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MINIMAL PRIME IDEALS AND SEMISTAR OPERATIONS

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Dedicated to Professor Rahim Zaare-Nahandi on the occasion of his 60th birthday

ABSTRACT. Let R be a commutative integral domain, let $\star$ be a semistar operation of finite type on R, and let I be a quasi-$\star$-ideal of R. We show that, if every minimal prime ideal of I is the radical of a $\star$-finite ideal, then the set $\operatorname{Min}(I)$ of minimal prime ideals over I is finite.

1. Introduction. In [**12**, Theorem 88], Kaplansky proved that: Let R be a commutative ring satisfying the *ascending chain condition* (a.c.c. for short) on radical ideals, and let I be an ideal of R. Then there are only a finite number of prime ideals minimal over I.

This result was generalized in [**9**, Theorem 1.6] by showing that (see also [**1**]): Let R be a commutative ring with identity, and let $I \neq R$ be an ideal of R. If every prime ideal minimal over I is the radical of a finitely generated ideal, then there are only finitely many prime ideals minimal over I.

In 1994, Okabe and Matsuda [**13**] introduced the concept of *semistar operation* to extend the notion of classical *star operations* as described in [**8**, Section 32]. Star operations have been proven to be an essential tool in *multiplicative ideal theory*, allowing one to study different classes of integral domains. Semistar operations, thanks to a higher flexibility than star operations, permit a finer study and new classifications of special classes of integral domains.

Throughout this note let R be a commutative integral domain, with identity, and let K be its quotient field.

The purpose of this note is to prove the semistar analogue of Kaplansky's [**12**, Theorem 88] and Gilmer and Heinzer's [**9**, Theorem 1.6] results. More precisely we prove the following theorem.

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Theorem. *Let $\star$ be a semistar operation of finite type on the integral domain R, and let I be a quasi-$\star$-ideal of R. If every minimal prime ideal of I is the radical of a $\star$-finite ideal, then I has finitely many minimal prime ideals.*

Now we recall some definitions and properties related to semistar operations. Let $\overline{\mathcal{F}}(R)$ denote the set of all nonzero R-submodules of K, and let $\mathcal{F}(R)$ be the set of all nonzero *fractional* ideals of R, i.e., $E \in \mathcal{F}(R)$ if $E \in \overline{\mathcal{F}}(R)$, and there exists a nonzero $r \in R$ with $rE \subseteq R$. Let $f(R)$ be the set of all nonzero finitely generated fractional ideals of R. Then, obviously $f(R) \subseteq \mathcal{F}(R) \subseteq \overline{\mathcal{F}}(R)$. A semistar operation on R is a map $\star : \overline{\mathcal{F}}(R) \to \overline{\mathcal{F}}(R)$, $E \to E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathcal{F}}(R)$, the following properties hold:

$\star_1$ $(xE)^\star = xE^\star$;

$\star_2$ $E \subseteq F$ implies that $E^\star \subseteq F^\star$;

$\star_3$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$,

cf., for instance, [**13**]. Recall that, given a semistar operation $\star$ on R, for all $E, F \in \overline{\mathcal{F}}(R)$, the following basic formulas follow easily from the axioms:

(1) $(EF)^\star = (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star$;

(2) $(E+F)^\star = (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star$;

(3) $(E \cap F)^\star \subseteq E^\star \cap F^\star = (E^\star \cap F^\star)^\star$, if $E \cap F \neq 0$.

cf., for instance, [**13**, Proposition 5].

A *(semi)star operation* is a semistar operation that, when restricted to $\mathcal{F}(R)$, is a star operation (in the sense of [**8**, Section 32]). It is easy to see that a semistar operation $\star$ on R is a (semi)star operation if and only if $R^\star = R$.

Let $\star$ be a semistar operation on the integral domain R. An ideal I of R is called a *quasi-$\star$-ideal* of R if $I^\star \cap R = I$. It is easy to see that, for any ideal I of R, the ideal $I^\star \cap R$ is a quasi-$\star$-ideal. An ideal is said to be a *quasi-$\star$-prime*, if it is prime and a quasi-$\star$-ideal. Let $\star$ be a semistar operation, put $E^{\star_f} = \cup F^\star$ where the union taken over all finitely generated $F \subseteq E$, for every $E \in \overline{\mathcal{F}}(R)$. It is easy to see that $\star_f$ defines a semistar operation on R called the *semistar operation of finite type associated to* $\star$. Note that there is the equality $(\star_f)_f = \star_f$. A semistar operation $\star$ is said to be *of finite type* if $\star = \star_f$; in particular, $\star_f$ is of finite type. An element $E \in \overline{\mathcal{F}}(R)$ is said to be *$\star$-finite* if $E^\star = F^\star$ for some $F \in f(R)$. Note that if I is an ideal of R, then $I^\star$ is an ideal of the overring $R^\star$ of R. Denote by $\star$-Min$\,(I)$ the set of *minimal quasi-$\star$-prime* ideals over I. So that when $\star = d$ is the identity semistar operation, then d-Min$\,(I) =$ Min$\,(I)$. It can be seen that if I is an ideal of R, then $\star$-Min$\,(I) \subseteq$ Min$\,(I^\star \cap R)$. Using [**4**, Lemma 2.3 (d)] we have each minimal prime over a quasi-$\star_f$-ideal is a quasi-$\star_f$-ideal. Therefore, If I is a quasi-$\star_f$-ideal, we have $\star_f$-Min$\,(I) =$ Min$\,(I)$.

The most widely studied (semi)star operations on R have been the identity d_R, v_R, and $t_R := (v_R)_f$ operations, where $E^{v_R} := (E^{-1})^{-1}$, with $E^{-1} := (R : E) := \{x \in K | xE \subseteq R\}$. Our terminology and notation come from [**8**].

2. Main result. Before proving the main result of this paper, we need a lemma.

Lemma 2.1. *Suppose that $\star$ is a semistar operation of finite type on the integral domain R, and that I is a quasi-$\star$-ideal. Then $\sqrt{I}^\star \cap R = \sqrt{I}$, that is, $\sqrt{I}$ is also a quasi-$\star$-ideal of R.*

Proof. Since $\sqrt{I} \subseteq \sqrt{I}^\star \cap R$, it is enough to show that $\sqrt{I}^\star \cap R \subseteq \sqrt{I}$. Let $x \in \sqrt{I}^\star \cap R$. Then, for every $P \in$ Min$\,(I)$, we have $xR^\star \subseteq \sqrt{I}^\star \subseteq P^\star$. Since P is a quasi-$\star$-prime ideal of R by [**4**, Lemma 2.3 (d)], we obtain that $x \in P$. Hence $x \in \sqrt{I}$ as desired. $\square$

Remark 2.2. Suppose that $\star$ is a semistar operation of finite type on the integral domain R, and that I is a quasi-$\star$-ideal. Then $\sqrt{I^\star} \cap R = \sqrt{I}$. Indeed, suppose that $x \in \sqrt{I^\star} \cap R$, then there is a positive integer n such that $x^n \in I^\star$. Since I is a quasi-$\star$-ideal, and $x \in R$, we obtain that $x^n \in I$. Hence, $x \in \sqrt{I}$. Consequently, we have $\sqrt{I^\star} \cap R \subseteq \sqrt{I} \subseteq \sqrt{I^\star \cap R} = \sqrt{I^\star} \cap R$, which gives us the desired equality.

We next give the main result of this paper.

Theorem 2.3. *Let $\star$ be a semistar operation of finite type on the integral domain R, and let I be a quasi-$\star$-ideal of R. If every minimal prime ideal of I is the radical of a $\star$-finite ideal, then I has finitely many minimal prime ideals.*

Proof. Note that $\sqrt{I}$ is a radical quasi-$\star$-ideal by Lemma 2.1. Hence since we have $\mathrm{Min}\,(I) = \mathrm{Min}\,(\sqrt{I})$, it is harmless to assume that I is a radical quasi-$\star$-ideal.

Let $S = \{P_1 \cdots P_n |$ each P_i is a prime ideal minimal over $I\}$. If for some $C = P_1 \cdots P_n \in S$ we have $C^\star \subseteq I^\star$, hence $C \subseteq C^\star \cap R \subseteq I^\star \cap R = I$. Then any prime ideal minimal over I contains some P_i, so $\{P_1, \cdots, P_n\}$ is the set of minimal prime ideals of I. Hence we may assume that $C^\star \nsubseteq I^\star$ for each $C \in S$. Consider the set $\mathcal{A}$ consisting of all radical quasi-$\star$-ideals J of R containing I such that $C^\star \nsubseteq J^\star$ for each $C \in S$. Since $I \in \mathcal{A}$ we have $\mathcal{A} \neq \varnothing$. The set $\mathcal{A}$ is partially ordered under inclusion $\subseteq$, and we show that it is inductive under this ordering. Let $\{J_\alpha\}_{\alpha\in\Gamma}$ be a chain in $\mathcal{A}$. Put $J = \cup J_\alpha$ so that J is a radical quasi-$\star$-ideal of R containing I such that $J^\star = (\cup J_\alpha)^\star = \cup J_\alpha^\star$, in which the second equality holds, since $\star$ is a semistar operation of finite type. Now assume that $C^\star \subseteq J^\star$ for some $C \in S$. Suppose that $C = P_1 \cdots P_n$, and that $P_i = \sqrt{L_i}$ for some $\star$-finite ideal L_i of R, for $i = 1, \dots, n$. Let F_is be finitely generated ideals of R, such that $L_i^\star = F_i^\star$, for $i = 1, \dots, n$. Now consider $(F_1 \cdots F_n)^\star = (L_1 \cdots L_n)^\star \subseteq (P_1 \cdots P_n)^\star \subseteq \cup J_\alpha^\star$, which implies that $F_1 \cdots F_n \subseteq \cup J_\alpha$. Therefore there exists an index $\alpha \in \Gamma$ such that $F_1 \cdots F_n \subseteq J_\alpha$ and hence $(L_1 \cdots L_n)^\star = (F_1 \cdots F_n)^\star \subseteq J_\alpha^\star$. Thus $L_1 \cdots L_n \subseteq J_\alpha$, and so we have $P_1 \cdots P_n \subseteq \sqrt{L_1 \cdots L_n} \subseteq \sqrt{J_\alpha} = J_\alpha$. Consequently $(P_1 \cdots P_n)^\star \subseteq J_\alpha^\star$ which is impossible. Now Zorn's lemma gives us a maximal element Q of $\mathcal{A}$. One can actually assume that $Q \neq R$. We show that Q is a prime ideal of R. To this end, let a, b be two elements of R such that $ab \in Q$ and assume that $a, b \notin Q$. Since $Q \subsetneq (Q + aR) \subseteq \sqrt{(Q + aR)^\star \cap R}$, and $\sqrt{(Q + aR)^\star \cap R}$ is a radical quasi-$\star$-ideal (by Lemma 2.1) containing I, there exists an element $C_1 \in S$ such that $C_1^\star \subseteq \sqrt{(J + aR)^\star \cap R}^\star$. By the same reason there exists again an element $C_2 \in S$ such that $C_2^\star \subseteq \sqrt{(Q + bR)^\star \cap R}^\star$.

Therefore we have

$$\begin{aligned}(C_1C_2)^\star = (C_1^\star C_2^\star)^\star &\subseteq (\sqrt{(Q+aR)^\star \cap R}^\star \sqrt{(Q+bR)^\star \cap R}^\star)^\star \\ &= (\sqrt{(Q+aR)^\star \cap R}\sqrt{(Q+bR)^\star \cap R})^\star \\ &= (\sqrt{((Q+aR)^\star \cap R)((Q+bR)^\star \cap R)})^\star \\ &\subseteq (\sqrt{((Q+aR)^\star (Q+bR)^\star) \cap R})^\star \\ &= (\sqrt{((Q^\star)^2 + aQ^\star + bQ^\star + abR^\star) \cap R})^\star \\ &\subseteq (\sqrt{Q^\star \cap R})^\star = (\sqrt{Q})^\star = Q^\star,\end{aligned}$$

which is a contradiction. Therefore Q is a prime ideal of R. But, since $I \subseteq Q$, it contains a prime ideal P minimal over I by [**12**, Theorem 10]. Thus $P \in S$ and $P^\star \subseteq Q^\star$, a contradiction. □

Defining different semistar operations, we can derive different corollaries.

Corollary 2.4 (Gilmer and Heinzer [**9**, Theorem 1.6] and Anderson [**1**]). *Let R be an integral domain, and let I be an ideal of R. If each minimal prime of the ideal I is the radical of a finitely generated ideal, then I has only finitely many minimal primes.*

The following result proved recently by El Baghdadi and Gabelli [**6**, Proposition 1.4] over PvMDs. They used the lattice isomorphism between the lattice of t-ideals of R and the lattice of ideals of the t-Nagata ring of R over PvMDs.

Corollary 2.5. *Let I be a t-ideal of the integral domain R. If each minimal prime ideal of I is the radical of a t-finite ideal, then* $\mathrm{Min}(I)$ *is finite.*

Corollary 2.6. *Suppose that R has a Noetherian overring S. If I is an ideal of R such that $IS \cap R = I$, then* $\mathrm{Min}(I)$ *is finite.*

Proof. Define a semistar operation $\star$, by $E^\star = ES$, for each $E \in \overline{\mathcal{F}}(R)$. Thus $I = IS \cap R = I^\star \cap R$ is a quasi-$\star$-ideal of R. Note that

$\star$ is a semistar operation of finite type. Let $P \in \text{Min}(I)$. Since S is a Noetherian ring, we have $P^\star = PS = (x_1, \dots, x_n)S = (x_1, \dots, x_n)^\star$, for some elements $x_1, \dots, x_n$ of P. This means that P is a $\star$-finite ideal. Now the result is clear from Theorem 2.3. □

Recall that a semistar operations $\star$ on the integral domain R is called *stable*, if $(E \cap F)^\star = E^\star \cap F^\star$ for every E, F in $\overline{\mathcal{F}}(R)$. Again recall from the introduction that $\star$-Min(I) is the set of quasi-$\star$-prime ideals minimal over I.

Lemma 2.7. *Suppose that $\star$ is a semistar operation stable and of finite type on the integral domain R, and that I is a nonzero ideal of R, such that $I^\star \cap R \neq R$. Then $\star$-Min$(I) = $ Min$(I^\star \cap R)$. In particular,*

$$\sqrt{I^\star} \cap R = \sqrt{I^\star \cap R} = \cap_{P \in \star\text{-Min}(I)} P.$$

Proof. One sees easily that $\star$-Min$(I) \subseteq \text{Min}(I^\star \cap R)$. For the reverse inclusion, let $P \in \text{Min}(I^\star \cap R)$. So that $I \subseteq I^\star \cap R \subseteq P$. Choose by [**12**, Theorem 10] a prime ideal Q minimal over I contained in P. Note that Q is a quasi-$\star$-ideal, since it is contained in P ([**7**, Corollary 3.9, Lemma 4.1, and Remark 4.5]). Then $I^\star \subseteq Q^\star \subseteq P^\star$ and so $I \subseteq I^\star \cap R \subseteq Q^\star \cap R = Q \subseteq P^\star \cap R = P$. Thus $Q = P$, since P is minimal over $I^\star \cap R$. □

Corollary 2.8. *Let $\star$ be a semistar operation stable and of finite type on the integral domain R, and let I be a nonzero ideal of R, such that $I^\star \cap R \neq R$. If every quasi-$\star$-prime ideal minimal over I is the radical of a $\star$-finite ideal, then $\star$-*Min(I) *is finite.*

Proof. By the above lemma we have $\star$-Min$(I) = \text{Min}(I^\star \cap R)$. Thus every prime ideal minimal over $I^\star \cap R$ is the radical of a $\star$-finite ideal. Noting that $I^\star \cap R$ is a quasi-$\star$-ideal of R, and using Theorem 2.3, we have $\star$-Min(I) is a finite set. □

In [**5**, Section 3] the authors defined and studied the *semistar Noetherian domains*, that is, domains having the ascending chain condition on quasi semistar ideals. In [**14**] Picozza generalizes several of the classical results that hold in Noetherian domains to the case of semistar operations stable and of finite type, for instance, Cohen's theorem, primary decomposition, principal ideal theorem, Krull intersection theorem, etc.

Corollary 2.9 ([**14**, Proposition 2.4 (2)]). *Suppose that $\star$ is a stable semistar operation of finite type on the integral domain R, and that R is a $\star$-Noetherian domain. Then $\star$-$\mathrm{Min}\,(I)$ is finite for every ideal I of D.*

Next we give equivalent conditions that every quasi-$\star$-prime of R is the radical of a $\star$-finite ideal.

Proposition 2.10 *Let $\star$ be a semistar operation of finite type on the integral domain R, the following then are equivalent:*

(1) *Each quasi-$\star$-prime is the radical of a $\star$-finite ideal.*

(2) *Each radical quasi-$\star$-ideal is the radical of a $\star$-finite ideal.*

(3) *R satisfies the a.c.c. on radical quasi-$\star$-ideals.*

Proof. (1) $\Rightarrow$ (2). Consider the following set.

$$\mathcal{A}=\{I | I=\sqrt{I}, I=I^{\star}\cap R, \text{ which is not the radical of a } \star\text{-finite ideal}\}.$$

If $\mathcal{A}\neq\varnothing$, let $\beta=\{I_\alpha\}$ be a chain of elements of $\mathcal{A}$. Put $I=\cup I_\alpha$. Hence I is a radical quasi-$\star$-ideal of R such that $I^\star=(\cup I_\alpha)^\star=\cup I_\alpha^\star$. Suppose that $I=\sqrt{L}$ for some $\star$-finite ideal L. Let $L^\star=F^\star$ for some $F\in f(R)$. So that $F^\star=L^\star\subseteq I^\star=\cup I_\alpha^\star$ which implies that $F\subseteq\cup I_\alpha$. Therefore, there is an index α such that $F\subseteq I_\alpha$. Consequently, $L^\star=F^\star\subseteq I_\alpha^\star$ and hence $L\subseteq I_\alpha$. So we obtain that $I_\alpha=\sqrt{L}$, which is impossible. Hence, $I\in\mathcal{A}$. Thus by Zorn's lemma $\mathcal{A}$ has a maximal element P. Let a, b be two elements of R such that $ab\in P$, and suppose that $a, b\notin P$. Since $P\subsetneq(P+aR)\subseteq\sqrt{(P+aR)^\star\cap R}$, and $\sqrt{(P+aR)^\star\cap R}$ is a radical quasi-$\star$-ideal (by Lemma 2.1), we have $\sqrt{(P+aR)^\star\cap R}=\sqrt{L}$, for some $\star$-finite ideal. By the same reason $\sqrt{(P+bR)^\star\cap R}=\sqrt{N}$, where N is a $\star$-finite ideal. The same proof as Theorem 2.3 shows that $P=\sqrt{LN}$, which is impossible, since LN is a $\star$-finite ideal. Hence, P is a quasi-$\star$-prime, a contradiction. Hence, $\mathcal{A}=\varnothing$.

(2) $\Rightarrow$ (3). Suppose that $(I_n)_{n\in\mathbf{N}}$ is an ascending chain of radical quasi-$\star$-ideals, and set $I = \cup_{n\in\mathbf{N}} I_n$. Then I is a radical quasi-$\star$-ideal. Hence $I = \sqrt{L}$ for some $\star$-finite ideal L. So, there is an integer n_0 such that $I_{n_0} = \sqrt{L} = I$. Hence $(I_n)_{n\in\mathbf{N}}$ is stationary.

(3) $\Rightarrow$ (1). Suppose that (1) is false. Then we can construct a chain $(I_n)_{n\in\mathbf{N}}$ of radical quasi-$\star$-ideals strictly ascending. Indeed, let P be a quasi-$\star$-prime ideal which is not the radical of a $\star$-finite ideal. Set $I_1 = \sqrt{(x)}$, where $0 \neq x \in P$. Given $I_n = \sqrt{(x_1, \ldots, x_n)^\star \cap R}$, where $x_1, \ldots, x_n \in P$, then $I_{n+1} = \sqrt{(x_1, \ldots, x_n, x_{n+1})^\star \cap R}$, where $x_{n+1} \in P \backslash I_n$. □

Corollary 2.11 (Kaplansky [**12**, Theorem 88]). *Suppose that $\star$ is a semistar operation of finite type on the integral domain R. If R satisfies the a.c.c. on radical quasi-$\star$-ideals, then $\star$-*Min(I) *is finite for every ideal I of R.*

Proof. Note $\star$-Min$(I) \subseteq$ Min$(I^\star \cap R)$ and use Theorem 2.3. □

Remark 2.12. (1) One can prove Theorem 2.3 for arbitrary rings with zero divisors. Let R be a commutative ring, with total quotient ring $T(R)$. Let $\mathcal{F}(R)$ denote the set of all R-submodules of $T(R)$. Suppose an operation $* : \mathcal{F}(R) \to \mathcal{F}(R)$, $E \to E^*$, satisfies, for all $E, F \in \mathcal{F}(R)$, and for all $x \in T(R)$, the following:

$*_1$ $xE^* \subseteq (xE)^*$ and if x is regular, then $xE^* = (xE)^*$;

$*_2$ $E \subseteq F$ implies that $E^* \subseteq F^*$;

$*_3$ $E \subseteq E^*$ and $E^{**} := (E^*)^* = E^*$.

Then from these axioms the following directly follow:

(i) $(EF)^* = (E^*F)^* = (EF^*)^* = (E^*F^*)^*$;

(ii) $(E+F)^* = (E^*+F)^* = (E+F^*)^* = (E^*+F^*)^*$;

(iii) $(E \cap F)^* \subseteq E^* \cap F^* = (E^* \cap F^*)^*$.

It is clear that any semistar operation satisfies these axioms.

It is routine to see that [**10**, Lemma 3.3] the v-operation satisfies, these axioms, where $E^v = (E^{-1})^{-1}$, in which $E^{-1} = (R : E) = \{x \in T(R) | xE \subseteq R\}$, for $E \in \mathcal{F}(R)$.

By this operation Theorem 2.3 is true for rings with zero divisors.

(2) It is interesting to note that if we take R to be the ring of all sequences from $\mathbf{Z}/2\mathbf{Z}$ that are eventually constant, with pointwise addition and multiplication, then R is a zero-dimensional Boolean ring with minimal prime ideals $P_i = \{\{a_n\} \in R | a_i = 0\}$ and $P_\infty = \{\{a_n\} \in R | a_n = 0 \text{ for large } n\}$ and each P_i is principal but P_∞ is not finitely generated. Thus, while R has infinitely many minimal prime ideals, only one is not the radical of a finitely generated ideal.

In the rest of the paper we will define a class of rings, that satisfy the a.c.c. on radical quasi-$\star$-ideals.

Let R be a commutative ring. An ideal I of R is called an *ideal of strong finite type* (*SFT-ideal* for short) if there exist a finitely generated ideal $J \subseteq I$ and a positive integer k such that $a^k \in J$ for each $a \in I$. The ring R is called an *SFT-ring* if each ideal of R is an SFT-ideal. These concepts were introduced by Arnold in [**2**]. The condition that R is an SFT-ring plays a key role in computing the Krull dimension of the power series ring $R[[X]]$ over R. In [**11**], Kang and Park defined and studied the $\star = t$ analogue of SFT-rings. Now we define the more general semistar-SFT-rings.

Let R be a domain and $\star$ a semistar operation on it. We define a nonzero ideal I of R to be a *$\star$-SFT-ideal* if there exist a finitely generated ideal $J \subseteq I$ and a positive integer k such that $a^k \in J^\star$ for each $a \in I^{\star_f}$. The ring R is said to be a *$\star$-SFT-ring* if each nonzero ideal of R is a $\star$-SFT-ideal. Obvious examples of a $\star$-SFT-ring are $\star$-Noetherian domains.

Proposition 2.13. *Suppose that $\star$ is a semistar operation of finite type on the integral domain R. If R is a $\star$-SFT-ring, then R satisfies the a.c.c. on radical quasi-$\star$-ideals.*

Proof. Let P be a quasi-$\star$-prime ideal. Since P is a $\star$-SFT-ideal, there is a finitely generated subideal $J \subseteq P$ such that $\sqrt{P^\star} = \sqrt{J^\star}$. Now consider

$$P = \sqrt{P} = \sqrt{P^\star \cap R} = \sqrt{P^\star} \cap R = \sqrt{J^\star} \cap R = \sqrt{J^\star \cap R}.$$

Since $J^\star \cap R$ is an $\star$-finite ideal, the result follows by Proposition 2.10. $\square$

Corollary 2.14. *Each quasi-$\star$-ideal of a $\star$-SFT-ring R, has only finitely many minimal prime ideals.*

We close the paper with the following characterization of $\star$-SFT-rings.

Proposition 2.15 ([**3**, Proposition 2.2]). *Suppose that $\star$ is a semistar operation of finite type on the integral domain R. Then R is a $\star$-SFT-ring if and only if each quasi-$\star$-prime ideal of R is a $\star$-SFT-ideal.*

Proof. Suppose that R is not a $\star$-SFT-ring. Therefore the set

$$\mathcal{A} = \{I | I = I^\star \cap R, \text{ and is not a } \star\text{-SFT-ideal}\},$$

is not an empty set. The set $\mathcal{A}$ is partially ordered under inclusion, and is inductive under this ordering. By Zorn's lemma, $\mathcal{A}$ contains a maximal element P. Assume that a_1, a_2 are two elements of R such that $a_1 a_2 \in P$ and $a_1, a_2 \notin P$. Since $P \subsetneq (P + a_i R)^\star \cap R$, $(P + a_i R)^\star \cap R$ is a $\star$-SFT-ideal of R. Consequently, there exist a finitely generated ideal $L_i \subseteq (P + a_i R)^\star \cap R$, and a positive integer k_i such that $c^{k_i} \in (L_i)^\star$ for each $c \in (P + a_i R)^\star$. Let $L = L_1 L_2$ and $k = k_1 + k_2$. Then L is a finitely generated subideal of P such that $c^k = c^{k_1} c^{k_2} \in (L_1)^\star (L_1)^\star \subseteq (L_1 L_2)^\star$, for each $c \in P^\star$. Thus P is a $\star$-SFT-ideal, a contradiction. Therefore, P is a quasi-$\star$-prime ideal which is not a $\star$-SFT-ideal. □

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