

$$G(x,y) = \lim_{n\to\infty} F_n(x,y) = \begin{cases} (1/y-x)\int_x^y f_n(t)\,dt & x,y\in[a,b],\ x\neq y,\\ f(x) & x=y\in[a,b], \end{cases}$$

is a convex function on $[a,b]^2$. Hence, we have

$$F\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) = G\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) \le \frac{G(x_1,y_1)+G(x_2,y_2)}{2} = \frac{F(x_1,y_1)+F(x_2,y_2)}{2}.$$

From the above inequality and Definition 1 we conclude that F is convex function on I^2.

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