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MULTIPLE POSITIVE SOLUTIONS OF SECOND-ORDER STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS FOR IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we study two types of impulsive Sturm-Liouville boundary value problems depending on the parameter λ. The existence of multiple positive solutions is obtained by applying a three critical points theorem given by Averna and Bonanno [**2**].

1. Introduction. In recent years, a great deal of work has been done on the study of the existence of solutions of boundary value problems for impulsive differential equations, by which the phenomena, such as many evolution processes, states changed at certain moments of time due to abrupt changes, are described. For relevant and recent references on impulsive differential equations, we refer to [**13, 20–22, 26, 27**]. For the background and applications of the theory of impulsive differential equations to different areas, we refer the reader to the monographs and some recent contributions as [**7, 9, 11, 14, 18, 19, 29, 33, 34, 36, 37**].

Some classical tools have been used to study impulsive differential equations in the literature. These classical tools include fixed point theorems in cones [**1, 8, 12, 15**] and the method of lower and upper solutions with monotone iterative technique, see [**10, 16**].

Critical point theory is a new method to deal with the existence of solutions for boundary value problems, please refer to [**2–6, 17, 23–25,**

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30, 31, 32]. The method has become a powerful tool. In critical point theory, Averna and Bonanno gave a definite interval, say

$$\left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[,$$

in which if λ lies, then $\Phi + \lambda\Psi$ has at least three critical points. Their result is as follows.

Theorem 1.1 [**2**]. *Let X be a reflexive real Banach space, let $\Phi : X \to R$ be a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|x\|\to+\infty}(\Phi(x) + \lambda\Psi(x)) = +\infty$ for all $\lambda \in [0, +\infty[$;

(ii) *there is an $r \in R$ such that*

$$\inf_X \Phi < r$$

and

$$\varphi_1(r) < \varphi_2(r),$$

where

$$\varphi_1(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi}{r - \Phi(x)},$$

$$\varphi_2(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \sup_{y\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)},$$

and $\overline{\Phi^{-1}(]-\infty, r[)}^w$ is the closure of $\Phi^{-1}(]-\infty, r[)$ in the weak topology.

Then, for each

$$\lambda \in \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[,$$

the functional $\Phi + \lambda\Psi$ has at least three critical points in X.

This theorem has been applied to Dirichlet, mixed and Sturm-Liouville boundary value problems to get interesting results, see [**2–4,**

31]. In [**31**], Tian and Ge studied the second-order Sturm-Liouville boundary value problem

$$(1.1)\qquad \begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = \lambda f(t,x(t)) & t \in [a,b], \\ \alpha x'(a) - \beta x(a) = A \quad \gamma x'(b) + \sigma x(b) = B, \end{cases}$$

where $p > 1$, $\Phi_p(x) := |x|^{p-2}x$, $\rho, s \in L^\infty[a,b]$ with $\operatorname{ess\,inf}_{[a,b]} \rho > 0$ and $\operatorname{ess\,inf}_{[a,b]} s > 0$, $\lambda \in]0,+\infty[$, A, B are constants, $\alpha, \beta, \gamma, \sigma > 0$ and f is an L^1-Carathéodory function. By using a three critical points theorem, the existence of three solutions was obtained.

As far as we know, there are few papers to study impulsive boundary value problems by using the critical point theorem. The aim of this paper is to apply Theorem 1.1 to more general fields. In this paper, we study impulsive boundary value problems

$$(1.2)\quad \begin{cases} -(\rho(t)\Phi_p(x'(t)))' = \lambda f(t,x(t)) & t \neq t_i,\ t \in [0,T], \\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) = \lambda I_i(x(t_i)) & i = 1,2,\dots,l, \\ \alpha x'(0) - \beta x(0) = 0 \quad \gamma x'(T) + \sigma x(T) = 0, \end{cases}$$

and

$$(1.3)\quad \begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = \lambda f(t,x(t)) & t \neq t_i,\ t \in [0,T], \\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) = \lambda I_i(x(t_i)) & i = 1,2,\dots,l, \\ \alpha x'(0) - \beta x(0) = 0, \quad \gamma x'(T) + \sigma x(T) = 0, \end{cases}$$

where $p > 1$, $\Phi_p(x) := |x|^{p-2}x$, $\rho, s \in L^\infty[0,T]$ with $\operatorname{ess\,inf}_{[0,T]} \rho > 0$ and $\operatorname{ess\,inf}_{[0,T]} s > 0$, $\lambda \in]0,+\infty[$, $0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = T$, $\Delta(\rho(t_i)\Phi_p(x'(t_i))) = \rho(t_i^+)\Phi_p(x'(t_i^+)) - \rho(t_i^-)\Phi_p(x'(t_i^-))$, where $x'(t_i^+)$ (respectively $x'(t_i^-)$) denotes the right limit (respectively left limit) of $x'(t)$ at $t = t_i$, $I_i \in C([0,+\infty),[0,+\infty))$, $i = 1,2,\dots,l$, $f \in C([0,T] \times [0,+\infty),[0,+\infty))$, $f(t,0) \not\equiv 0$ for $t \in [0,T]$, $\alpha, \beta, \gamma, \sigma > 0$. Under suitable hypotheses, we prove that problem (1.2) ((1.3)) has at least three positive solutions when λ lies in an explicitly determined open interval.

For impulsive problems (1.2) and (1.3), the construction of corresponding functionals $\Phi, \overline{\Phi}$ are different, which yields the difficulty of verification of assumptions in Theorem 1.1, for example, Φ^{-1} is continuous on X^*. Lemma 2.6, Lemma 2.7 and Lemma 2.9 are very important in overcoming these difficulties.

This paper is organized as follows. In Section 2, to (1.2), the variational approach is justified and the regularity of an appropriate functional involved is proved. In Section 3, existence results of (1.2) are given in Theorem 3.1 and Corollary 3.2. In Section 4, to (1.3), the variational approach is justified and the regularity of an appropriate functional involved is proved. At the same time, existence results of (1.3) are given in Theorem 4.6 and Corollary 4.7. Besides, some examples are presented in each section to illustrate the results obtained.

2. Related lemmas for (1.2). Let $W^{1,p}([0,T]) = \{x \in C([0,T]) : x' \in L^p([0,T])\}$ with the norm

$$\|x\|_{W^{1,p}} = \left(\int_0^T |x'(t)|^p + |x(t)|^p dt \right)^{1/p}.$$

Lemma 2.1 [**32**]. *For $x \in W^{1,p}([0,T])$, let $x^{\pm} = \max\{\pm x, 0\}$. Then the following six properties hold*:

(i) $x \in W^{1,p}([0,T]) \Rightarrow x^+, x^- \in W^{1,p}([0,T])$;

(ii) $x = x^+ - x^-$;

(iii) $\|x^+\|_{W^{1,p}} \leq \|x\|_{W^{1,p}}$;

(iv) *if (x_n) uniformly converges to x in $C([0,T])$, then (x_n^+) uniformly converges to x^+ in $C([0,T])$*;

(v) $x^+(t)x^-(t) = 0, (x^+)'(t)(x^-)'(t) = 0$ *for* $t \in [0,T]$;

(vi) $\Phi_p(x)x^+ = |x^+|^p$, $\Phi_p(x)x^- = -|x^-|^p$.

In order to obtain the existence of positive solutions for (1.2), now we consider the problem

$$\begin{cases} -(\rho(t)\Phi_p(x'(t)))' = \lambda f(t, x^+(t)) & t \neq t_i, t \in [0,T], \\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) = \lambda I_i(x^+(t_i)) & i = 1, 2, \dots, l, \\ \alpha x'(0) - \beta x(0) = 0, \quad \gamma x'(T) + \sigma x(T) = 0. \end{cases} \tag{2.1}$$

Lemma 2.2. *If $x \in C([0,T])$ is a solution of (2.1), then $x(t) \geq 0, x(t) \not\equiv 0$, $t \in [0,T]$, and hence it is a positive solution of (1.2).*

Proof. If $x \in C([0,T])$ is a solution of (2.1), then by Lemma 2.1,

$$\begin{aligned}
0 &= \int_0^T \left[(\rho(t)\Phi_p(x'(t)))' + \lambda f(t, x^+(t))\right] x^-(t)\, dt \\
&= \sum_{i=0}^{l} \rho(t)\Phi_p(x'(t))x^-(t)\big|_{t=t_i^+}^{t_{i+1}} \\
&\quad - \int_0^T \rho(t)\Phi_p(x'(t))(x^-)'(t)\, dt \\
&\quad + \lambda \int_0^T f(t, x^+(t))x^-(t)\, dt \\
&= -\sum_{i=1}^{l} \Delta(\rho(t_i)\Phi_p(x'(t_i)))x^-(t_i) \\
&\quad - \rho(0)\Phi_p\left(\frac{\beta}{\alpha}x(0)\right)x^-(0) \\
&\quad - \rho(T)\Phi_p\left(\frac{\sigma}{\gamma}x(T)\right)x^-(T) \\
&\quad + \int_0^T \left[\rho(t)|(x^-)'(t)|^p + \lambda f(t, x^+(t))x^-(t)\right]\, dt \\
&\geq \sum_{i=1}^{l} \lambda I_i(x^+(t_i))x^-(t_i) + \int_0^T \lambda f(t, x^+(t))x^-(t)\, dt \\
&\quad + \int_0^T \rho(t)|(x^-)'(t)|^p dt \\
&\geq \int_0^T \rho(t)|(x^-)'(t)|^p dt.
\end{aligned}$$

So $(x^-)'(t) = 0$ for almost every $t \in [0,T]$, i.e., $x^-(t) = c \geq 0$ for $t \in [0,T]$. So $x(t) \equiv -c$. By boundary condition, $\alpha x'(0) - \beta x(0) = 0 - \beta(-c) = \beta c = 0$, we have $c = 0$. So $x(t) \geq 0$. If $x(t) \equiv 0$ for $t \in [0,T]$, the fact $f(t,0) \not\equiv 0$ gives a contradiction. □

Now we define the space $X = W^{1,p}([0,T])$ equipped with the norm

$$\|x\|_X = \left(\int_0^T \rho(t)|x'(t)|^p + |x(t)|^p dt\right)^{1/p},$$

which is clearly equivalent to the usual one $\|x\|_{W^{1,p}}$; F is the real function

$$F(t,\xi)=\int_0^\xi f(t,x)\,dx.$$

We denote $\|x\|_\infty := \max_{t\in[0,T]}|x(t)|$ to be the norm in $C^0([0,T])$.

For each $x\in X$, put

$$\begin{aligned}\Phi(x) := \frac{1}{p}\int_0^T \rho(t)|x'(t)|^p dt + \frac{\rho(T)}{p}\left(\frac{\sigma}{\gamma}\right)^{p-1}|x(T)|^p \\ + \frac{\rho(0)}{p}\left(\frac{\beta}{\alpha}\right)^{p-1}|x(0)|^p,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\Psi(x) := -\int_0^T \left[F(t,x^+(t)) - f(t,0)x^-(t)\right]\,dt \\ -\sum_{i=1}^{l}\left[\int_0^{x^+(t_i)} I_i(s)\,ds - I_i(0)x^-(t_i)\right].\end{aligned} \tag{2.3}$$

Clearly, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x\in X$ is the functional $\Phi'(x)\in X^*$, given by

$$\begin{aligned}\langle\Phi'(x),v\rangle = \int_0^T \rho(t)\Phi_p(x'(t))v'(t)\,dt + \rho(T)\Phi_p\left(\frac{\sigma}{\gamma}x(T)\right)v(T) \\ + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}x(0)\right)v(0)\end{aligned} \tag{2.4}$$

for every $v\in X$, and $\Phi' : X\to X^*$ is continuous. Moreover, taking into account that Φ is convex, from Proposition 25.20 (i) of [**35**], we obtain that Φ is a sequentially weakly lower semi-continuous functional.

It is easy to see that $\Psi : X\to R$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x\in X$ is the functional $\Psi'(x)\in X^*$ given by

$$\langle\Psi'(x),v\rangle = -\int_0^T f(t,x^+(t))v(t)\,dt - \sum_{i=1}^{l} I_i(x^+(t_i))v(t_i) \tag{2.5}$$

for every $v\in X$.

Lemma 2.3 [**31**]. *If the function $x \in X$ is a critical point of the functional $\Phi + \lambda\Psi$, then x is a solution of* (2.1).

Remark 2.1. By Lemma 2.2 and Lemma 2.3, if $x \in X$ is a critical point of the functional $\Phi + \lambda\Psi$, then x is a positive solution of (1.2).

Lemma 2.4. *If $x \in X$, and there exists an $r > 0$ such that $\Phi(x) \le r$, then*

$$\|x\|_\infty \le \sqrt[p]{pr}\left[\left(\frac{\alpha}{\beta}\right)^{1/q}(\rho(0))^{-1/p} + T^{1/q}\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{-1/p}\right] := \Theta(r).$$

Proof. If $\Phi(x) \le r$, then

$$\int_0^T \rho(t)|x'(t)|^p dt \le pr, \tag{2.6}$$

$$\rho(0)\left(\frac{\beta}{\alpha}\right)^{p-1}|x(0)|^p \le pr. \tag{2.7}$$

By the mean value theorem and (2.6), (2.7), we have

$$\begin{aligned}|x(t)| &= \left|x(0) + \int_0^t x'(s)\,ds\right| \le |x(0)| + \left(\int_0^T |x'(s)|^p ds\right)^{1/p} T^{1/q}\\ &\le \sqrt[p]{pr}\left(\frac{\alpha}{\beta}\right)^{1/q}(\rho(0))^{-1/p} + \frac{\left(\int_0^T \rho(t)|x'(t)|^p dt\right)^{1/p}}{\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{1/p}}T^{1/q}\\ &\le \sqrt[p]{pr}\left(\frac{\alpha}{\beta}\right)^{1/q}(\rho(0))^{-1/p} + \sqrt[p]{pr}T^{1/q}\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{-1/p}\\ &= \Theta(r). \qquad \square\end{aligned}$$

Lemma 2.5. *If $x \in X$, then*

$$\begin{aligned}&\int_0^T |x(t)|^p dt\\ &\le T\max\{2^{p-1},1\}\left[|x(0)|^p + T^{p/q}\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{-1}\int_0^T \rho(s)|x'(s)|^p ds\right].\end{aligned}$$

Proof. For $x \in X$, we have

$$\begin{aligned}
\int_0^T |x(t)|^p dt &= \int_0^T \left| x(0) + \int_0^t x'(s)\, ds \right|^p dt \\
&\le \max\{2^{p-1}, 1\} \int_0^T |x(0)|^p + \left(\int_0^t |x'(s)|\, ds \right)^p dt \\
&\le T \max\{2^{p-1}, 1\} \Big[|x(0)|^p + T^{p/q} \int_0^T |x'(s)|^p ds \Big] \\
&\le T \max\{2^{p-1}, 1\} \Big[|x(0)|^p + T^{p/q} \left(\operatorname{ess} \inf_{[0,T]} \rho \right)^{-1} \\
&\qquad \times \int_0^T \rho(s) |x'(s)|^p ds \Big]. \qquad \square
\end{aligned}$$

Lemma 2.6. *If $x \in X \setminus \{0\}$, then there exists a $\theta_1 > 0$ satisfying*

$$\begin{aligned}
&\int_0^T \rho(t)|x'(t)|^p dt + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)|x(0)|^p \\
&\qquad > \theta_1 \int_0^T \left(\rho(t)|x'(t)|^p + |x(t)|^p\right)\, dt := \theta_1 \|x\|_X^p.
\end{aligned}$$

Proof. Let

$$\begin{aligned}
0 < \theta_1 < \min \Bigg\{ &\left[1 + T^p \max\{2^{p-1}, 1\} \left(\operatorname{ess} \inf_{[0,T]} \rho \right)^{-1} \right]^{-1}, \\
&\frac{\rho(0)\Phi_p\,(\beta/\alpha)}{T \max\{2^{p-1}, 1\}} \Bigg\}.
\end{aligned}$$

Then by Lemma 2.5 we have

$$\begin{aligned}
&(1-\theta_1) \int_0^T \rho(t)|x'(t)|^p dt + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)|x(0)|^p \\
&\quad - \theta_1 \int_0^T |x(t)|^p dt
\end{aligned}$$

$$
\begin{aligned}
&\geq (1-\theta_1)\int_0^T \rho(t)|x'(t)|^p dt + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)|x(0)|^p \\
&- \theta_1 T \max\{2^{p-1},1\}\Big[|x(0)|^p + T^{p/q}\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{-1} \\
&\qquad\qquad \times \int_0^T \rho(s)|x'(s)|^p ds\Big] \\
&= \Big\{1-\theta_1\Big[1+T^p\max\{2^{p-1},1\}\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{-1}\Big]\Big\} \\
&\qquad\qquad \times \int_0^T \rho(t)|x'(t)|^p\,dt \\
&\quad + |x(0)|^p\Big[\rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right) - \theta_1 T\max\{2^{p-1},1\}\Big] \\
&> 0.
\end{aligned}
$$

The proof is complete. □

Lemma 2.7. *For $p \geq 2$, there exists a $\theta_2 > 0$ satisfying*

$$
\begin{aligned}
&\int_0^T \rho(t)\left[\Phi_p(u'(t)) - \Phi_p(v'(t))\right](u'(t)-v'(t))\,dt \\
&\quad + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)\left[\Phi_p(u(0)) - \Phi_p(v(0))\right](u(0)-v(0)) > \theta_2\|u-v\|_X^p
\end{aligned}
$$

for $u, v \in X, u \not\equiv v$.

Proof. By (2.2) of [**28**], there exists a $c_p > 0$ such that

$$
\begin{aligned}
&\int_0^T \rho(t)\left[\Phi_p(u'(t)) - \Phi_p(v'(t))\right](u'(t)-v'(t))\,dt \\
&\quad + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)\left[\Phi_p(u(0)) - \Phi_p(v(0))\right](u(0)-v(0)) \\
&\geq c_p\int_0^T \rho(t)|u'(t)-v'(t)|^p dt + c_p\rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)|u(0)-v(0)|^p.
\end{aligned}
\tag{2.8}
$$

By Lemma 2.6,

$$\text{the right hand of (2.8)} \ > c_p\theta_1\|u-v\|_X^p.$$

Let $\theta_2 = c_p\theta_1$; the proof is complete. $\square$

Lemma 2.8. *If $1 < p < 2$, $K > 0$, then there exists a $\theta_3 > 0$ such that*

$$\left(\int_0^T \rho(t)|x'(t)|^p dt\right)^{2/p} + K|x(0)|^2 > \theta_3\|x\|_X^2$$

for $x \in X \setminus \{0\}$.

Proof. Let

$$0 < \theta_3 < \min\left\{\left[2^{2/p-1} + T^2 2^{2/p}\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{-2/p}\right]^{-1},\ K(2T)^{-2/p}\right\}.$$

Then

$$\begin{aligned}(2.9)\quad \left(\int_0^T \rho(t)|x'(t)|^p dt\right)^{2/p} &+ K|x(0)|^2 \\ &- \theta_3\left(\int_0^T \rho(t)|x'(t)|^p + |x(t)|^p dt\right)^{2/p} \\ &\geq \left(\int_0^T \rho(t)|x'(t)|^p dt\right)^{2/p} + K|x(0)|^2 \\ &- \theta_3 2^{(2/p)-1}\left(\int_0^T \rho(t)|x'(t)|^p dt\right)^{2/p} \\ &- \theta_3 2^{(2/p)-1}\left(\int_0^T |x(t)|^p dt\right)^{2/p}.\end{aligned}$$

By Lemma 2.5,

$$\begin{aligned}&\text{the right hand of (2.9)} \\ &\geq \left(1-\theta_3 2^{(2/p)-1}\right)\left(\int_0^T \rho(t)|x'(t)|^p dt\right)^{2/p} + K|x(0)|^2 \\ &\quad - \theta_3 2^{(2/p)-1} T^{2/p}\left(\max\left\{2^{p-1},1\right\}\right)^{2/p}\end{aligned}$$

$$
\begin{aligned}
&\quad \times \left[|x(0)|^p + T^{p/q}\left(\text{ess}\inf_{[0,T]}\rho\right)^{-1}\int_0^T \rho(s)|x'(s)|^p ds\right]^{2/p}\\
&\geq \left(1-\theta_3 2^{2/p-1}\right)\left(\int_0^T \rho(t)|x'(t)|^p\right)^{2/p} + K|x(0)|^2\\
&\quad - \theta_3(2T)^{2/p}\left[|x(0)|^2 + T^{2/q}\left(\text{ess}\inf_{[0,T]}\rho\right)^{-2/p}\right.\\
&\quad \left.\times\left(\int_0^T \rho(t)|x'(t)|^p dt\right)^{2/p}\right]\\
&= \left[1-\theta_3 2^{(2/p)-1} - \theta_3 2^{2/p}T^2\left(\text{ess}\inf_{[0,T]}\rho\right)^{-2/p}\right]\\
&\quad \times\left(\int_0^T \rho(t)|x'(t)|^p dt\right)^{2/p}\\
&\quad + \left[K-\theta_3(2T)^{2/p}\right]|x(0)|^2\\
&> 0.
\end{aligned}
$$

The proof is complete. □

Lemma 2.9. *For $1 < p < 2$, there exists $\theta_4 > 0$ satisfying*

$$
\begin{aligned}
&\int_0^T \rho(t)\left[\Phi_p(u'(t)) - \Phi_p(v'(t))\right](u'(t)-v'(t))\,dt\\
&\qquad + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)\left[\Phi_p(u(0)) - \Phi_p(v(0))\right](u(0)-v(0))\\
&\qquad > \theta_4 \frac{\|u-v\|_X^2}{(\|u\|_X + \|v\|_X)^{2-p}}
\end{aligned}
$$

for $u, v \in X$, $u \not\equiv v$.

Proof. By (2.2) of [**28**], there exists a $d_p > 0$ satisfying

$$\begin{aligned}(2.10)\quad &\int_0^T \rho(t)\left[\Phi_p(u'(t)) - \Phi_p(v'(t))\right](u'(t) - v'(t))\,dt \\ &+ \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)\left[\Phi_p(u(0)) - \Phi_p(v(0))\right](u(0) - v(0)) \\ &\geq d_p \int_0^T \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\,dt + d_p\rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)\frac{|u(0) - v(0)|^2}{(|u(0)| + |v(0)|)^{2-p}}.\end{aligned}$$

By the Hölder inequality, we obtain that

$$\begin{aligned}(2.11)\quad &\int_0^T \rho(t)|u'(t) - v'(t)|^p dt \\ &\leq \left(\int_0^T \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\,dt\right)^{p/2} \\ &\qquad \times \left[\int_0^T \rho(t)(|u'(t)| + |v'(t)|)^p dt\right]^{(2-p)/2} \\ &\leq \left(\int_0^T \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\,dt\right)^{p/2} 2^{((p-1)(2-p))/2} \\ &\times \left(\int_0^T \rho(t)|u'(t)|^p + \rho(t)|v'(t)|^p\,dt\right)^{(2-p)/2} \\ &\leq M\left(\int_0^T \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\,dt\right)^{p/2} (\|u\|_X + \|v\|_X)^{((2-p)p)/2}\end{aligned}$$

holds for some $M > 0$.

By the mean value theorem, there exists a $\tau \in (0, T)$ such that

$1/T\int_0^T u(s)\,ds = u(\tau)$. Then

$$
\begin{aligned}
|u(0)| &= \left|u(\tau) - \int_0^\tau u'(s)\,ds\right| \\
&\le \frac{1}{T}\int_0^T |u(s)|\,ds + \int_0^T |u'(s)|\,ds \\
&\le T^{-1/p}\left(\int_0^T |u(s)|^p ds\right)^{1/p} + T^{1/q}\left(\int_0^T |u'(s)|^p ds\right)^{1/p} \qquad (2.12)\\
&\le \left[T^{-1/p} + T^{1/q}\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{-1/p}\right]\|u\|_X \\
&:= \Delta\|u\|_X.
\end{aligned}
$$

By (2.10), (2.11) and (2.12), we have

$$
\begin{aligned}
&\int_0^T \rho(t)\left[\Phi_p(u'(t)) - \Phi_p(v'(t))\right](u'(t) - v'(t))\,dt \\
&\qquad + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)\left[\Phi_p(u(0)) - \Phi_p(v(0))\right](u(0) - v(0)) \\
&\quad \ge d_p\left[\frac{\int_0^T \rho(t)|u'(t) - v'(t)|^p dt}{M(\|u\|_X + \|v\|_X)^{((2-p)p)/2}}\right]^{2/p} \\
&\qquad + d_p\rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)\frac{|u(0) - v(0)|^2}{\Delta^{2-p}(\|u\|_X + \|v\|_X)^{2-p}} \\
&\quad = \frac{K_1}{(\|u\|_X + \|v\|_X)^{2-p}}\left\{\left(\int_0^T \rho(t)|u'(t) - v'(t)|^p dt\right)^{2/p}\right. \\
&\qquad\qquad \left. + K_2|u(0) - v(0)|^2\right\}
\end{aligned}
$$

for some $K_1, K_2 > 0$.

Applying Lemma 2.8, the result follows as $\theta_4 = K_1\theta_3$. $\square$

Lemma 2.10. *Φ' admits a continuous inverse on X^*.*

Proof. First we will show that Φ' is coercive.

For every $x \in X\backslash\{0\}$, it follows from (2.4) and Lemma 2.6 that

$$\begin{aligned}
&\lim_{\|x\|_X \to \infty} \frac{\langle \Phi'(x), x\rangle}{\|x\|_X} \\
&\qquad = \lim_{\|x\|_X \to \infty} \frac{\int_0^T \rho(t)|x\prime(t)|^p dt + \rho(T)\Phi_p\left(\sigma/\gamma\right)|x(T)|^p}{\|x\|_X} \\
&\qquad\quad + \frac{\rho(0)\Phi_p\left(\beta/\alpha\right)|x(0)|^p}{\|x\|_X} \\
&\qquad \geq \lim_{\|x\|_X \to \infty} \frac{\int_0^T \rho(t)|x'(t)|^p dt + \rho(0)\Phi_p\left(\beta/\alpha\right)|x(0)|^p}{\|x\|_X} \\
&\qquad \geq \lim_{\|x\|_X \to \infty} \frac{\theta_1\|x\|_X^p}{\|x\|_X} = +\infty,
\end{aligned}$$

that is, Φ' is coercive.

Moreover, given $u, v \in X$, we have by (2.4),

$$\begin{aligned}
&\langle \Phi'(u) - \Phi'(v), u - v\rangle \\
&\qquad = \int_0^T \rho(t)\left[\Phi_p(u'(t)) - \Phi_p(v'(t))\right](u'(t) - v'(t))\, dt \\
&\qquad\quad + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)\left[\Phi_p(u(0)) - \Phi_p(v(0))\right](u(0) - v(0)) \\
&\qquad\quad + \rho(T)\Phi_p\left(\frac{\sigma}{\gamma}\right)\left[\Phi_p(u(T)) - \Phi_p(v(T))\right](u(T) - v(T)).
\end{aligned}$$

If $p \geq 2$, by Lemma 2.7, we have

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle > \theta_2\|u - v\|_X^p,$$

so Φ' is uniformly monotone. By Theorem 26.A (d) of [**35**], we have that $(\Phi')^{-1}$ exists and is continuous on X^*.

If $1 < p < 2$, by Lemma 2.9, we have

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle > \theta_4 \frac{\|u - v\|_X^2}{(\|u\|_X + \|v\|_X)^{2-p}}; \tag{2.13}$$

therefore, Φ' is strictly monotone. By Theorem 26.A (d) of [**35**] we obtain that $(\Phi')^{-1}$ exists and is bounded. Furthermore, given $g_1, g_2 \in X^*$, by (2.13) we have

$$\begin{aligned}\|(\Phi')^{-1}(g_1) - (\Phi')^{-1}(g_2)\|_X \le \frac{1}{\theta_4}\big(\|(\Phi')^{-1}(g_1)\|_X \\ +\|(\Phi')^{-1}(g_2)\|_X\big)^{2-p}\|g_1 - g_2\|_{X^*},\end{aligned}$$

so $(\Phi')^{-1}$ is Lipschitz continuous for $1 < p < 2$. Thus Φ' admits a continuous inverse on X^*. The proof is complete. $\square$

Lemma 2.11. $\Psi' : X \to X^*$ *is a continuous and compact operator.*

Proof. First we will show that Ψ' is strongly continuous on X. For this, let $u_n \rightharpoonup u$ as $n \to \infty$ on X; by [**35**] we have u_n converges uniformly to u on $[0,T]$ as $n \to \infty$. Since $f, I_i,\ i = 1, 2, \dots, l$ are continuous, one has $f(t, u_n) \to f(t, u)$ and $I_i(u_n) \to I_i(u)$ as $n \to \infty$. So $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus, we have shown that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 [**35**]. Moreover, Ψ' is continuous since it is strongly continuous. The proof is complete. $\square$

3. Existence results for (1.2). In the following results, we will use the following notations:

$$\begin{aligned}L := \int_0^T \rho(t)\,dt\Big[\Big(\frac{\beta}{\alpha}\Big)^p + \Big(\frac{\sigma}{\gamma + T\sigma}\Big)^p + \Big(\frac{2\sigma}{\gamma + T\sigma} + \frac{\beta}{\alpha}\Big)^p\Big] \\ + \rho(T)\Phi_p\Big(\frac{\sigma}{\gamma}\Big)\Big(\frac{\gamma}{\gamma + T\sigma}\Big)^p + \rho(0)\Phi_p\Big(\frac{\beta}{\alpha}\Big),\end{aligned}$$

$$y(t) = \begin{cases} y_1(t) = \left(\frac{\beta}{\alpha}t + 1\right)k & t \in \left[0, \frac{T}{3}\right]; \\ y_2(t) = -kt\left(\frac{2\sigma}{\gamma+T\sigma} + \frac{\beta}{\alpha}\right) + \frac{2Tk}{3}\left(\frac{\sigma}{\gamma+T\sigma} + \frac{\beta}{\alpha}\right) & t \in \left[\frac{T}{3}, \frac{2T}{3}\right]; \\ y_3(t) = \left(-\frac{\sigma}{\gamma+T\sigma}t + 1\right)k & t \in \left[\frac{2T}{3}, T\right]. \end{cases}$$

$$\Gamma := \int_0^{T/3} F(t, y_1(t))\, dt + \int_{T/3}^{(2T)/3} F(t, y_2(t))\, dt + \int_{(2T)/3}^{T} F(t, y_3(t))\, dt + \sum_{i=1}^{l} \int_0^{y(t_i)} I_i(s)\, ds.$$

Theorem 3.1. *Assume that there exist positive constants* $k, d, l, l_i, \mu_i > 0$ *with* $l < p$, $l_i < p$, $i = 1, 2, \dots, l$, $k^p L > pd$ *and a positive function* $\mu \in C([0,T])$ *such that*:

(H1)

$$\frac{\int_0^T F(t, \Theta(d))\, dt + \sum_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\, ds}{d} < \frac{p\left\{\Gamma - \left(\int_0^T F(t, \Theta(d))\, dt + \sum_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\, ds\right)\right\}}{k^p L};$$

(H2) $F(t,\xi) \le \mu(t)(1+|\xi|^l)$ *for* $t \in [0,T]$ *and* $\xi \in [0,\infty)$; $\int_0^\xi I_i(s)\, ds \le \mu_i(1 + |\xi|^{l_i})$ *for* $\xi \in [0, \infty)$.

Then for each $\lambda \in]\lambda_1, \lambda_2[$, *the problem* (1.2) *has at least three positive solutions, where*

$$\lambda_1 = \frac{k^p L}{p\left\{\Gamma - \left(\int_0^T F(t, \Theta(d))\, dt + \sum_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\, ds\right)\right\}}$$

and

$$\lambda_2 = \frac{d}{\int_0^T F(t, \Theta(d))\, dt + \sum_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\, ds}.$$

Proof. From Section 2 we have seen that $\Phi : X \to R$ is a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by (2.4). $(\Phi')^{-1}$ exists and is continuous on X^*. $\Psi : X \to R$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative given by (2.5) is compact.

Now we will apply Theorem 1.1 to Φ and Ψ.

For (i) in Theorem 1.1, by condition (H2) and Lemma 2.6 we have

$$\begin{aligned}
&\lim_{\|x\|_X\to+\infty}(\Phi(x)+\lambda\Psi(x))\\
&\qquad\geq \lim_{\|x\|_X\to+\infty}\bigg(\frac{1}{p}\int_0^T\rho(t)|x'(t)|^p dt+\frac{\rho(0)}{p}\left(\frac{\beta}{\alpha}\right)^{p-1}|x(0)|^p\\
&\qquad\quad-\lambda\int_0^T\mu(t)(1+|x(t)|^l)\,dt-\lambda\sum_{i=1}^{l}\mu_i(1+|x(t_i)|^{l_i})\,dt\bigg)\\
&\qquad\geq \lim_{\|x\|_X\to+\infty}\frac{\theta_1}{p}\|x\|_X^p-\lambda(M_1+\|x\|_X^{\bar{l}})=+\infty,
\end{aligned}$$

where M_1 is a positive constant and $0<\bar{l}<p$. So (i) is satisfied.

To prove (ii) in Theorem 1.1, first we claim that

(A1) $\varphi_1(r)\leq(\int_0^T F(t,\Theta(r))\,dt+\sum_{i=1}^{l}\int_0^{\Theta(r)}I_i(s)\,ds)/r$ for each $r>0$ and

(A2)

$$\begin{aligned}
&\varphi_2(r)\\
&\geq p\Bigg[\frac{\int_0^T[F(t,y^+(t))-f(t,0)y^-(t)]\,dt}{\int_0^T\rho(t)|y'(t)|^p dt+\rho(T)\Phi_p\left(\sigma/\gamma\right)|y(T)|^p+\rho(0)\Phi_p\left(\beta/\alpha\right)|y(0)|^p}\\
&\quad+\frac{\sum_{i=1}^{l}\left[\int_0^{y^+(t_i)}I_i(s)\,ds-I_i(0)y^-(t_i)\right]}{\int_0^T\rho(t)|y'(t)|^p dt+\rho(T)\Phi_p\left(\sigma/\gamma\right)|y(T)|^p+\rho(0)\Phi_p\left(\beta/\alpha\right)|y(0)|^p}\\
&\quad-\frac{\left(\int_0^T F(t,\Theta(d))\,dt+\sum_{i=1}^{l}\int_0^{\Theta(d)}I_i(s)\,ds\right)}{\int_0^T\rho(t)|y'(t)|^p dt+\rho(T)\Phi_p\left(\sigma/\gamma\right)|y(T)|^p+\rho(0)\Phi_p\left(\beta/\alpha\right)|y(0)|^p}\Bigg]
\end{aligned}$$

for each $r>0$ and every $y\in X$ such that

$$\Phi(y)\geq r \tag{3.1}$$

$$\begin{aligned}
&\int_0^T[F(t,y^+(t))-f(t,0)y^-(t)]\,dt\\
&\qquad+\sum_{i=1}^{l}\left[\int_0^{y^+(t_i)}I_i(s)\,ds-I_i(0)y^-(t_i)\right]\\
&\qquad\qquad\geq\int_0^T F(t,\Theta(r))\,dt+\sum_{i=1}^{l}\int_0^{\Theta(r)}I_i(s)\,ds.
\end{aligned}\tag{3.2}$$

In fact, for $r > 0$, taking into account that $\overline{\Phi^{-1}(]-\infty, r[)}^{\omega} = \Phi^{-1}(]-\infty, r])$ and $x \equiv 0$ on $[0, T]$ obviously belongs to $\Phi^{-1}(]-\infty, r[)$ and that $\Psi(0) = 0$, we have

$$\varphi_1(r) \leq \frac{\sup_{x\in\Phi^{-1}(]-\infty,r])} \left\{\int_0^T F(t, x^+(t))\, dt + \sum_{i=1}^l \int_0^{x^+(t_i)} I_i(s)\, ds\right\}}{r}.$$

Thus, since $x \in \Phi^{-1}(]-\infty, r])$, that is, $\Phi(x) \leq r$, by Lemma 2.4 we have

$$\|x\|_\infty \leq \Theta(r). \tag{3.3}$$

As a consequence,

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} \left(\int_0^T F(t, x^+(t))\, dt + \sum_{i=1}^l \int_0^{x^+(t_i)} I_i(s)\, ds\right) \leq \int_0^T F(t, \Theta(r))\, dt + \sum_{i=1}^l \int_0^{\Theta(r)} I_i(s)\, ds.$$

So (A1) is proved.

Moreover, for each $r > 0$ and each $y \in X$ such that (3.1) holds, we have

$$\begin{aligned}\varphi_2(r) &\geq \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)} \\ &= \inf_{x\in\Phi^{-1}(]-\infty,r[)} \left(\frac{\int_0^T [F(t,y^+(t)) - f(t,0)y^-(t)]\, dt}{\Phi(y)-\Phi(x)}\right. \\ &\qquad \left.+ \frac{\sum_{i=1}^l \left[\int_0^{y^+(t_i)} I_i(s)\, ds - I_i(0)y^-(t_i)\right]}{\Phi(y)-\Phi(x)}\right) \\ &\qquad - \left(\frac{\int_0^T [F(t,x^+(t)) - f(t,0)x^-(t)]\, dt}{\Phi(y)-\Phi(x)}\right. \\ &\qquad \left.+ \frac{\sum_{i=1}^l \left[\int_0^{x^+(t_i)} I_i(s)\, ds - I_i(0)x^-(t_i)\right]}{\Phi(y)-\Phi(x)}\right).\end{aligned}$$

Since (3.3) holds for $x \in \Phi^{-1}(]-\infty, r])$, we obtain

$$\varphi_2(r) \geq \inf_{x\in\Phi^{-1}(]-\infty,r[)} \left(\frac{\int_0^T [F(t,y^+(t)) - f(t,0)y^-(t)]\,dt}{\Phi(y)-\Phi(x)} + \frac{\sum_{i=1}^l \left[\int_0^{y^+(t_i)} I_i(s)\,ds - I_i(0)y^-(t_i)\right]}{\Phi(y)-\Phi(x)} - \frac{\int_0^T F(t,\Theta(r))\,dt + \sum_{i=1}^l \int_0^{\Theta(r)} I_i(s)\,ds}{\Phi(y)-\Phi(x)} \right),$$

and under further condition (3.2) we can write

$$\varphi_2(r) \geq \frac{\int_0^T [F(t,y^+(t)) - f(t,0)y^-(t)]\,dt}{1/p\int_0^T \rho(t)|y'(t)|^p dt + (\rho(T)/p)\Phi_p(\sigma/\gamma)|y(T)|^p + (\rho(0)/p)\Phi_p(\beta/\alpha)|y(0)|^p}$$
$$+ \frac{\sum_{i=1}^l \left[\int_0^{y^+(t_i)} I_i(s)\,ds - I_i(0)y^-(t_i)\right]}{1/p\int_0^T \rho(t)|y'(t)|^p dt + (\rho(T)/p)\Phi_p(\sigma/\gamma)|y(T)|^p + (\rho(0)/p)\Phi_p(\beta/\alpha)|y(0)|^p}$$
$$- \frac{\left(\int_0^T F(t,\Theta(r))\,dt + \sum_{i=1}^l \int_0^{\Theta(r)} I_i(s)\,ds\right)}{1/p\int_0^T \rho(t)|y'(t)|^p dt + (\rho(T)/p)\Phi_p(\sigma/\gamma)|y(T)|^p + (\rho(0)/p)\Phi_p(\beta/\alpha)|y(0)|^p}.$$

So (A2) is proved.

Now in order to prove (ii) in Theorem 1.1, taking into account (A1) and (A2), it suffices to find $r > 0$, $y \in X$ such that (3.1), (3.2) and

$$\frac{\int_0^T F(t,\Theta(r))\,dt + \sum_{i=1}^l \int_0^{\Theta(r)} I_i(s)\,ds}{r} \tag{3.4}$$
$$< p\left(\frac{\int_0^T [F(t,y^+(t)) - f(t,0)y^-(t)]\,dt + \sum_{i=1}^l \left[\int_0^{y^+(t_i)} I_i(s)\,ds - I_i(0)y^-(t_i)\right]}{\int_0^T \rho(t)|y'(t)|^p dt + \rho(T)\Phi_p(\sigma/\gamma)|y(T)|^p + \rho(0)\Phi_p(\beta/\alpha)|y(0)|^p} - \frac{\int_0^T F(t,\Theta(r))\,dt + \sum_{i=1}^l \int_0^{\Theta(r)} I_i(s)\,ds}{\int_0^T \rho(t)|y'(t)|^p dt + \rho(T)\Phi_p(\sigma/\gamma)|y(T)|^p + \rho(0)\Phi_p(\beta/\alpha)|y(0)|^p}\right)$$

hold. To this end, we define

$$y(t) = \begin{cases} y_1(t) = \left(\frac{\beta}{\alpha}t + 1\right)k & t \in \left[0, \frac{T}{3}\right]; \\ y_2(t) = -kt\left(\frac{2\sigma}{\gamma+T\sigma} + \frac{\beta}{\alpha}\right) + \frac{2Tk}{3}\left(\frac{\sigma}{\gamma+T\sigma} + \frac{\beta}{\alpha}\right) + k & t \in \left[\frac{T}{3}, \frac{2T}{3}\right]; \\ y_3(t) = \left(-\frac{\sigma}{\gamma+T\sigma}t + 1\right)k & t \in \left[\frac{2T}{3}, T\right], \end{cases}$$

and $r := d$. Clearly $y(t) > 0$, $y \in X$. The assumption $k^p L > pd$ means (3.1). From (H1), it follows that (3.4) holds, which means (3.2) holds, too. Applying Theorem 1.1, $\Phi + \lambda\Psi$ has at least three critical points. By Lemma 2.2, Lemma 2.3, problem (1.2) has at least three positive solutions. $\square$

By Theorem 3.1, it is easy to obtain the following corollary

Corollary 3.2. *Assume that $g : [0, +\infty) \to [0, +\infty)$ is a continuous function, and put $G(\xi) = \int_0^\xi g(s)\, ds$. Besides assume there exist positive constants $k, d, l, l_i, \mu, \mu_i > 0$ with $l < p$, $l_i < p$, $i = 1, 2, \dots, l$, and $k^p L > pd$ such that the following conditions hold:*

(H3)

$$\left[G(\Theta(d)) + \sum_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\, ds\right]\left(\frac{1}{d} + \frac{p}{k^p L}\right)$$
$$< \frac{p}{k^p L}\left[\int_{\frac{2T}{3}}^{T} G\left(\left(-\frac{\sigma}{\gamma + T\sigma} t + 1\right) k\right) dt + \sum_{t_i \in [(2/3)T, T]} \int_0^{y_3(t_i)} I_i(s)\, ds\right];$$

(H4) $G(\xi) \le \mu(1 + |\xi|^l)$, $\int_0^\xi I_i(s)\, ds \le \mu_i(1 + |\xi|^{l_i})$ *for* $\xi \in [0, \infty)$.
Then for each $\lambda \in]\lambda_1, \lambda_2[$, the problem

$$\begin{cases} -(\rho(t)\Phi_p(x'(t)))' = \lambda g(x(t)) & t \neq t_i,\ t \in [0, T], \\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) = \lambda I_i(x(t_i)) & i = 1, 2, \dots, l, \\ \alpha x'(0) - \beta x(0) = 0, \quad \gamma x'(T) + \sigma x(T) = 0 & \end{cases}$$

has at least two positive solutions, where

$$\lambda_1 = \frac{k^p L}{p\left[\int_{\frac{2T}{3}}^{T} G\left(\left(-\frac{\sigma}{\gamma + T\sigma} t + 1\right) k\right) dt + \sum_{t_i \in [(2/3)T, T]} \int_0^{y_3(t_i)} I_i(s)\, ds - G(\Theta(d)) - \sum_{i=1}^{l} I_i(\Theta(d))\right]},$$

$$\lambda_2 = \frac{d}{G(\Theta(d)) + \sum_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\, ds}.$$

Example 3.1. The problem

$$
\begin{cases}
-((1+t)\Phi_3(x'(t)))' = \lambda g(x(t)) & t \neq t_1,\ t \in [0,1], \\
-\Delta((1+t_1)\Phi_3(x'(t_1))) = \lambda I_1(x(t_1)) & t_1 = 1/2, \\
x'(0) - x(0) = 0, \quad 3x'(1) + x(1) = 0,
\end{cases}
\tag{3.5}
$$

where $T = 1, t_1 = 1/2$, $p = 3$, $\rho(t) = 1 + t$, $\alpha = 1$, $\beta = 1$, $\gamma = 3$, $\sigma = 1$,

$$
g(x) = I_1(x) = \begin{cases} x/4 & x \leq 4, \\ 1 + 10^4(x-4) & x > 4, \end{cases}
$$

admits at least two nontrivial positive solutions for each $\lambda \in\,]0.16, 0.3[$.

In fact, the functions

$$
G(x) = \int_0^x I_1(s)\, ds = \begin{cases} x^2/8 & x \leq 4; \\ x - 2 + (10^4/2)(x-4)^2 & x > 4, \end{cases}
$$

satisfy all the assumptions of Corollary 3.2 by choosing $k = 10$, $d = 1/3$.

4. Existence results for (1.3). In order to obtain the existence of positive solutions for (1.3), now we consider the problem

$$
\begin{cases}
-(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = \lambda f(t, x^+(t)) & t \neq t_i,\ t \in [0,T], \\
-\Delta(\rho(t_i)\Phi_p(x'(t_i))) = \lambda I_i(x^+(t_i)) & i = 1, 2, \dots, l, \\
\alpha x'(0) - \beta x(0) = 0, \quad \gamma x'(T) + \sigma x(T) = 0.
\end{cases}
$$

Lemma 4.1. *If $x \in C([0,T])$ is a solution of* (4.1)*, then $x(t) \geq 0$, $x(t) \not\equiv 0$, $t \in [0,T]$, and hence it is a positive solution of* (1.3).

Proof. The proof is similar to Lemma 2.2; we omit it here. $\square$

Now we define the space $Y = W^{1,p}([0,T])$ equipped with the norm

$$
\|x\|_Y = \left(\int_0^T \rho(t)|x'(t)|^p + s(t)|x(t)|^p dt \right)^{1/p},
$$

which is clearly equivalent to the usual one $\|x\|_{W^{1,p}}$; F, $\|x\|_\infty$ are defined in Section 2.

For each $x \in Y$, put

$$\overline{\Phi}(x) := \frac{\|x\|_Y^p}{p} + \frac{\rho(T)}{p}\left(\frac{\sigma}{\gamma}\right)^{p-1}|x(T)|^p + \frac{\rho(0)}{p}\left(\frac{\beta}{\alpha}\right)^{p-1}|x(0)|^p, \tag{4.2}$$

$$\begin{aligned}\Psi(x) := &-\int_0^T \left[F(t, x^+(t)) - f(t,0)x^-(t)\right]\, dt \\ &-\sum_{i=1}^{l}\left[\int_0^{x^+(t_i)} I_i(s)\, ds - I_i(0)x^-(t_i)\right].\end{aligned} \tag{4.3}$$

Clearly, $\overline{\Phi}$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in Y$ is the functional $\overline{\Phi}'(x) \in Y^*$, given by

$$\begin{aligned}\langle\overline{\Phi}'(x), v\rangle = &\int_0^T \rho(t)\Phi_p(x'(t))v'(t) + s(t)\Phi_p(x(t))v(t)\, dt \\ &+ \rho(T)\Phi_p\left(\frac{\sigma}{\gamma}\right)|x(T)|^p + \rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)|x(0)|^p\end{aligned} \tag{4.4}$$

for every $v \in Y$, and $\overline{\Phi}' : Y \to Y^*$ is continuous. Moreover, taking into account that $\overline{\Phi}$ is convex, from Proposition 25.20 (i) of [**35**], we obtain that $\overline{\Phi}$ is a sequentially weakly lower semi-continuous functional.

It is easy to see that $\Psi : Y \to R$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in Y$ is the functional $\Psi'(x) \in Y^*$ given by

$$\langle\Psi'(x), v\rangle = -\int_0^T f(t, x^+(t))v(t)\, dt - \sum_{i=1}^{l} I_i(x^+(t_i))v(t_i) \tag{4.5}$$

for every $v \in Y$.

Lemma 4.2 [**31**]. *If the function $x \in Y$ is a critical point of the functional $\overline{\Phi} + \lambda\Psi$, then x is a solution of* (1.3).

Remark 4.1. By Lemma 4.1 and Lemma 4.2, if $x \in X$ is a critical point of the functional $\overline{\Phi} + \lambda\Psi$, then x is the positive solution of (1.3).

Lemma 4.3. *If $x \in Y$, and there exists an $r > 0$ such that $\overline{\Phi}(x) \leq r$, then*

$$\|x\|_\infty \leq \sqrt[p]{pr}\left[\left(\frac{\alpha}{\beta}\right)^{1/q}(\rho(0))^{-1/p} + T^{1/q}\left(\operatorname{ess}\inf_{[0,T]}\rho\right)^{-1/p}\right] := \Theta(r).$$

Proof. The proof is the same as that of Lemma 2.4. □

Lemma 4.4. $\overline{\Phi}' : Y \to Y^*$ *admits a continuous inverse on* Y^*.

Lemma 4.5. $\overline{\Phi}' : Y \to Y^*$ *is a continuous and compact operator.*

Proof. Since $\|x\|_Y$ is equivalent to $\|x\|_{W^{1,p}}$, the proof is the same as Lemma 2.11. □

In the following results, we will use the following notations: L, $y(t)$ and Γ are defined in Section 3. Besides, we define

$$Q = \int_0^T s(t)\,dt\left(\frac{\beta T + 3\alpha}{3\alpha}\right)^p.$$

Theorem 4.6. *Assume that there exist positive constants k, d, l, l_i, $\mu_i > 0$ with $l < p$, $l_i < p$, $k^p L > pd$ and a positive function $\mu \in C([0,T])$ such that*:

(L1)

$$\frac{\int_0^T F(t,\Theta(d))\,dt + \sum_{i=1}^{l}\int_0^{\Theta(d)} I_i(s)\,ds}{d} < \frac{p\left\{\Gamma - \left(\int_0^T F(t,\Theta(d))\,dt + \sum_{i=1}^{l}\int_0^{\Theta(d)} I_i(s)\,ds\right)\right\}}{k^p(L+Q)};$$

(L2) $F(t,\xi) \leq \mu(t)(1+|\xi|^l)$ *for* $t \in [0,T]$ *and* $\xi \in [0,\infty)$; $\int_0^\xi I_i(s)\,ds \leq \mu_i(1+|\xi|^{l_i})$ *for* $\xi \in [0,\infty)$.

Then for each $\lambda \in]\lambda_1, \lambda_2[$, *the problem* (1.3) *has at least three positive solutions, where*

$$\lambda_1 = \frac{k^p(L+Q)}{p\left\{\Gamma - \left(\int_0^T F(t,\Theta(d))\,dt + \sum\limits_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\,ds\right)\right\}}$$

and

$$\lambda_2 = \frac{d}{\int_0^T F(t,\Theta(d))\,dt + \sum\limits_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\,ds}.$$

Proof. From the above arguments, we have seen that $\overline{\Phi} : Y \to R$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative at the point $u \in Y$ is the functional $\overline{\Phi}'(u) \in Y^*$, given by (4.5). $(\overline{\Phi}')^{-1}$ exists and is continuous on Y^*. $\Psi : Y \to R$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative given by (4.5) is compact.

Now we will apply Theorem 1.1 to $\overline{\Phi}$ and Ψ.

For (i) in Theorem 1.1, by (4.2) and condition (L2), we have

$$\begin{aligned}
&\lim_{\|x\|_Y \to +\infty} (\overline{\Phi}(x) + \lambda\Psi(x)) \\
&\geq \lim_{\|x\|_Y \to +\infty} \left(\frac{\|x\|_Y^p}{p} - \lambda\int_0^T \mu(t)(1+|x(t)|^l)\,dt - \lambda\sum_{i=1}^{l} \mu_i\left[1+|x(t_i)|^{l_i}\right]\right) \\
&\geq \lim_{\|x\|_Y \to +\infty} \frac{\|x\|_Y^p}{p} - \lambda(M_2 + \|x\|_Y^{\bar{l}}) \\
&= +\infty,
\end{aligned}$$

where M_2 is a positive constant and $0 < \bar{l} < p$. So (i) is satisfied.

To prove (ii) in Theorem 1.1, first we claim that (A1) (Section 3, Theorem 3.1) holds for each $r > 0$ and

$$\text{(B)}\quad \begin{aligned}\varphi_2(r) \geq p\, &\frac{\int_0^T \left[F(t,y^+(t)) - f(t,0)y^-(t)\right]\, dt}{\|y\|_Y^p + \rho(T)\Phi_p\left(\sigma/\gamma\right)|y(T)|^p + \rho(0)\Phi_p\left(\beta/\alpha\right)|y(0)|^p} \\ &+ \frac{\sum_{i=1}^{l}\left[\int_0^{y^+(t)} I_i(s)\, ds - I_i(0)y^-(t_i)\right]}{\|y\|_Y^p + \rho(T)\Phi_p\left(\sigma/\gamma\right)|y(T)|^p + \rho(0)\Phi_p\left(\beta/\alpha\right)|y(0)|^p} \\ &- \frac{\left(\int_0^T F(t,\Theta(d))\, dt + \sum_{i=1}^{l}\int_0^{\Theta(d)} I_i(s)\, ds\right)}{\|y\|_Y^p + \rho(T)\Phi_p\left(\sigma/\gamma\right)|y(T)|^p + \rho(0)\Phi_p\left(\beta/\alpha\right)|y(0)|^p}\end{aligned}$$

for each $r > 0$ and every $y \in Y$ such that

$$\overline{\Phi}(y) \geq r \tag{4.6}$$

and (3.2) hold.

It is the same to the proof in Section 3, (A1) holds.

Moreover, for each $r > 0$ and each $y \in Y$ such that (4.6) holds, we have

$$\begin{aligned}\varphi_2(r) &\geq \inf_{x\in\overline{\Phi}^{-1}(]-\infty,r[)} \frac{\Psi(x) - \Psi(y)}{\overline{\Phi}(y) - \overline{\Phi}(x)} \\ &\geq \inf_{x\in\overline{\Phi}^{-1}(]-\infty,r[)} \left(\frac{\int_0^T [F(t,y^+(t)) - f(t,0)y^-(t)]\, dt}{\overline{\Phi}(y) - \overline{\Phi}(x)} \right. \\ &\qquad + \frac{\sum_{i=1}^{l}\left[\int_0^{y^+(t_i)} I_i(s)\, ds - I_i(0)y^-(t_i)\right]}{\overline{\Phi}(y) - \overline{\Phi}(x)} \\ &\qquad \left. - \frac{\int_0^T \left[F(t,x^+(t)) - f(t,0)x^-(t)\right] dt + \sum_{i=1}^{l}\left[\int_0^{x(t_i)} I_i(s)\, ds - I_i(0)x^-(t_i)\right]}{\overline{\Phi}(y) - \overline{\Phi}(x)} \right).\end{aligned}$$

Since $\|x\|_\infty \leq \Theta(r)$ holds for $x \in \overline{\Phi}^{-1}(]-\infty, r])$ and (3.1) holds, we obtain

$$\varphi_2(r) \geq p\left(\frac{\int_0^T [F(t,y^+(t))-f(t,0)y^-(t)]\,dt+\sum_{i=1}^{l}\left[\int_0^{y^+(t_i)} I_i(s)\,ds-I_i(0)y^-(t_i)\right]}{\|y\|_Y^p+\rho(T)\Phi_p(\sigma/\gamma)|y(T)|^p+\rho(0)\Phi_p(\beta/\alpha)|y(0)|^p}\right.$$
$$\left.-\frac{\int_0^T F(t,\Theta(r))\,dt+\sum_{i=1}^{l}\int_0^{\Theta(r)} I_i(s)\,ds}{\|y\|_Y^p+\rho(T)\Phi_p(\sigma/\gamma)|y(T)|^p+\rho(0)\Phi_p(\beta/\alpha)|y(0)|^p}\right).$$

So (B) is proved.

Now in order to prove (ii) in Theorem 1.1, taking into account (A1) (B), it suffices to find $r > 0$, $y \in Y$ such that (3.2), (4.6) and

$$\frac{\int_0^T F(t,\Theta(r))\,dt+\sum_{i=1}^{l}\int_0^{\Theta(r)} I_i(s)\,ds}{r}$$
$$< p\left(\frac{\int_0^T [F(t,y^+(t))-f(t,0)y^-(t)]\,dt}{\|y\|_Y^p+\rho(T)\Phi_p(\sigma/\gamma)|y(T)|^p+\rho(0)\Phi_p(\beta/\alpha)|y(0)|^p}\right.$$
$$+\frac{\sum_{i=1}^{l}\left[\int_0^{y^+(t_i)} I_i(s)ds-I_i(0)y^-(t_i)\right]}{\|y\|_Y^p+\rho(T)\Phi_p(\sigma/\gamma)|y(T)|^p+\rho(0)\Phi_p(\beta/\alpha)|y(0)|^p}$$
$$\left.-\frac{\int_0^T F(t,\Theta(r))\,dt+\sum_{i=1}^{l}\int_0^{\Theta(r)} I_i(s)\,ds}{\|y\|_Y^p+\rho(T)\Phi_p(\sigma/\gamma)|y(T)|^p+\rho(0)\Phi_p(\beta/\alpha)|y(0)|^p}\right)$$

hold. To this end, we define $y(t)$ as in Section 3 and $r := d$. Clearly $y \in Y$, and

(4.8)
$$k^pL < \|y\|_Y^p+\rho(T)\Phi_p\left(\frac{\gamma}{\sigma}\right)|y(T)|^p+\rho(0)\Phi_p\left(\frac{\beta}{\alpha}\right)|y(0)|^p = k^p(L+Q).$$

The assumption $k^pL > pd$ means that $\overline{\Phi}(y) \geq (\|y\|_Y^p/p) > (k^pL/p) > d$. From condition (L1), it follows that (4.7) holds, which means that (3.2) holds, too. Applying Theorem 1.1, $\overline{\Phi}+\lambda\Psi$ has at least three critical points. By Lemma 4.1 and Lemma 4.2, problem (1.3) has at least three positive solutions. □

By Theorem 4.6, we have the following Corollary.

Corollary 4.7. *Assume that* $g : [0,\infty) \to [0,\infty)$ *is a continuous function, and put* $G(\xi) = \int_0^\xi g(s)\,ds$. *Besides assume there exist positive constants* $k, d, l, \mu, l_i, \mu_i > 0$ *with* $l < p$, ${}_i < p$, $i = 1, 2, \dots, l$, *such that* $k^p L > pd$ *and the following conditions hold*

(L3)

$$\left[G(\Theta(d)) + \sum_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\,ds\right]\left(\frac{1}{d} + \frac{p}{k^p(L+Q)}\right) < \frac{p}{k^p(L+Q)}\left[\int_{(2T)/3}^{T} G\left(\left(-\frac{\sigma}{\gamma+T\sigma}t+1\right)k\right)dt + \sum_{t_i\in[(2/3)T,T]} \int_0^{y_3(t_i)} I_i(s)\,ds\right];$$

(L4) $G(\xi) \le \mu(1+|\xi|^l)$, $\int_0^\xi I_i(s)\,ds \le \mu_i(1+|\xi|^{l_i})$ *for* $\xi \in [0,\infty)$. *Then for each* $\lambda \in]\lambda_1, \lambda_2[$, *the problem*

(4.9)
$$\begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = \lambda g(x(t)) & t \ne t_i,\ t \in [0,T],\\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) = \lambda I_i(x(t_i)) & i = 1, 2, \dots, l,\\ \alpha x'(0) - \beta x(0) = 0, \quad \gamma x'(T) + \sigma x(T) = 0, \end{cases}$$

has at least two nontrivial positive solutions, where

$$\lambda_1 = \frac{k^p(L+Q)}{p\left(\int_{\frac{2T}{3}}^{T} G\left(\left(-\frac{\sigma}{\gamma+T\sigma}t+1\right)k\right)dt + \sum_{t_i\in\left[\frac{2T}{3},T\right]} \int_0^{y_3(t_i)} I_i(s)ds - G(\Theta(d)) - \sum_{i=1}^{l}\int_0^{\Theta(d)} I_i(s)ds\right)}$$

and

$$\lambda_2 = \frac{d}{G(\Theta(d)) + \sum_{i=1}^{l} \int_0^{\Theta(d)} I_i(s)\,ds}.$$

Example 4.1. The problem

$$\begin{cases} -((1+t)\Phi_3(x'(t)))' + (1+t)\Phi_3(x(t)) = \lambda g(x(t)) & t \neq t_1,\, t \in [0,1], \\ -\Delta((1+t_1)\Phi_3(x'(t_1))) = \lambda I_1(x(t_1)) & t_1 = 1/2, \\ x'(0) - x(0) = 0, \quad 3x'(1) + x(1) = 0, \end{cases} \tag{4.10}$$

where $T = 1$, $t_1 = 1/2$, $p = 3$, $\rho(t) = 1 + t$, $s(t) = 1 + t$, $\alpha = 1$, $\beta = 1$, $\gamma = 3$, $\sigma = 1$,

$$g(x) = I_1(x) = \begin{cases} x/4 & x \leq 4, \\ 1 + 10^4(x-4) & x > 4 \end{cases}$$

admits at least two nontrivial positive solutions for each $\lambda \in]0.0008, 0.3[$.

In fact, the function

$$G(x) = \int_0^x I_1(s)\, ds = \begin{cases} x^2/8 & x \leq 4; \\ x - 2 + (10^4/2)(x-4)^2 & x > 4 \end{cases}$$

satisfy all the assumptions of Corollary 3.2 by choosing $k = 10$, $d = 1/3$.

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