

SOME REMARKS ON THE GENERALIZED TANAKA-WEBSTER CONNECTION OF A CONTACT METRIC MANIFOLD

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ABSTRACT. We find necessary and sufficient conditions for the bi-Legendrian connection ∇ associated to a bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ on a contact metric manifold (M, ϕ, ξ, η, g) being a metric connection with respect to the associated metric g , and then we give conditions ensuring that ∇ coincides with the (generalized) Tanaka-Webster connection of (M, ϕ, ξ, η, g) . Using these results, we give some interpretations of the Tanaka-Webster connection and we study the interplays between the Tanaka-Webster, the bi-Legendrian and the Levi Civita connection in a Sasakian manifold.

1. Introduction. In this paper we study some properties of the (generalized) Tanaka-Webster connection of a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$. This connection has been introduced by Tanno, cf. [15] as a generalization of the well-known connection defined at the end of the 1970's by Tanaka in [14] and, independently, by Webster in [17], in the context of CR-geometry. We put in relation the (generalized) Tanaka-Webster connection with the theory of Legendrian foliations on contact metric manifolds, cf. [10, 11, 13]. In particular, in [4] the author has attached to any Legendrian foliation a canonical connection, called *bi-Legendrian connection*, and in [5] he has found many applications of this connection in the theory of Legendrian foliations. In this paper we find conditions for which the Tanaka-Webster connection and the bi-Legendrian connection associated to a given Legendrian foliation coincide. We discuss some consequences of these results and give new interpretations both of Tanaka-Webster and of bi-Legendrian connections. For the latter, more precisely, we prove that the bi-Legendrian connection associated to a given Legendrian foliation on a contact manifold (M^{2n+1}, η) can be viewed as the Tanaka-Webster connection of a suitable Sasakian structure (ϕ, ξ, η, g) on

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M^{2n+1} , and they are *contact metric connections* in the sense of [12]. By virtue of this and other theorems which we will prove in Sections 3 and 4, compared with the analogous results in even dimension, we see that the Tanaka-Webster connection of a Sasakian manifold plays the role of the Levi Civita connection on a Kählerian manifold. Finally, in Section 5, we present some examples and counterexamples; for instance, we construct a Sasakian structure on S^3 , endowed with a nonflat bi-Legendrian structure, for which the Tanaka-Webster connection and the bi-Legendrian connection do not coincide.

The framework of this paper are contact metric manifolds. Recall that a *contact structure* on an odd-dimensional smooth manifold M^{2n+1} is given by a 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1} . It is well known that, given η , there exists a unique vector field ξ , called *Reeb vector field*, such that $d\eta(\xi, \cdot) = 0$ and $\eta(\xi) = 1$. The distribution defined by $\ker(\eta)$ is called the *contact distribution* and is denoted by \mathcal{D} . Then the tangent bundle of M^{2n+1} splits as the direct sum $TM^{2n+1} = \mathcal{D} \oplus \mathbf{R}\xi$. A Riemannian metric g is an *associated metric* for a contact form η if the following two conditions hold:

(i) $g(V, \xi) = \eta(V)$ for all $V \in \Gamma(TM^{2n+1})$, that is, ξ is orthogonal to \mathcal{D} ;

(ii) there exists a tensor field ϕ of type $(1, 1)$ on M^{2n+1} such that $\phi^2 = -I + \eta \otimes \xi$ and $d\eta(V, W) = g(V, \phi W)$ for all $V, W \in \Gamma(TM^{2n+1})$.

Moreover, from (i) and (ii), one can prove the following well-known relations (cf. [1]):

$$\begin{aligned} \phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W), \\ g(\phi V, W) = -g(V, \phi W) \end{aligned}$$

for all $V, W \in \Gamma(TM^{2n+1})$. We refer to (ϕ, ξ, η, g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold*. A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called a *Sasakian manifold* if it is *normal*, i.e., if the tensor field $N := [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically. In terms of the covariant derivative of ϕ the Sasakian condition is

$$(1) \quad (\widehat{\nabla}_V \phi)W = g(V, W)\xi - \eta(W)V,$$

where $\widehat{\nabla}$ denotes, and will denote in all this paper, the Levi Civita connection. In the study of contact metric manifolds it is useful to

define a tensor field h by $h = (1/2)\mathcal{L}_\xi\phi$. The operator h is symmetric, anti-commutes with ϕ , satisfies $h\xi = 0$ and it vanishes if and only if ξ is a Killing vector field (in this case the contact metric manifold in question is said to be *K-contact*; it is easy to show that a Sasakian manifold is also *K-contact*). Moreover,

$$(2) \quad \widehat{\nabla}_V \xi = -\phi V - \phi hV$$

holds for all $V \in \Gamma(TM^{2n+1})$. For the proofs of all these properties and more details on contact metric manifolds, we refer the reader to [1].

Given a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, there is defined on M^{2n+1} a canonical connection, called the *generalized Tanaka-Webster connection* or, simply, the Tanaka-Webster connection of the contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$. This connection is defined by the following formula:

$$(3) \quad \begin{aligned} * \nabla_V W &:= \widehat{\nabla}_V W + \eta(V)\phi W + \eta(W)(\phi V + \phi hV) \\ &\quad + d\eta(V + hV, W)\xi, \end{aligned}$$

for all $V, W \in \Gamma(TM^{2n+1})$. The torsion tensor of this connection has the following expression:

$$(4) \quad *T(V, W) = \eta(W)\phi hV - \eta(V)\phi hW + 2g(V, \phi W)\xi.$$

Tanno ([15]) found a characterization of this connection. He proved that the Tanaka-Webster connection $*\nabla$ is the unique linear connection on M^{2n+1} such that

$$(5) \quad \begin{aligned} (i) \quad &* \nabla g = 0, \quad * \nabla \eta = 0, \quad * \nabla \xi = 0, \\ (ii) \quad &>(* \nabla_V \phi)W = (\widehat{\nabla}_V \phi)W - g(V + hV, W)\xi + \eta(W)(V + hV), \\ (iii) \quad &*T(\xi, \phi V) = -\phi *T(\xi, V), \\ (iv) \quad &*T(Z, Z') = 2d\eta(Z, Z')\xi \quad \text{for all } Z, Z' \in \Gamma(\mathcal{D}). \end{aligned}$$

This connection agrees with the connection of Tanaka in [14] when the contact metric manifold is a strongly pseudo-convex (integrable) CR-manifold.

All manifolds considered here are assumed to be smooth, i.e., of the class C^∞ , and connected; we denote by $\Gamma(\cdot)$ the set of all sections of a corresponding bundle. We use the convention that $2u \wedge v = u \otimes v - v \otimes u$.

2. Bi-Legendrian connections. The contact condition $\eta \wedge (d\eta)^n \neq 0$ can be interpreted geometrically saying that the contact distribution is as far from being integrable as possible. One can prove that the maximal dimension of an integrable subbundle L of \mathcal{D} is n . In this case L necessarily satisfies the condition $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$, since $2d\eta(X, X') = X(\eta(X')) - X'(\eta(X)) - \eta([X, X']) = 0$, L being integrable. This motivates the following definition.

Definition 2.1. A *Legendrian distribution* on a contact manifold (M^{2n+1}, η) is an n -dimensional subbundle L of the contact distribution such that $d\eta(X, X') = 0$ for all $X, X' \in \Gamma(L)$. When L is integrable, it defines a *Legendrian foliation* of (M^{2n+1}, η) . Equivalently, a Legendrian foliation of (M^{2n+1}, η) is a foliation of M^{2n+1} whose leaves are n -dimensional C -totally real submanifolds of (M^{2n+1}, η) .

Legendrian foliations have been extensively investigated in recent years from various points of views, cf. [10, 11, 12]. In particular, Pang provided a classification of Legendrian foliations by means of a bilinear symmetric form $\Pi_{\mathcal{F}}$ on the tangent bundle of the foliation, defined by $\Pi_{\mathcal{F}}(X, X') = -(\mathcal{L}_X \mathcal{L}_{X'} \eta)(\xi)$. He called a Legendrian foliation \mathcal{F} *nondegenerate*, *degenerate* or *flat* according to the circumstance that the bilinear form $\Pi_{\mathcal{F}}$ is nondegenerate, degenerate or vanishes identically, respectively. A geometrical interpretation of this classification is given in the following lemma.

Lemma 2.2 [10]. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold foliated by a Legendrian foliation \mathcal{F} . Then*

- (a) \mathcal{F} is flat if and only if $[\xi, X] \in \Gamma(T\mathcal{F})$ for all $X \in \Gamma(T\mathcal{F})$,
- (b) \mathcal{F} is degenerate if and only if there exist $X \in \Gamma(T\mathcal{F})$ such that $[\xi, X] \in \Gamma(T\mathcal{F})$,
- (c) \mathcal{F} is nondegenerate if and only if $[\xi, X] \notin \Gamma(T\mathcal{F})$ for all $X \in \Gamma(T\mathcal{F})$.

Lemma 2.2 also allows us to extend the notion nondegenerateness, degenerateness and flatness to Legendrian distributions. Thus, we say that a Legendrian distribution L is *flat* if $[\xi, X] \in \Gamma(L)$ for all $X \in \Gamma(L)$, *degenerate* if there exist $X \in \Gamma(L)$ such that $[\xi, X] \in \Gamma(L)$, and *nondegenerate* if $[\xi, X] \notin \Gamma(L)$ for all $X \in \Gamma(L)$.

By an *almost bi-Legendrian manifold* we mean a contact manifold (M^{2n+1}, η) endowed with two transversal Legendrian distributions L_1 and L_2 . Thus, in particular, the tangent bundle of M^{2n+1} splits up as the direct sum $TM^{2n+1} = L_1 \oplus L_2 \oplus \mathbf{R}\xi$. When both L_1 and L_2 are integrable we speak of *bi-Legendrian manifold* ([4]). An (almost) bi-Legendrian manifold is said to be flat, degenerate or nondegenerate if and only if both the Legendrian distributions are flat, degenerate or nondegenerate, respectively.

In [4] it has been attached to any almost bi-Legendrian manifold a canonical connection which plays an important role in the study of almost bi-Legendrian manifolds.

Theorem 2.3 [4]. *Let $(M^{2n+1}, \eta, L_1, L_2)$ be an almost bi-Legendrian manifold. There exists a unique linear connection ∇ on M^{2n+1} such that*

$$\begin{aligned}
 & \text{(i) } \nabla L_1 \subset L_1, \quad \nabla L_2 \subset L_2, \quad \nabla(\mathbf{R}\xi) \subset \mathbf{R}\xi; \\
 & \text{(ii) } \nabla d\eta = 0; \\
 \text{(6) } & \text{(iii) } T(X, Y) = 2d\eta(X, Y)\xi, \text{ for all } X \in \Gamma(L_1), Y \in \Gamma(L_2), \\
 & \quad T(V, \xi) = [\xi, V_{L_1}]_{L_2} + [\xi, V_{L_2}]_{L_1}, \text{ for all } V \in \Gamma(TM^{2n+1}),
 \end{aligned}$$

where T denotes the torsion tensor of ∇ and X_{L_1} and X_{L_2} the projections of X onto the subbundles L_1 and L_2 of TM^{2n+1} , respectively.

Such a connection is called the *bi-Legendrian connection* of the almost bi-Legendrian manifold $(M^{2n+1}, \eta, L_1, L_2)$. We recall the explicit construction of this connection. First, for any two vector fields V and W on M^{2n+1} , let $H(V, W)$ be the unique section of \mathcal{D} such that

$$\text{(7) } \quad i_{H(V,W)}d\eta|_{\mathcal{D}} = (\mathcal{L}_V i_W d\eta)|_{\mathcal{D}},$$

that is, for every $Z \in \Gamma(\mathcal{D})$, $d\eta(H(V, W), Z) = V(d\eta(W, Z)) - d\eta(W, [V, Z])$. The existence and uniqueness of this section depend on

the fact that the 2-form $d\eta$ is nondegenerate on \mathcal{D} . The main properties of operator H are collected in the following lemma.

Lemma 2.4 [4]. *For every $f \in C^\infty(M^{2n+1})$ and $V, V', W, W' \in \Gamma(TM^{2n+1})$, we have:*

1. $H(V+V', W) = H(V, W) + H(V', W)$, $H(V, W+W') = H(V, W) + H(V, W')$,
2. $H(V, fW) = fH(V, W) + V(f)W_{\mathcal{D}}$,
3. $H(fV, W) = fH(V, W)$, if $d\eta(V, W) = 0$,
4. $H(V, \xi) = 0$, $H(\xi, W) = [\xi, W]_{\mathcal{D}}$,

where $W_{\mathcal{D}}$ denotes the projection of W onto the subbundle \mathcal{D} of TM^{2n+1} .

Using Lemma 2.4, one can define a connection ∇^{L_1} on bundle L_1 and a connection ∇^{L_2} on bundle L_2 setting, for all $W \in \Gamma(TM^{2n+1})$, $X \in \Gamma(L_1)$, $Y \in \Gamma(L_2)$,

$$\begin{aligned}\nabla_W^{L_1} X &:= H(W_{L_1}, X)_{L_1} + [W_{L_2}, X]_{L_1} + [W_{\mathbf{R}\xi}, X]_{L_1}, \\ \nabla_W^{L_2} Y &:= H(W_{L_2}, Y)_{L_2} + [W_{L_1}, Y]_{L_2} + [W_{\mathbf{R}\xi}, Y]_{L_2}.\end{aligned}$$

Moreover, we define a connection $\nabla^{\mathbf{R}\xi}$ on the line bundle $\mathbf{R}\xi$ requiring that $\nabla^{\mathbf{R}\xi}\xi = 0$, thus setting

$$\nabla_W^{\mathbf{R}\xi} Z := W(\eta(Z))\xi$$

for all $Z \in \Gamma(\mathbf{R}\xi)$. Then, from ∇^{L_1} , ∇^{L_2} and $\nabla^{\mathbf{R}\xi}$, one can define a global connection on M^{2n+1} by putting for any $V, W \in \Gamma(TM^{2n+1})$,

$$\nabla_W V := \nabla_W^{L_1} V_{L_1} + \nabla_W^{L_2} V_{L_2} + \nabla_W^{\mathbf{R}\xi} V_{\mathbf{R}\xi}.$$

In particular it follows that, for all $W \in \Gamma(TM^{2n+1})$, $\nabla_\xi W = [\xi, W_{L_1}]_{L_1} + [\xi, W_{L_2}]_{L_2} + \xi(\eta(W))\xi$. The above connection is called the bi-Legendrian connection associated to the almost bi-Legendrian manifold $(M^{2n+1}, \eta, L_1, L_2)$, and it can be characterized as the unique linear connection on M^{2n+1} satisfying (6). Further properties of this connection are collected in the following propositions.

Proposition 2.5 [4]. *The torsion tensor field of the bi-Legendrian connection of an almost bi-Legendrian manifold $(M^{2n+1}, \eta, L_1, L_2)$ is given by*

- (i) $T(X, X') = -[X, X']_{L_2}$ for $X, X' \in \Gamma(L_1)$,
- (ii) $T(Y, Y') = -[Y, Y']_{L_1}$ for $Y, Y' \in \Gamma(L_2)$,
- (iii) $T(X, Y) = 2d\eta(X, Y)\xi$ for $X \in \Gamma(L_1), Y \in \Gamma(L_2)$,
- (iv) $T(W, \xi) = [\xi, W_{L_1}]_{L_2} + [\xi, W_{L_2}]_{L_1}$ for $W \in \Gamma(TM^{2n+1})$.

In particular, if L_1 and L_2 are flat, then the terms $T(W, \xi)$ vanish, and if L_1 and L_2 are integrable then ∇ is torsion free along the leaves of the Legendrian foliations defined by L_1 and L_2 .

Proposition 2.6 [5]. *Let $(M^{2n+1}, \eta, L_1, L_2)$ be an almost bi-Legendrian manifold, and let ∇ denote the corresponding bi-Legendrian connection. Then the 1-form η is ∇ -parallel, the parallel transport along curves preserves the distributions L_1 and L_2 and, if L_1, L_2 are integrable and flat, the curvature tensor field of ∇ vanishes along the leaves of the foliations defined by $L_1, L_1 \oplus \mathbf{R}\xi, L_2$ and $L_2 \oplus \mathbf{R}\xi$.*

Proposition 2.6 gives a further geometrical interpretation of the flatness of a bi-Legendrian structure. It implies that the leaves of the Legendrian foliations in question admit a canonical flat affine structure. This always holds in symplectic geometry: for a symplectic manifold foliated by a Lagrangian foliation, Weinstein proved that each leaf possesses a natural flat connection; moreover, in case the symplectic manifold in question admits two transversal Lagrangian foliations, Hess proved that this connection extends to a symplectic connection on the tangent bundle called bi-Lagrangian connection (see the Appendix for more details).

Thus, the flatness of a bi-Legendrian structure, and more in general of a Legendrian distribution, seems to be a quite natural condition for comparing Legendrian and Lagrangian foliations. This can also be seen in the following results.

Proposition 2.7. *Every contact manifold (M^{2n+1}, η) endowed with a Legendrian distribution L embeds into a symplectic manifold (C, ω) endowed with a Lagrangian distribution L^C . Furthermore, L^C is integrable if and only if L is integrable and flat.*

Proof. Let $C = M^{2n+1} \times \mathbf{R}^+$ be the cone on M^{2n+1} , and let ω be the symplectic form on C defined by $\omega = e^t(d\eta - \eta \wedge dt) = d\lambda$, $\lambda = e^t\eta$. We set $L^C := L \oplus \mathbf{R}\xi$, considered as a distribution on C . Then for all $X, X' \in \Gamma(L)$ we have $\omega(X, X') = e^t d\eta(X, X') = 0$ and $\omega(X, \xi) = e^t d\eta(X, \xi) = 0$, from which it follows that L^C is Lagrangian. The final part of the statement is then a direct consequence of the definition of L^C . \square

Theorem 2.8 [5]. *Let (M^{2n+1}, η) be a regular contact manifold endowed with a Lagrangian distribution L . Then L projects onto a Lagrangian distribution on the space of leaves of M^{2n+1} by the 1-dimensional foliation defined by ξ if and only if L is flat. Furthermore, if M^{2n+1} is endowed with a flat almost bi-Legendrian structure (L_1, L_2) , the bi-Legendrian connection associated to (L_1, L_2) projects to the bi-Lagrangian connection associated to the projection of (L_1, L_2) on the space of leaves of M^{2n+1} .*

On the other hand, the flatness of Legendrian foliations implies also some strong topological obstructions, such as a vanishing phenomenon for the characteristic classes ([5]). Moreover, we remark that there are also several examples of nonflat Legendrian foliations (see for instance the following Example 2.10).

Any contact manifold (M^{2n+1}, η) endowed with a Legendrian distribution L admits a canonical almost bi-Legendrian structure. Indeed, let (ϕ, ξ, η, g) be a compatible contact metric structure. Then, from the relation $d\eta(\phi V, \phi W) = d\eta(V, W)$, it easily follows that $Q := \phi L$ is a Legendrian distribution on M^{2n+1} which is orthogonal to L . Thus, the tangent bundle of M^{2n+1} splits as the orthogonal sum $TM^{2n+1} = L \oplus Q \oplus \mathbf{R}\xi$. Q is called the *conjugate Legendrian distribution* of L , and in general is not integrable even if L is. Some conditions ensuring the integrability of the conjugate Legendrian distribution of a Legendrian foliation of a contact metric manifold are given in [10].

In this article we mainly study the bi-Legendrian connection ∇ associated to the almost bi-Legendrian structure (L, Q) , with $Q = \phi L$, on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$. We start by finding conditions which ensure that ∇ is a metric connection with respect to the associated metric g .

Proposition 2.9. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold endowed with a Legendrian distribution L . Let $Q := \phi L$ be the conjugate Legendrian distribution of L and ∇ the bi-Legendrian connection associated to (L, Q) . Then the following statements are equivalent:*

- (i) $\nabla g = 0$;
- (ii) $\nabla \phi = 0$;
- (iii) $\nabla_X X' = -(\phi[X, \phi X'])_L$ for all $X, X' \in \Gamma(L)$, $\nabla_Y Y' = -(\phi[Y, \phi Y'])_Q$ for all $Y, Y' \in \Gamma(Q)$ and the tensor h maps the subbundle L onto L and the subbundle Q onto Q ;
- (iv) g is a bundle-like metric with respect both to the distribution $L \oplus \mathbf{R}\xi$ and to the distribution $Q \oplus \mathbf{R}\xi$.

Furthermore, assuming L and Q integrable, (i)–(iv) are equivalent to the total geodesicity of the Legendrian foliations defined by L and Q .

Proof. In order to prove the equivalence of (i), (ii), (iii) and (iv), it is sufficient to prove the following implications: (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).

(i) \Rightarrow (ii). Since $d\eta$ is ∇ -parallel and $d\eta(\cdot, \cdot) = g(\cdot, \phi\cdot)$, under the assumption that the bi-Legendrian connection is metric, we have easily that

$$\begin{aligned} 0 &= (\nabla_V d\eta)(W, W') = (\nabla_V g)(W, \phi W') + g(W, (\nabla_V \phi)W') \\ &= g(W, (\nabla_V \phi)W') \end{aligned}$$

for all $V, W, W' \in \Gamma(TM^{2n+1})$, from which (ii) holds.

(ii) \Rightarrow (iii). Assuming $\nabla \phi = 0$, it follows that, for all $X, X' \in \Gamma(L)$, $0 = (\nabla_X \phi)X' = \nabla_X \phi X' - \phi \nabla_X X'$, from which, applying ϕ and taking into account that $\nabla \eta = 0$, we get $\nabla_X X' = \eta(\nabla_X X')\xi - \phi \nabla_X \phi X' = X(\eta(X'))\xi - \phi([X, \phi X']_Q) = -(\phi[X, \phi X'])_L$. In the same way one finds $\nabla_Y Y' = -(\phi[Y, \phi Y'])_Q$ for all $Y, Y' \in \Gamma(Q)$. Next, for all $X \in \Gamma(L)$, we have

$$\begin{aligned} (8) \quad 2(hX)_Q &= [\xi, \phi X]_Q - \phi([\xi, X]_L) = \nabla_\xi \phi X - \phi \nabla_\xi X = (\nabla_\xi \phi)X = 0, \end{aligned}$$

and, analogously, $2(hY)_L = (\nabla_\xi \phi)Y = 0$ for all $Y \in \Gamma(Q)$.

(iii) \Rightarrow (iv). Let us suppose that (iii) holds. Then for all $X, X', X'' \in \Gamma(L)$, we have

$$\begin{aligned}
(\mathcal{L}_X g)(\phi X', \phi X'') &= X(g(\phi X', \phi X'')) - g([X, \phi X'], \phi X'') - g(\phi X', [X, \phi X'']) \\
&= X(g(\phi X', \phi X'')) - g([X, \phi X']_Q, \phi X'') + g(X', (\phi[X, \phi X''])_L) \\
&= X(g(\phi X', \phi X'')) - g(\nabla_X \phi X', \phi X'') - g(X', \nabla_X X'') \\
&= X(d\eta(\phi X', X'')) - d\eta(\nabla_X \phi X', X'') - d\eta(\phi X', \nabla_X X'') \\
&= (\nabla_X d\eta)(\phi X', X'') = 0,
\end{aligned}$$

since $d\eta$ is ∇ -parallel. Next, note that by (8) we get $(\nabla_\xi \phi)X = 2(hX)_Q = 0$ for all $X \in \Gamma(L)$ and, analogously, $(\nabla_\xi \phi)Y = 2(hY)_L = 0$ for all $Y \in \Gamma(Q)$. Using this, we have for all $X', X'' \in \Gamma(L)$,

$$\begin{aligned}
(\mathcal{L}_\xi g)(\phi X', \phi X'') &= \xi(g(\phi X', \phi X'')) - g([\xi, \phi X']_Q, \phi X'') - g(\phi X', [\xi, \phi X'']_Q) \\
&= \xi(g(\phi X', \phi X'')) - g(\nabla_\xi \phi X', \phi X'') - g(\phi X', \nabla_\xi \phi X'') \\
&= \xi(g(\phi X', \phi X'')) - g(\nabla_\xi \phi X', \phi X'') - g(\phi X', \phi \nabla_\xi X'') \\
&= (\nabla_\xi d\eta)(\phi X', X'') = 0.
\end{aligned}$$

Arguing in a similar way, one can prove that $(\mathcal{L}_Y g)(X', X'') = 0$ and $(\mathcal{L}_\xi g)(X', X'') = 0$ for all $Y \in \Gamma(Q)$ and $X', X'' \in \Gamma(L)$.

(iv) \Rightarrow (i). Since the bi-Legendrian connection ∇ preserves the orthogonal decomposition $TM^{2n+1} = L \oplus Q \oplus \mathbf{R}\xi$, in order to prove that ∇ is metric, it is enough to check that $(\nabla_V g)(X', X'') = 0$ and $(\nabla_V g)(Y', Y'') = 0$ for all $V \in \Gamma(TM^{2n+1})$, $X', X'' \in \Gamma(L)$ and $Y', Y'' \in \Gamma(Q)$. Using (iv) we get

$$\begin{aligned}
(9) \quad (\nabla_X g)(X', X'') &= X(g(X', X'')) - g(\nabla_X X', X'') - g(X', \nabla_X X'') \\
&= X(g(X', X'')) - g(H(X, X')_L, X'') - g(X', H(X, X'')_L) \\
&= -X(d\eta(X', \phi X'')) + d\eta(H(X, X'), \phi X'') \\
&\quad + d\eta(H(X, X''), \phi X') \\
&= -X(d\eta(X', \phi X'')) + X(d\eta(X', \phi X'')) - d\eta(X', [X, \phi X'']) \\
&\quad + X(d\eta(X'', \phi X')) - d\eta(X'', [X, \phi X'])
\end{aligned}$$

$$\begin{aligned}
 &= -X(g(X', X'')) - g(X', \phi[X, \phi X'']) - g(X'', \phi[X, \phi X']) \\
 &= -X(g(\phi X', \phi X'')) + g([X, \phi X'], \phi X'') \\
 &\quad + g(\phi X', [X, \phi X'']) \\
 &= -(\mathcal{L}_X g)(\phi X', \phi X'') = 0,
 \end{aligned}$$

$$\begin{aligned}
 (\nabla_Y g)(X', X'') &= Y(g(X', X'')) - g([Y, X']_L, X'') \\
 &\quad - g(X', [Y, X'']_L) \\
 &= Y(g(X', X'')) - g([Y, X'], X'') - g(X', [Y, X'']) \\
 &= (\mathcal{L}_Y g)(X', X'') = 0
 \end{aligned}$$

and

$$\begin{aligned}
 (\nabla_\xi g)(X', X'') &= \xi(g(X', X'')) - g([\xi, X']_L, X'') - g(X', [\xi, X'']_L) \\
 &= \xi(g(X', X'')) - g([\xi, X'], X'') - g(X', [\xi, X'']) \\
 &= (\mathcal{L}_\xi g)(X', X'') = 0
 \end{aligned}$$

for all $X, X', X'' \in \Gamma(L)$ and $Y \in \Gamma(Q)$. Analogously, one can prove that $(\nabla_V g)(Y', Y'') = 0$ for all $V \in \Gamma(TM^{2n+1})$ and $Y', Y'' \in \Gamma(Q)$.

Now we prove the last part of the theorem. We prove that, under the assumption of the integrability of L and Q , (i) is equivalent to the total geodesicity of the foliations defined by L and Q . Let X, X' be sections of L . Then, for any $Y \in \Gamma(Q)$, the Koszul formula for the Levi Civita connection yields

$$\begin{aligned}
 (10) \quad &2g(\widehat{\nabla}_X X', Y) \\
 &= X(g(X', Y)) + X'(g(X, Y)) - Y(g(X, X')) + g([X, X'], Y) \\
 &\quad + g([Y, X], X') - g([X', Y], X) \\
 &= -Y(g(X, X')) + g([Y, X]_L, X') + g([Y, X']_L, X) \\
 &= -Y(g(X, X')) + g(\nabla_Y X, X') + g(X, \nabla_Y X') \\
 &= -(\nabla_Y g)(X, X')
 \end{aligned}$$

and, in the same way,

$$(11) \quad 2g(\widehat{\nabla}_X X', \xi) = -(\nabla_\xi g)(X, X'),$$

from which it follows that, if the bi-Legendrian connection is metric, then the foliation defined by L is totally geodesic. A similar argument

also works for Q . Conversely, if L and Q define two totally geodesic foliations, by (10)–(11), one has $(\nabla_Y g)(X, X') = (\nabla_X g)(Y, Y') = (\nabla_\xi g)(X, X') = (\nabla_\xi g)(Y, Y') = 0$ for any $X, X' \in \Gamma(L)$, $Y, Y' \in \Gamma(Q)$. Moreover, for all $X, X', X'' \in \Gamma(L)$, using the same computations in (9),

$$\begin{aligned} & (\nabla_X g)(X', X'') \\ &= -X(g(X', X'')) - g(X', \phi[X, \phi X'']) - g(X'', \phi[X, \phi X']) \\ &= \phi X'(g(\phi X'', X)) + \phi X''(g(\phi X', X)) - X(g(\phi X', \phi X'')) \\ &\quad + g([\phi X', \phi X''], X) + g([X, \phi X'], \phi X'') - g([\phi X'' X], \phi X') \\ &= 2g(\widehat{\nabla}_{\phi X'} \phi X'', X) = 0 \end{aligned}$$

because of the total geodesicity of the foliation defined by Q . Analogously one can prove that $(\nabla_Y g)(Y', Y'') = 0$ for all $Y, Y', Y'' \in \Gamma(Q)$. Hence, $\nabla g = 0$. \square

Example 2.10. A class of examples of bi-Legendrian structures verifying one of the equivalent conditions stated in Proposition 2.9 is given by contact (κ, μ) -manifolds, i.e., contact metric manifolds such that the Reeb vector field satisfies

$$\widehat{R}(V, W)\xi = \kappa(\eta(W)V - \eta(V)W) + \mu(\eta(W)hV - \eta(V)hW)$$

for some constants $\kappa, \mu \in \mathbf{R}$. This class of contact metric manifolds has been introduced in [2] and then extensively studied by several authors. It is well known that $\kappa \leq 1$, and when $\kappa < 1$ the contact metric manifold in question admits two mutually orthogonal and integrable Legendrian distributions $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ determined by the eigenspaces of the operator h , where $\lambda = \sqrt{1 - \kappa}$. Moreover, these Legendrian foliations are totally geodesic, hence they verify (i)–(iv) of Proposition 2.9. This bi-Legendrian structure and the corresponding bi-Legendrian connection have been studied in detail in [6] where in particular it is proved that $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are never both flat.

3. The bi-Legendrian and the Tanaka-Webster connection.

In this section we consider a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ endowed with a Legendrian distribution L . We denote, as usual, by Q the conjugate Legendrian distribution of L and by ∇ the bi-Legendrian

connection corresponding to (L, Q) . We assume that the pair (L, Q) is flat, that is, both L and Q are flat Legendrian distributions, and satisfies one of the equivalent four properties of Proposition 2.9. Under these assumptions we study the relationship between ∇ and the Tanaka-Webster connection ${}^*\nabla$ of $(M^{2n+1}, \phi, \xi, \eta, g)$.

Theorem 3.1. *Under the notation and the assumptions above, the bi-Legendrian connection ∇ coincides with the Tanaka-Webster connection ${}^*\nabla$ if and only if L and Q are integrable and $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold.*

Proof. Suppose that $\nabla = {}^*\nabla$. Then the torsion tensor field T of the bi-Legendrian connection must satisfy (iv) of (5). In particular, $[X, X']_Q = -T(X, X') = -2d\eta(X, X')\xi = 0$ for all $X, X' \in \Gamma(L)$ and $[Y, Y']_L = -T(Y, Y') = -2d\eta(Y, Y')\xi = 0$ for all $Y, Y' \in \Gamma(Q)$, from which we deduce the integrability of L and Q . Moreover, from (4) it follows that $[\xi, X]_Q = T(X, \xi) = {}^*T(X, \xi) = \eta(\xi)\phi hX - \eta(X)\phi h\xi + 2g(X, \phi\xi)\xi = -h\phi X$. So, for all $X \in \Gamma(L)$,

$$(12) \quad [\xi, X]_Q = -h\phi X$$

and, in the same way,

$$(13) \quad [\xi, Y]_L = -h\phi Y$$

for all $Y \in \Gamma(Q)$. By (12) and (13) we see that the flatness of L and Q is equivalent to the vanishing of h . With this remark we can prove that $(M^{2n+1}, \phi, \xi, \eta, g)$ is Sasakian. Indeed, since $\nabla g = 0$, by Proposition 2.9 we have $\nabla\phi = 0$. Moreover, ∇ satisfies (ii) of (5), so for all $V, W \in \Gamma(TM)$

$$\begin{aligned} (\widehat{\nabla}_V\phi)W &= g(V + hV, W)\xi - \eta(W)(V + hV) \\ &= g(V, W)\xi - \eta(W)V, \end{aligned}$$

since $h = 0$. Now we prove the converse, showing that ∇ satisfies (5). We already know that ∇ satisfies $\nabla\xi = 0$, $\nabla\eta = 0$ and, by hypothesis, $\nabla g = 0$. Moreover, ∇ also satisfies $T(X, Y) = 2d\eta(X, Y)\xi$ for all $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$, so in order to check (iv) it is sufficient to prove that $T(X, X') = T(Y, Y') = 0$ for all $X, X' \in \Gamma(L)$,

$Y, Y' \in \Gamma(Q)$. But this is true because, by the assumption of the integrability of L and Q , we have $T(X, X') = -[X, X']_Q = 0$ and $T(Y, Y') = -[Y, Y']_L = 0$. Moreover, since $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold and, in particular, a K -contact manifold, we have, for all $V, W \in \Gamma(TM)$,

$$\begin{aligned} &(\widehat{\nabla}_V \phi)W - g(V + hV, W) \xi + \eta(W) (V + hV) \\ &= (\widehat{\nabla}_V \phi)W - g(V, W) \xi + \eta(W) V = 0 = (\nabla_V \phi) W \end{aligned}$$

because of Proposition 2.9. So ∇ also satisfies (ii). Finally, since $h = 0$ and L, Q are flat, we have, for all $X \in \Gamma(L)$, $T(\xi, \phi X) = [\phi X, \xi]_L = 0 = -\phi([X, \xi]_Q) = -\phi T(\xi, X)$, and, similarly, for all $Y \in \Gamma(Q)$, $T(\xi, \phi Y) = 0 = -\phi T(\xi, Y)$, hence (iii) is also satisfied. Thus, by the uniqueness of the Tanaka-Webster connection, we conclude that $\nabla = {}^* \nabla$. \square

Remark 3.2. In the proof of Theorem 3.1, we have found the following expression for the tensor field h :

$$hX = [\xi, \phi X]_L = -(\phi[\xi, X])_L, \quad hY = [\xi, \phi Y]_Q = -(\phi[\xi, Y])_Q,$$

for all $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$. In particular, as we already know by Proposition 2.9, h preserves the distributions L and Q .

As immediate consequences of Theorem 3.1 and Proposition 2.9, we have:

Corollary 3.3. *Under the assumptions of Theorem 3.1, the Tanaka-Webster connection of $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfies*

$${}^* \nabla_X X' = -(\phi[X, \phi X'])_L \quad \text{for all } X, X' \in \Gamma(L)$$

and

$${}^* \nabla_Y Y' = -(\phi[Y, \phi Y'])_Q \quad \text{for all } Y, Y' \in \Gamma(Q).$$

Corollary 3.4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian manifold foliated by a flat Legendrian foliation \mathcal{F} such that the conjugate Legendrian*

distribution is integrable. Let ∇ be the corresponding bi-Legendrian connection, and suppose that $\nabla g = 0$. Let S be the tensor field of type $(1, 2)$ defined by $S(V, W) = \nabla_V W - \widehat{\nabla}_V W$. Then we have $S(V, \xi) = S(\xi, V) = \phi V$ for all $V \in \Gamma(TM)$ and $S(Z, Z') = d\eta(Z, Z')\xi$ for all $Z, Z' \in \Gamma(\mathcal{D})$. In particular, for all $X, X' \in \Gamma(L)$ and for all $Y, Y' \in \Gamma(Q)$, we have

$$(14) \quad \nabla_X X' = \widehat{\nabla}_X X', \quad \nabla_Y Y' = \widehat{\nabla}_Y Y'.$$

Proof. Indeed, by Theorem 3.1, ∇ coincides with the Tanaka-Webster connection of $(M^{2n+1}, \phi, \xi, \eta, g)$. Then, by (3) we deduce the following relations:

$$\begin{aligned} \nabla_X X' - \widehat{\nabla}_X X' &= 0, \quad \nabla_X Y' - \widehat{\nabla}_X Y' = d\eta(X, Y)\xi, \quad \nabla_X \xi - \widehat{\nabla}_X \xi = \phi X, \\ \nabla_Y X - \widehat{\nabla}_Y X &= d\eta(Y, X)\xi, \quad \nabla_Y Y' - \widehat{\nabla}_Y Y' = 0, \quad \nabla_Y \xi - \widehat{\nabla}_Y \xi = \phi Y, \\ \nabla_\xi X - \widehat{\nabla}_\xi X &= \phi X, \quad \nabla_\xi Y - \widehat{\nabla}_\xi Y = \phi Y, \quad \nabla_\xi \xi = \widehat{\nabla}_\xi \xi = 0 \end{aligned}$$

for all $X, X' \in \Gamma(L)$, $Y, Y' \in \Gamma(Q)$, from which the assertion follows. \square

Remark 3.5. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian manifold, and let \mathfrak{S}_M be the set of all flat Legendrian foliations on M^{2n+1} such that the conjugate Legendrian distribution is integrable and $\nabla g = 0$. Take two elements \mathcal{F}_1 and \mathcal{F}_2 of \mathfrak{S}_M . \mathcal{F}_1 and \mathcal{F}_2 are flat Legendrian foliations on M^{2n+1} such that $\nabla^1 g = 0$ and $\nabla^2 g = 0$, where ∇^1 and ∇^2 denote the bi-Legendrian connections associated to \mathcal{F}_1 and \mathcal{F}_2 , respectively. Then, by Theorem 3.1, $\nabla^1 = \nabla^2$ because they both coincide with the Tanaka-Webster connection ${}^*\nabla$. In particular, we have that $\nabla^1 \mathcal{F}_2 \subset \mathcal{F}_2$ and $\nabla^2 \mathcal{F}_1 \subset \mathcal{F}_1$. Moreover, we deduce that the Tanaka-Webster connection preserves all the Legendrian foliations belonging to \mathfrak{S}_M .

A variation of Theorem 3.1 is the following Theorem 3.7. But, before proving it, we need a preliminary lemma.

Lemma 3.6. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a K -contact manifold endowed with a flat Legendrian distribution L . Then its conjugate Legendrian distribution $Q = \phi L$ is also flat.*

Proof. Indeed, as ξ is Killing, we have $h = 0$, so that, for all $X \in \Gamma(L)$, $0 = 2hX = [\xi, \phi X] - \phi[\xi, X]$, from which $[\xi, \phi X] = \phi[\xi, X] \in \Gamma(Q)$, because L is flat. \square

Theorem 3.7. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian manifold endowed with a flat Legendrian distribution L . Let $Q = \phi L$ be its conjugate Legendrian distribution. If the Tanaka-Webster connection ${}^*\nabla$ preserves the distribution L , then L and Q are integrable and ${}^*\nabla$ coincides with the bi-Legendrian connection ∇ associated to the almost bi-Legendrian structure (L, Q) .*

Proof. First of all, we prove that ${}^*\nabla L \subset L$ implies the integrability of L . Let $X, X' \in \Gamma(L)$. Then $[X, X'] = {}^*\nabla_X X' - {}^*\nabla_{X'} X - 2d\eta(X, X')\xi = {}^*\nabla_X X' - {}^*\nabla_{X'} X \in \Gamma(L)$. Now we show that ${}^*\nabla Q \subset Q$. Let $Y \in \Gamma(Q)$. Then, by ${}^*\nabla g = 0$ and ${}^*\nabla L \subset L$, we get for all $V \in \Gamma(TM^{2n+1})$ and $X \in \Gamma(L)$

$$\begin{aligned} 0 &= ({}^*\nabla_V g)(X, Y) = V(g(X, Y)) - g({}^*\nabla_V X, Y) - g(X, {}^*\nabla_V Y) \\ &= -g(X, {}^*\nabla_V Y), \end{aligned}$$

so that ${}^*\nabla_V Y \in \Gamma(Q \oplus \mathbf{R}\xi)$. Moreover, since ${}^*\nabla \xi = 0$, $0 = ({}^*\nabla_V g)(\xi, Y) = V(g(\xi, Y)) - g({}^*\nabla_V \xi, Y) - g(\xi, {}^*\nabla_V Y) = -g(\xi, {}^*\nabla_V Y)$, from which ${}^*\nabla_V Y \in \Gamma(Q)$. Then, arguing in the same way as for L , one can prove that Q is integrable. Note also that, since M^{2n+1} is Sasakian and in particular K-contact, by Lemma 3.6, also Q is flat. Finally, we prove that ${}^*\nabla$ coincides with the bi-Legendrian connection corresponding to (L, Q) , that is, ${}^*\nabla$ verifies (ii) and (iii) in (6). The relations ${}^*T(X, \xi) = [\xi, X]_Q$ for $X \in \Gamma(L)$ and ${}^*T(Y, \xi) = [\xi, Y]_L$ for $Y \in \Gamma(Q)$ hold because L and Q are flat and, on the other hand, ${}^*T(X, \xi) = \phi hX = 0$, ${}^*T(Y, \xi) = \phi hY = 0$. In order to prove ${}^*\nabla d\eta = 0$, we firstly show that ${}^*\nabla \phi = 0$. Indeed, since M^{2n+1} is Sasakian,

$$({}^*\nabla_V \phi)W = (\widehat{\nabla}_V \phi)W - g(V, W)\xi + \eta(W)V = 0$$

for all $V, W \in \Gamma(TM^{2n+1})$, so that ${}^*\nabla \phi = 0$. Now we can prove that $({}^*\nabla_V d\eta)(W, W') = 0$ for all $V, W, W' \in \Gamma(TM^{2n+1})$. This equality holds immediately for $W, W' \in \Gamma(L)$ and for $W, W' \in \Gamma(Q)$ because L and Q are preserved by ${}^*\nabla$. Also the case $W' = \xi$ is obvious since

$*\nabla\xi = 0$. So it remains to show that $(*\nabla_V d\eta)(X, Y) = 0$ for $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$. In fact, using $*\nabla\phi = 0$,

$$\begin{aligned} (*\nabla_V d\eta)(X, Y) &= V(g(X, \phi Y)) - g(*\nabla_V X, \phi Y) - g(X, \phi*\nabla_V Y) \\ &= V(g(X, \phi Y)) - g(*\nabla_V X, \phi Y) - g(X, *\nabla_V \phi Y) \\ &= (*\nabla_V g)(X, \phi Y) = 0, \end{aligned}$$

since $*\nabla g = 0$. Thus, $*\nabla$ satisfies all the properties which characterize the bi-Legendrian connection associated to (L, Q) . \square

4. An interpretation of the Tanaka-Webster connection. In

Section 3 we have found that, under certain assumptions, the Tanaka-Webster connection of a Sasakian manifold foliated by a Legendrian foliation \mathcal{F} coincides with the bi-Legendrian connection associated to \mathcal{F} (Theorem 3.7). This result has an analogue in even dimension: Etayo and Santamaria proved in [7] that, under suitable assumptions, the Levi Civita connection of a Kählerian manifold foliated by a Lagrangian foliation \mathcal{F}' coincides with the bi-Lagrangian connection associated to \mathcal{F}' . Therefore, it seems that the Tanaka-Webster connection plays the same role in contact Riemannian geometry of that one played by the Levi Civita connection for symplectic or Kählerian manifolds. This is not surprising since it is a well-known fact that the Tanaka-Webster connection of a Sasakian manifold which is a circle bundle over a Kählerian manifold can be viewed as the lift of the Levi Civita connection of the Kählerian base manifold. Now we prove this property for any, in general nonregular, Sasakian manifold.

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian manifold. It is well known that the Reeb vector field ξ defines a transversely Kählerian foliation, that is, this foliation, which we denote by \mathcal{F}_ξ , can be defined by local submersions $f_i : U_i \rightarrow M^{2n}$ from an open set U_i of M^{2n+1} , with $\{U_i\}_{i \in I}$ an open covering of M^{2n+1} , onto a Kählerian manifold (M^{2n}, J, ω, G) , where J, ω and G are the projection of $\phi, d\eta$ and g , respectively. Moreover, any two of these submersions f_i and f_j , with $U_i \cap U_j \neq \emptyset$, are connected by local diffeomorphisms $\gamma_{ij} : f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$ satisfying, on $U_i \cap U_j$, the relation $\gamma_{ij} \circ f_j = f_i$, and preserving the Kählerian structure. Let $\widehat{\nabla}'$ be the Levi Civita connection of (M^{2n}, G) and define a connection ∇^i , locally on each U_i , as the lift of $\widehat{\nabla}'$ under the submersion f_i . More precisely, for any basic vector

fields Z_1, Z_2 , we define $\nabla_{Z_1}^i Z_2$ as the unique basic vector field on U_i such that $f_{i*}(\nabla_{Z_1}^i Z_2) = \widehat{\nabla}'_{f_{i*}(Z_1)} f_{i*}(Z_2)$. Moreover, we put, by definition, $\nabla^i \xi = 0$ and, for any vector field V on U_i , $\nabla_\xi^i V = [\xi, V]$. Note that these last definitions imply that, for any basic vector field Z , $f_{i*}(\nabla_Z^i \xi) = 0 = \widehat{\nabla}'_{f_{i*}(Z)} f_{i*}(\xi)$ and $f_{i*}(\nabla_\xi^i Z) = f_{i*}([\xi, Z]) = 0 = \widehat{\nabla}'_{f_{i*}(\xi)} f_{i*}(Z)$. Note also that ∇^i preserves the “horizontal” distribution \mathcal{D} . We have the following result:

Proposition 4.1. *The above connection ∇^i coincides with the Tanaka-Webster connection of $(M^{2n+1}, \phi, \xi, \eta, g)$ restricted to U_i .*

Proof. It is sufficient to show that ∇^i satisfies all the properties which characterize the Tanaka-Webster connection of $(M^{2n+1}, \phi, \xi, \eta, g)$. First of all, by definition, $\nabla^i \xi = 0$. Next, from $\nabla^i \xi = 0$ and $\nabla^i \mathcal{D} \subset \mathcal{D}$, we deduce $\nabla^i \eta = 0$. Furthermore, since $\widehat{\nabla}' G = 0$ and f_i is a Riemannian submersion, we get $(\nabla_Z^i g)(Z_1, Z_2) = 0$ for all Z, Z_1, Z_2 basic vector fields on U_i , and, since $\nabla^i \mathcal{D} \subset \mathcal{D}$, also $(\nabla_Z^i g)(Z_1, \xi) = 0$. So it remains to prove that $(\nabla_\xi^i g)(Z_1, Z_2) = 0$ for Z_1, Z_2 basic vector fields. Indeed,

$$\begin{aligned} (\nabla_\xi^i g)(Z_1, Z_2) &= \xi(g(Z_1, Z_2)) - g([\xi, Z_1], Z_2) - g(Z_1, [\xi, Z_2]) \\ &= (\mathcal{L}_\xi g)(Z_1, Z_2) = 0 \end{aligned}$$

because ξ is Killing. In the same way, since $\widehat{\nabla}' J = 0$ and $f_{i*} \circ \phi = J \circ f_{i*}$, we get $(\nabla_{Z_1}^i \phi)Z_2 = 0$ for all Z_1, Z_2 basic vector fields on U_i . Next, for any basic vector field Z on U_i we have $(\nabla_\xi^i \phi)Z = [\xi, \phi Z] - \phi[\xi, Z] = 2hZ = 0$ because $h = 0$, M^{2n+1} being Sasakian. Thus, in order to conclude the proof, it remains to check the properties involving the torsion. Let Z be a basic vector field defined on U_i . Then $T^i(\xi, \phi Z) = \nabla_\xi^i \phi Z - \nabla_{\phi Z}^i \xi - [\xi, \phi Z] = [\xi, \phi Z] - [\xi, \phi Z] = 0$ and $T^i(\xi, Z) = \nabla_\xi^i Z - \nabla_Z^i \xi - [\xi, Z] = [\xi, Z] - [\xi, Z] = 0$, so that $T^i(\xi, \phi Z) = 0 = -\phi T^i(\xi, Z)$. Finally, for any Z_1, Z_2 basic vector fields, we have $f_{i*}(T^i(Z_1, Z_2)) = T'(f_{i*}(Z_1), f_{i*}(Z_2)) = 0$ and so $T^i(Z_1, Z_2)$ is vertical. Hence, $T^i(Z_1, Z_2) = \eta(T^i(Z_1, Z_2))\xi = -\eta([Z_1, Z_2])\xi = 2d\eta(Z_1, Z_2)\xi$. \square

Now we prove that this family of connections gives rise to a well-defined global connection on M^{2n+1} .

Proposition 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a Sasakian manifold, and let $\{U_i, f_i, \gamma_{ij}\}$ be a cocycle defining the foliation \mathcal{F}_ξ . Then the family of connections $(\nabla^i)_{i \in I}$ defined above gives rise to a global connection on M^{2n+1} which coincides with the Tanaka-Webster connection of $(M^{2n+1}, \phi, \xi, \eta, g)$.*

Proof. We have to prove that, for any $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, $\nabla^i = \nabla^j$. Firstly note that $\gamma_{ij} : f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$ is a (local) affine transformation with respect to the Levi Civita connection, because it is a (local) isometry. Now let Z'_1 and Z'_2 be vector fields on M^{2n} , and let Z_1^i, Z_1^j and Z_2^i, Z_2^j be the basic vector fields f_i -related and f_j -related, respectively, to Z'_1 and Z'_2 . Note that Z_1^i is also the basic vector field f_j -related to $\gamma_{ij*}(Z'_1)$ because it is horizontal also for f_j , as $\ker(f_{i*}) = \mathbf{R}\xi = \ker(f_{j*})$, and, for all $p \in M^{2n+1}$, $f_{j*}(Z_{1p}^i) = (\gamma_{ij*})_{f_j(p)}(Z'_{1f_j(p)})$, since $f_i(p) = \gamma_{ij}(f_j(p))$ and $\gamma_{ij} = \gamma_{ji}^{-1}$. Then we get $f_{i*}(\nabla_{Z_1^i}^i Z_2^i) = \gamma_{ij*}(f_{j*}(\nabla_{Z_1^j}^j Z_2^j)) = f_{i*}(\nabla_{Z_1^i}^j Z_2^i)$, which implies that $\nabla_{Z_1^i}^i Z_2^i - \nabla_{Z_1^i}^j Z_2^i$ is vertical. Since it is also horizontal, we get $\nabla_{Z_1^i}^i Z_2^i = \nabla_{Z_1^i}^j Z_2^i$. Moreover, clearly, $\nabla^i \xi = 0 = \nabla^j \xi$ and, on $U_i \cap U_j$, $\nabla_\xi^i V = [\xi, V] = \nabla_\xi^j V$. Finally, the last part of the statement follows from Proposition 4.1. \square

More in general, for any contact metric manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$, we can define a connection on M^{2n+1} setting, for all $Z \in \Gamma(\mathcal{D})$,

$$(15) \quad \tilde{\nabla}_Z Z' = (\hat{\nabla}_Z Z')_{\mathcal{D}}, \quad \tilde{\nabla}_\xi Z = [\xi, Z], \quad \tilde{\nabla}_\xi \xi = 0.$$

That $\tilde{\nabla}$ is a connection on M^{2n+1} preserving the contact distribution \mathcal{D} is easy to check. Moreover, we can give an interesting characterization of this connection:

Theorem 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold, and let $\tilde{\nabla}$ be the connection on M^{2n+1} defined by (15). Then $\tilde{\nabla}$ is the unique connection on M^{2n+1} satisfying the following properties:*

- (i) $\tilde{\nabla} \xi = 0$,
- (ii) $\tilde{T}(V, W) = 2d\eta(V, W)\xi$ for all $V, W \in \Gamma(TM)$,
- (iii) $(\tilde{\nabla}_Z g)(Z', Z'') = 0$ for all $Z, Z', Z'' \in \Gamma(\mathcal{D})$.

Furthermore, M^{2n+1} is K -contact if and only if $\tilde{\nabla}g = 0$, and M^{2n+1} is Sasakian if and only if $\tilde{\nabla}\phi = 0$ and in this case $\tilde{\nabla}$ coincides with the Tanaka-Webster connection of $(M^{2n+1}, \phi, \xi, \eta, g)$.

Proof. Firstly we prove that $\tilde{\nabla}$ satisfies (i), (ii) and (iii). By definition we have $\tilde{\nabla}\xi = 0$. Next, for all $Z, Z' \in \Gamma(\mathcal{D})$, $\tilde{T}(Z, Z') = (\widehat{\nabla}_Z Z')_{\mathcal{D}} - (\widehat{\nabla}_{Z'} Z)_{\mathcal{D}} - [Z, Z'] = (\widehat{T}(Z, Z'))_{\mathcal{D}} - [Z, Z']_{\mathbf{R}\xi} = -\eta([Z, Z'])\xi = 2d\eta(Z, Z')\xi$, and $\tilde{T}(Z, \xi) = -\tilde{\nabla}_\xi Z - [Z, \xi] = [Z, \xi] - [Z, \xi] = 0 = 2d\eta(Z, \xi)\xi$. Then, since on the contact distribution $\tilde{\nabla}$ coincides with the projection on \mathcal{D} of the Levi Civita connection, we get (iii). Now let ∇ be any connection on M^{2n+1} satisfying (i), (ii) and (iii). Then by (i) and (ii) we have $\nabla_\xi Z = \nabla_Z \xi + [\xi, Z] + T(\xi, Z) = [\xi, Z] + 2d\eta(\xi, Z)\xi = [\xi, Z]$ for all $Z \in \Gamma(\mathcal{D})$. So it remains to prove that $\nabla_Z Z' = (\widehat{\nabla}_Z Z')_{\mathcal{D}}$ for all $Z, Z' \in \Gamma(\mathcal{D})$. For this purpose, let $\overline{\nabla}$ be the connection given by

$$\overline{\nabla}_V W = \nabla_{V_{\mathcal{D}}} W_{\mathcal{D}} + (\widehat{\nabla}_{V_{\mathcal{D}}} W_{\mathcal{D}})_{\mathbf{R}\xi} + \widehat{\nabla}_{V_{\mathbf{R}\xi}} W + \widehat{\nabla}_V W_{\mathbf{R}\xi}.$$

Then, if we prove that $\overline{\nabla}$ coincides with the Levi Civita connection of M^{2n+1} , we would have that $\nabla_Z Z' = (\widehat{\nabla}_Z Z')_{\mathcal{D}}$ for all $Z, Z' \in \Gamma(\mathcal{D})$. It is enough to verify that $\overline{\nabla}$ is metric and torsion free on the subbundle \mathcal{D} . That $\overline{\nabla}$ is metric on \mathcal{D} is ensured by (iii); then, $\overline{T}(Z, Z') = T(Z, Z') + \eta(\widehat{\nabla}_Z Z')\xi - \eta(\widehat{\nabla}_{Z'} Z)\xi = 2d\eta(Z, Z')\xi + \eta([Z, Z'])\xi = 0$. For proving the second part of the theorem, note that

$$(\tilde{\nabla}_\xi g)(Z, Z') = \xi(g(Z, Z')) - g([\xi, Z], Z') - g(Z, [\xi, Z']) = (\mathcal{L}_\xi g)(Z, Z'),$$

from which we deduce that M^{2n+1} is a K -contact manifold if and only if $\tilde{\nabla}$ is a metric connection with respect to the associated metric g . Finally, if $\tilde{\nabla}\phi = 0$ we have, first of all,

$$(16) \quad 0 = (\tilde{\nabla}_\xi \phi)Z = [\xi, \phi Z] - \phi[\xi, Z] = (\mathcal{L}_\xi \phi)Z = 2hZ,$$

from which M^{2n+1} is K -contact and by (2) $\phi Z = -\widehat{\nabla}_Z \xi$. Then, for all $Z, Z' \in \Gamma(\mathcal{D})$, $(\widehat{\nabla}_Z \phi)Z' = ((\widehat{\nabla}_Z \phi)Z')_{\mathcal{D}} + ((\widehat{\nabla}_Z \phi)Z')_{\mathbf{R}\xi} = (\tilde{\nabla}_Z \phi)Z' + \eta((\widehat{\nabla}_Z \phi)Z')\xi = g(\widehat{\nabla}_Z \phi Z', \xi)\xi = -g(\phi Z', \widehat{\nabla}_Z \xi)\xi = g(\phi Z, \phi Z')\xi = g(Z, Z')\xi$, and (1) is satisfied. Moreover, $(\widehat{\nabla}_\xi \phi)Z = \widehat{\nabla}_{\phi Z} \xi + [\xi, \phi Z] - \phi \widehat{\nabla}_Z \xi - \phi[\xi, Z] = -\phi^2 Z + \phi^2 Z + (\mathcal{L}_\xi \phi)Z = 0$ and $(\widehat{\nabla}_Z \phi)\xi = -\phi \widehat{\nabla}_Z \xi = \phi^2 Z = -Z$, so that (1) holds in any case. Conversely, if M^{2n+1}

is Sasakian, then it is K-contact; hence, $(\tilde{\nabla}_\xi\phi)Z = (\mathcal{L}_\xi\phi)Z = 0$. Moreover, for any $Z, Z' \in \Gamma(\mathcal{D})$, $(\tilde{\nabla}_Z\phi)Z' = (g(Z, Z')\xi - \eta(Z')Z)_\mathcal{D} = 0$. Finally, Proposition 4.2 implies that $\tilde{\nabla}$ is the Tanaka-Webster connection of the Sasakian manifold $(M^{2n+1}, \phi, \xi, \eta, g)$. \square

In the context of symplectic geometry, in the Appendix we shall prove the following result.

Theorem 4.4. *Let (M^{2n}, ω) be a symplectic manifold endowed with a bi-Lagrangian structure $(\mathcal{F}, \mathcal{G})$ such that $T\mathcal{G}$ is an affine transversal distribution for \mathcal{F} . Then there exists a Kählerian structure on (M^{2n}, ω) whose Levi Civita connection coincides with the bi-Lagrangian connection of $(M^{2n}, \omega, \mathcal{F}, \mathcal{G})$.*

Now we prove the analogue of Theorem 4.4 in odd dimension. As is expected, the role played in Theorem 4.4 by the Levi Civita connection is played now by the Tanaka-Webster connection:

Theorem 4.5. *Let (M^{2n+1}, η) be a contact manifold endowed with a flat bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ such that $T\mathcal{G}$ is an affine transversal distribution for \mathcal{F} . Then there exists a Sasakian structure on (M^{2n+1}, η) whose Tanaka-Webster connection coincides with the bi-Legendrian connection of $(M^{2n+1}, \eta, \mathcal{F}, \mathcal{G})$.*

Proof. The assumption of $T\mathcal{G}$ being an affine transversal distribution for \mathcal{F} means that the curvature tensor field of the corresponding bi-Legendrian connection satisfies $R(X, Y) = 0$ for $X \in \Gamma(T\mathcal{F})$, $Y \in \Gamma(T\mathcal{G})$, cf. [5]. So this assumption and the flatness of the bi-Legendrian structure imply that the curvature R of the bi-Legendrian connection ∇ associated to $(\mathcal{F}, \mathcal{G})$ vanishes identically, cf. Proposition 2.6. Now let p be a point of M^{2n+1} . Since $d\eta_p$ is a symplectic form on the subspace $\mathcal{D}_p \subset T_pM^{2n+1}$, it follows that there exists a basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}, \xi_p\}$ of T_pM^{2n+1} such that $\{e_1, \dots, e_n\}$ is a basis of $T_p\mathcal{F}$, $\{e_{n+1}, \dots, e_{2n}\}$ is a basis of $T_p\mathcal{G}$ and

$$(17) \quad d\eta_p(e_i, e_j) = d\eta_p(e_{n+i}, e_{n+j}) = 0, \quad d\eta_p(e_i, e_{n+j}) = -\frac{1}{2}\delta_{ij}$$

for all $i, j \in \{1, \dots, n\}$. For each $k \in \{1, \dots, 2n\}$ we define vector fields E_k on M^{2n+1} by the ∇ -parallel transport along curves of the vector e_k . More precisely, for any $q \in M^{2n+1}$ we consider a curve $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = p$, $\gamma(1) = q$, and we define $E_k(q) := \tau_\gamma(e_k)$, $\tau_\gamma : T_p M^{2n+1} \rightarrow T_q M^{2n+1}$ being the parallel transport along γ . Note that $E_k(q)$ does not depend on the curve joining p and q , since $R \equiv 0$. Setting $X_i := E_{n+i}$ and $Y_i := E_i$, we obtain $2n$ vector fields on M^{2n+1} such that, for each $i \in \{1, \dots, n\}$, $Y_i \in \Gamma(T\mathcal{F})$ and $X_i \in \Gamma(T\mathcal{G})$, since the parallel transport preserves the distributions $T\mathcal{F}$ and $T\mathcal{G}$. Moreover, (17) holds at any point of M^{2n+1} , that is, for any $q \in M^{2n+1}$ and $i, j \in \{1, \dots, n\}$,

$$(18) \quad \begin{aligned} d\eta_q(Y_i(q), Y_j(q)) &= d\eta_q(X_i(q), X_j(q)) = 0, \\ d\eta_q(Y_i(q), X_j(q)) &= -\frac{1}{2}\delta_{ij}. \end{aligned}$$

Indeed, since $d\eta$ is parallel with respect to ∇ , for all $h, k \in \{1, \dots, 2n\}$,

$$\begin{aligned} \frac{d}{dt}d\eta_{\gamma(t)}(E_h(\gamma(t)), E_k(\gamma(t))) \\ = d\eta_{\gamma(t)}(\nabla_{\gamma'} E_h, E_k) + d\eta_{\gamma(t)}(E_h, \nabla_{\gamma'} E_k) = 0 \end{aligned}$$

so that $d\eta_p(e_k, e_k) = d\eta_q(E_h(q), E_k(q))$, for all $q \in M^{2n+1}$. Note that, by construction, we have $\nabla_{E_h} E_k = 0$ and $\nabla_{\xi} E_k = 0$ for all $h, k \in \{1, \dots, 2n\}$. From this and the expression of the torsion of the bi-Legendrian connection, cf. Section 2, we get

$$(19) \quad [Y_i, Y_j] = [X_i, X_j] = [Y_i, \xi] = [X_i, \xi] = 0$$

$$(20) \quad [Y_i, X_j] = -T(Y_i, X_j) = -2d\eta(Y_i, X_j)\xi = \delta_{ij}\xi,$$

for all $i, j \in \{1, \dots, n\}$. Equations (19)–(20) imply that there exist local coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$ such that $Y_i = \partial/\partial y_i$, $X_j = (\partial/\partial x_j) + y_j(\partial/\partial z)$, $\xi = \partial/\partial z$, for any $i \in \{1, \dots, n\}$. Note that, from (18) it follows that, with respect to these coordinates, $d\eta = \sum_{i=1}^n dx_i \wedge dy_i$ from which we have $d(\eta + \sum_{i=1}^n y_i dx_i) = 0$ and so $\eta = df - \sum_{i=1}^n y_i dx_i$, for some $f \in C^\infty(M^{2n+1})$. But $\eta(Y_j) = 0$, $\eta(X_j) = 0$ and $\eta(\xi) = 1$ imply $\partial f/\partial y_j = 0$, $\partial f/\partial x_j = 0$ and $\partial f/\partial z = 1$, respectively. So $df = dz$, and in this coordinate system we have that $T\mathcal{F}$ is spanned by $Y_i = \partial/\partial y_i$, $T\mathcal{G}$ by $X_i = (\partial/\partial x_i) + y_i(\partial/\partial z)$,

$i \in \{1, \dots, n\}$, and the 1-form η is given by $\eta = dz - \sum_{i=1}^n y_i dx_i$. Now we define a tensor field ϕ and a Riemannian metric g on M^{2n+1} , putting $\phi\xi = 0$, $\phi Y_i = X_i$, $\phi X_i = -Y_i$ and $g(Z, Z') = -d\eta(Z, \phi Z')$ for all $Z, Z' \in \Gamma(\mathcal{D})$, $g(V, \xi) = \eta(V)$ for all $V \in \Gamma(TM^{2n+1})$. A straightforward computation shows that (ϕ, ξ, η, g) is indeed a Sasakian structure. Finally, since, by construction, $\nabla_{X_j} X_i = \nabla_{Y_j} X_i = \nabla_{\xi} X_i = 0$ and $\nabla_{X_j} Y_i = \nabla_{Y_j} Y_i = \nabla_{\xi} Y_i = 0$, we deduce easily that $\nabla\phi = 0$, and by Theorem 3.1 we get that $\nabla = *\nabla$. \square

Remark 4.6. Assuming in Theorems 4.4 and 4.5 that the manifold is also simply connected, we have that (M^{2n+1}, η) and (M^{2n}, ω) coincide with \mathbf{R}^{2n+1} and \mathbf{R}^{2n} with their usual contact and symplectic structure, respectively.

Removing the initial hypothesis of $T\mathcal{G}$ being an affine transversal distribution for \mathcal{F} , we have the following result, from a theorem of Jayne [9].

Theorem 4.7. *Let (M^{2n+1}, η) be a contact manifold foliated by a flat Legendrian foliation \mathcal{F} . Then there exists a Sasakian structure (ϕ, ξ, η, g) on (M^{2n+1}, η) whose Tanaka-Webster connection coincides with the bi-Legendrian connection associated to the almost bi-Legendrian structure (L, Q) , where $L = T\mathcal{F}$ and $Q = \phi L$.*

Proof. In [9] it has been proved that, given a flat Legendrian foliation \mathcal{F} of a contact manifold (M^{2n+1}, η) , there exists a canonical contact metric structure (ϕ, ξ, η, g) such that $(M^{2n+1}, \phi, \xi, \eta, g)$ is a Sasakian manifold. This Sasakian structure is defined in the following way. By the Darboux theorem for Legendrian foliations, cf. [13], for any point of M^{2n+1} there exists an open neighborhood with local coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$ such that $\eta = dz - \sum_{i=1}^n y_i dx_i$, $\xi = \partial/\partial z$, and \mathcal{F} is locally spanned by the vector fields $Y_i := \partial/\partial y_i$, $i \in \{1, \dots, n\}$. Now consider the contact metric structure (ϕ_U, ξ, η, g_U) on U given by

$$\phi_U = \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}, \quad g_U = \begin{pmatrix} \delta_{ij} + y_i y_j & 0 & -y_i \\ 0 & \delta_{ij} & 0 \\ -y_i & 0 & 1 \end{pmatrix},$$

where Y is the $(1 \times n)$ -matrix given by $Y = (y_1, \dots, y_n)$. It is known,

cf. [19], that (ϕ_U, ξ, η, g_U) is a Sasakian structure on U . Next, we consider an open covering of M^{2n+1} by Darboux neighborhoods as above and, using the fact that the leaves of \mathcal{F} have a natural flat affine structure, it can be proved (cf. [9]) that these Sasakian structures fit together to give rise to a global Sasakian structure (ϕ, ξ, η, g) on M^{2n+1} . Now consider the conjugate Legendrian distribution Q of \mathcal{F} , which by Lemma 3.6 is also flat and which is generated by the vector fields $X_i := \phi Y_i = (\partial/\partial x_i) + y_i(\partial/\partial z)$, $i \in \{1, \dots, n\}$. Applying [5, Proposition 5.1], we get $\nabla_{X_j} X_i = \nabla_{Y_j} X_i = \nabla_\xi X_i = 0$ and $\nabla_{X_j} Y_i = \nabla_{Y_j} Y_i = \nabla_\xi Y_i = 0$, from which $\nabla\phi = 0$. Then, again applying Theorem 3.1, we conclude that ∇ coincides with the Tanaka-Webster connection of $(M^{2n+1}, \phi, \xi, \eta, g)$. \square

Remark 4.8. Note that the Legendrian distribution Q of Theorem 4.7 is, a posteriori, integrable because of Theorem 3.7.

Remark 4.9. It should be noted that, by Corollary 3.4, in Theorems 4.5 and 4.7 the connections induced on the leaves of \mathcal{F} and \mathcal{G} by the Levi Civita, the Tanaka-Webster and the bi-Legendrian connection coincide.

5. Examples and remarks.

Example 5.1. Consider \mathbf{R}^{2n+1} with its standard Sasakian structure (ϕ, ξ, η, g) where

$$\eta = dz - \sum_{k=1}^n y_k dx_k, \quad \xi = \frac{\partial}{\partial z},$$

$$g = \eta \otimes \eta + \frac{1}{2} \sum_{k=1}^n \left((dx_k)^2 + (dy_k)^2 \right)$$

and ϕ is represented by the $(2n + 1) \times (2n + 1)$ matrix

$$\begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & y_1 & \cdots & y_n & 0 \end{pmatrix}.$$

The standard bi-Legendrian structure (L, Q) on $(\mathbf{R}^{2n+1}, \phi, \xi, \eta, g)$ is given by $L = \text{span}\{X_1, \dots, X_n\}$ and $Q = \text{span}\{Y_1, \dots, Y_n\}$ where,

for all $i \in \{1, \dots, n\}$, $X_i := \partial/\partial y_i$ and $Y_i := (\partial/\partial x_i) + y_i(\partial/\partial z)$. It is easy to check that $\phi X_i = Y_i$ for all $i \in \{1, \dots, n\}$ and that L and Q define two orthogonal flat Legendrian foliations on \mathbf{R}^{2n+1} . Let ∇ be the corresponding bi-Legendrian connection. A straightforward computation shows that $\nabla_{X_i} X_j = \nabla_{Y_i} X_j = \nabla_{\xi} X_j = 0$ and $\nabla_{X_i} Y_j = \nabla_{Y_i} Y_j = \nabla_{\xi} Y_j = 0$. Using these relations, we have $\nabla\phi = 0$ and so, by Proposition 2.9, $\nabla g = 0$. Then, by Theorem 3.1, the bi-Legendrian connection ∇ coincides with the Tanaka-Webster connection on $(\mathbf{R}^{2n+1}, \phi, \xi, \eta, g)$. In particular, with the notation of Remark 3.5, $L \in \mathfrak{S}_{\mathbf{R}^{2n+1}}$. Another consequence is that the Tanaka-Webster connection on \mathbf{R}^{2n+1} is everywhere flat since ∇ is flat, cf. [4].

Corollary 5.2. *Let \mathcal{F}' be any Legendrian foliation on \mathbf{R}^{2n+1} belonging to $\mathfrak{S}_{\mathbf{R}^{2n+1}}$. Then the curvature of the corresponding bi-Legendrian connection vanishes identically.*

Proof. \mathcal{F}' is a flat Legendrian foliation on \mathbf{R}^{2n+1} such that its conjugate Legendrian distribution is integrable and $\nabla'g = 0$, where ∇' denotes the bi-Legendrian connection associated to \mathcal{F}' . So, by Remark 3.5 we have $\nabla = \nabla'$, ∇ denoting the bi-Legendrian connection associated to the standard bi-Legendrian structure on \mathbf{R}^{2n+1} . In particular, the curvature tensor fields of the two connections must coincide and the result follows from the flatness of ∇ . \square

Now we give an example of a Sasakian manifold endowed with a non-flat bi-Legendrian structure for which the corresponding bi-Legendrian connection is metric but does not coincide with the Tanaka-Webster connection.

Example 5.3. Consider the sphere $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ with the following Sasakian structure:

$$\begin{aligned} \eta &= x_3 dx_1 + x_4 dx_2 - x_1 dx_3 - x_2 dx_4, \\ \xi &= x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} - x_2 \frac{\partial}{\partial x_4}, \end{aligned}$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Set $X := x_2\partial/\partial x_1 - x_1(\partial/\partial x_2) - x_4(\partial/\partial x_3) + x_3(\partial/\partial x_4)$ and $Y := \phi X = x_4(\partial/\partial x_1) - x_3(\partial/\partial x_2) + x_2(\partial/\partial x_3) - x_1(\partial/\partial x_4)$, and consider the one-dimensional distributions L and Q on S^3 generated by X and Y , respectively. An easy computation shows that $[X, \xi] = -2Y$, $[Y, \xi] = 2X$, $[X, Y] = 2\xi$. Thus, L and Q define two Legendrian foliations on the Sasakian manifold $(S^3, \phi, \xi, \eta, g)$ which are orthogonal and not flat. For the bi-Legendrian connection corresponding to this bi-Legendrian structure, we have, after a straightforward computation, $\nabla_X X = \nabla_X Y = \nabla_X \xi = \nabla_Y X = \nabla_Y Y = \nabla_Y \xi = 0$. Therefore, $\nabla \phi = 0$. But $T(\xi, \phi V) = -\phi T(\xi, V)$ for all $V \in \Gamma(TS^3)$ is not satisfied; indeed $T(\xi, \phi Y) = -T(\xi, X) = [\xi, X] = 2Y$ and, on the other hand, $\phi T(\xi, Y) = -\phi[\xi, Y] = 2\phi X = 2Y$, so that $T(\xi, \phi Y) = -\phi T(\xi, X)$ holds if and only if $Y = 0$.

We conclude with an example of a bi-Legendrian structure on a non-Sasakian manifold.

Example 5.4. Let \mathfrak{g} be a $(2n+1)$ -dimensional Lie algebra with basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$. The Lie bracket is defined in the following way:

$$\begin{aligned} [X_i, X_j] &= 0 \text{ for any } i, j \in \{1, \dots, n\}, \quad [Y_i, Y_j] = 0 \text{ for any } i \neq 2, \\ [Y_2, Y_j] &= 2Y_j \text{ for any } j \neq 2, \quad [X_1, Y_1] = 2\xi - 2X_2, \quad [X_1, Y_j] = 0 \\ &\text{for any } j \geq 2, \\ [X_h, Y_k] &= \delta_{hk} (2\xi - 2X_2) \text{ for any } h, k \geq 3, \quad [X_2, Y_j] = 2X_j \\ &\text{for any } j \neq 2, \\ [X_2, Y_2] &= 2\xi, \quad [X_k, Y_1] = [X_k, Y_2] = 0 \text{ for any } k \geq 3, \\ [\xi, X_j] &= 0 \text{ and } [\xi, Y_j] = 2X_j \text{ for any } j \in \{1, \dots, n\}. \end{aligned}$$

Let G be a Lie group whose Lie algebra is \mathfrak{g} . On G one can define a contact metric structure by defining $\phi\xi = 0$, $\phi X_i = Y_i$ and $\phi Y_i = -X_i$, for all $i \in \{1, \dots, n\}$, considering the left invariant Riemannian metric g such that $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$ is an orthonormal frame and,

finally, defining the 1-form η as the dual 1-form of the vector field ξ with respect to the metric g . It can be proved, cf. [3], that (G, ϕ, ξ, η, g) is a contact (κ, μ) -manifold with $\kappa = 0$ and $\mu = 4$ and so it is certainly non-Sasakian. Let L and Q be the n -dimensional distributions generated, respectively, by X_1, \dots, X_n and Y_1, \dots, Y_n . They can also be viewed as the eigenspaces of the eigenvectors λ and $-\lambda$ of the operator h , where $\lambda = \sqrt{1 - \kappa} = 1$. As remarked in Example 2.10, L and Q define two orthogonal Legendrian foliations of the contact metric manifold (G, ϕ, ξ, η, g) , and the corresponding bi-Legendrian connection satisfies $\nabla g = 0$, $\nabla \phi = 0$. Nevertheless it does not coincide with the Tanaka-Webster connection of (G, ϕ, ξ, η, g) . Indeed, $T(\xi, \phi X_1) = -T(Y_1, \xi) = -[\xi, Y_1]_L = -2X_1$ and, on the other hand, $T(\xi, X_1) = -T(X_1, \xi) = -[\xi, X_1]_Q = 0$, so $T(\xi, \phi X_1) \neq -\phi T(\xi, X_1)$.

APPENDIX

Recall that a Lagrangian foliation of a symplectic manifold (M^{2n}, ω) is an n -dimensional foliation \mathcal{F} of M^{2n} such that $\omega(X, X') = 0$ for any $X, X' \in \Gamma(T\mathcal{F})$. A bi-Lagrangian structure on (M^{2n}, ω) is nothing but a pair of transversal Lagrangian foliations $(\mathcal{F}, \mathcal{G})$ on (M^{2n}, ω) . In [8] Hess proved that, given two transversal Lagrangian distributions L and Q on M^{2n} , there exists a unique symplectic connection ∇ on M^{2n} preserving the distributions L and Q and whose torsion tensor field satisfies

$$(21) \quad T(X, Y) = 0$$

for all $X \in \Gamma(L)$ and $Y \in \Gamma(Q)$. This connection is called the *bi-Lagrangian connection* associated to (L, Q) , and if L and Q are integrable, i.e., if they define a bi-Lagrangian structure on M^{2n} , ∇ is torsion free and it is flat along the leaves of the foliations. In this Appendix we prove the already stated Theorem 4.4, which, to the knowledge of the author, has not yet been proved elsewhere.

Lemma 5.5 [7]. *Let $(\mathcal{F}, \mathcal{G})$ be a bi-Lagrangian structure on the symplectic manifold (M^{2n}, ω) . Let (J, ω, g) be a Hermitian structure on (M^{2n}, ω) . Then, for the bi-Lagrangian connection associated to $(\mathcal{F}, \mathcal{G})$, we have $\nabla g = 0$ if and only if $\nabla J = 0$.*

Proof of Theorem 4.4. First of all note that, as in Theorem 4.5, the assumption of $T\mathcal{G}$ being an affine transversal distribution implies that ∇ is everywhere flat. Fixing a point x of M^{2n} , there exists a basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ of $T_x M^{2n}$ such that $\{e_1, \dots, e_n\}$ is a basis of $T_x \mathcal{F}$, $\{e_{n+1}, \dots, e_{2n}\}$ is a basis of $T_x \mathcal{G}$ and

$$(22) \quad \omega_x(e_i, e_j) = \omega_x(e_{n+i}, e_{n+j}) = 0, \quad \omega_x(e_i, e_{n+j}) = -\frac{1}{2}\delta_{ij}$$

for all $i, j \in \{1, \dots, n\}$. For each $k \in \{1, \dots, 2n\}$, we define a vector field E_k on M^{2n} by the ∇ -parallel transport along curves of the vector e_k . Note that, for all $y \in M^{2n}$, $E_k(y)$ does not depend on the curve joining x and y , since $R \equiv 0$. Setting $X_i := E_{n+i}$ and $Y_i := E_i$, we obtain $2n$ vector fields on M^{2n} such that, for each $i \in \{1, \dots, n\}$, $Y_i \in \Gamma(T\mathcal{F})$ and $X_i \in \Gamma(T\mathcal{G})$, because the parallel transport preserves the distributions $T\mathcal{F}$ and $T\mathcal{G}$. Moreover, since $\nabla\omega = 0$ and (22) hold at any point of M^{2n} , that is,

$$(23) \quad \begin{aligned} \omega_y(Y_i(y), Y_j(y)) &= \omega_y(X_i(y), X_j(y)) = 0, \\ \omega_y(Y_i(y), X_j(y)) &= -\frac{1}{2}\delta_{ij} \end{aligned}$$

for any $y \in M^{2n}$ and $i, j \in \{1, \dots, n\}$. Note that, by construction, we have $\nabla_{E_n} E_k = 0$ for all $h, k \in \{1, \dots, 2n\}$. From this and (21), we get

$$(24) \quad [Y_i, Y_j] = [X_i, X_j] = [Y_i, X_j] = 0$$

for all $i, j \in \{1, \dots, n\}$, and (24) implies the existence of coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ such that, for each $i \in \{1, \dots, n\}$, $Y_i = (\partial/\partial y_i)$ and $X_j = (\partial/\partial x_j)$. So in this coordinate system we have that $T\mathcal{F}$ is spanned by $Y_i = (\partial/\partial y_i)$, $T\mathcal{G}$ by $X_i = (\partial/\partial x_i)$, $i \in \{1, \dots, n\}$, and, moreover, by (23), $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. Now we define a tensor field J and a Riemannian metric g on M^{2n} putting, for each $i \in \{1, \dots, n\}$, $JY_i = X_i$, $JX_i = -Y_i$ and $g(V, W) = -\omega(V, JW)$ for all $V, W \in \Gamma(TM^{2n})$. A straightforward computation shows that (J, ω, g) is indeed a Kählerian structure. Finally, since, by construction, $\nabla_{X_j} X_i = \nabla_{Y_j} X_i = \nabla_{X_j} Y_i = \nabla_{Y_j} Y_i = 0$, we deduce easily that $\nabla J = 0$, which, by Lemma 5.5, imply that $\nabla g = 0$. Thus, ∇ coincides with the Levi Civita connection of (M^{2n}, J, ω, g) . \square

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