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THE cd-INDEX OF THE POSET OF INTERVALS AND E_t-CONSTRUCTION

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ABSTRACT. Given a graded poset P, let $I(P)$ denote the associated poset of intervals and $E_t(P)$ the poset obtained from P by the E_t-construction of Paffenholz and Ziegler [**7**]. We analyze how the **ab**-index behaves under those operations and prove that its change is expressed in terms of certain, quite explicit, recursively defined linear operators. If the poset P is Eulerian, the recursive relations for those linear operators are interpreted inside the coalgebra spanned by **c** and **d**. We use these relations to prove that the **cd**-index of the dual of the poset of intervals of the simplest Eulerian poset is the same as the **cd**-index of appropriate Tchebyshev poset defined by Hetyei in [**5**].

1. Introduction. Throughout this paper, we will consider graded posets with rank function r. We refer to [**8**] as a good general reference for the poset terminology. For a poset P of rank $n+1$ and $S \subseteq [n] = \{1, 2, \dots, n\}$, let $f_S(P)$ denote the number of chains $x_1 < x_2 < \cdots < x_{|S|}$ such that $S = \{r(x_1), r(x_2), \dots, r(x_{|S|})\}$. The sequence $(f_S(P))_{S \subseteq [n]}$ is called the *flag f-vector* of P.

The flag f-vector of P can be encoded as a homogenous noncommutative polynomial in the variables **a** and **b**. Let P be a poset of rank $n+1$. To every chain

$$c = \left\{\hat{0} < x_1 < x_2 < \cdots < x_k < \hat{1}\right\}$$

of P we associate a *weight* $\mathrm{wt}\,(c) = w_1 w_2 \cdots w_n$ where

$$w_i = \begin{cases} \mathbf{b} & \text{if } i \in \{r(x_1), r(x_2), \dots, r(x_k)\}; \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

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Now, the **ab**-index of P is defined as

$$\Psi_P = \sum_{c \text{ chain in } P} \mathrm{wt}\,(c). \tag{1}$$

A finite graded poset is *Eulerian* if every interval of rank at least one contains as many elements of even rank as of odd rank. The face lattices of polytopes, and more generally, of regular CW-spheres are Eulerian. The linear span of the flag f-vectors of all polytopes (and all Eulerian posets) is described by generalized Dehn-Sommerville equations ([**1**, Theorem 2.1]).

Bayer and Klapper proved in [**2**] that the **ab**-index of an Eulerian poset can be written as a polynomial in the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial Φ_P is called the **cd**-index, and this is the most efficient way to encode the flag f-vector of Eulerian posets.

Ehrenborg and Readdy in [**4**] used some coalgebra techniques to determine the changes of the **cd**-index of a polytope (more generally, of an Eulerian poset) under certain geometric operations, such as taking a pyramid or prism.

Let $\mathcal{P}$ denote the vector space over $\mathbf{Q}$ spanned by all isomorphism types of graded posets. If $\overline{P}$ denotes the isomorphism type of P, the coproduct $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ is defined on generators by

$$\Delta(\overline{P}) = \sum_{\hat{0} < x < \hat{1}} \overline{[\hat{0}, x]} \otimes \overline{[x, \hat{1}]}.$$

Using the Sweedler notation [**9**], we write $\Delta(\overline{P}) = \sum_P P_{(1)} \otimes P_{(2)}$. The above-defined coproduct is coassociative because it satisfies $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. Therefore, we can define $\Delta^{k+1} = (\Delta^k \circ \mathrm{id}) \circ \Delta$. Note that $\Delta^2 = \Delta$, and using Sweedler notation we can write

$$\begin{aligned}\Delta^k(\overline{P}) &= \sum_{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}} \overline{[x_0, x_1]} \otimes \cdots \otimes \overline{[x_{k-1}, x_k]} \\ &= \sum_P P_{(1)} \otimes P_{(2)} \otimes \cdots \otimes P_{(k)}.\end{aligned}$$

There is a natural coproduct Δ on the algebra $\mathbf{Q}\langle \mathbf{a}, \mathbf{b} \rangle$. For an **ab**-monomial $u = u_1 u_2 \cdots u_n$, let $\Delta(u) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n$, and extend Δ by linearity to $\mathbf{Q}\langle\mathbf{a},\mathbf{b}\rangle$. Obviously, Δ is coassociative and $\Delta^k(u)=0$ when the degree of an **ab**-monomial u is less than k.

Proposition 1 [**4**]. *The* **ab***-index of posets is a coalgebra homomorphism. That is, for a poset P we have*

$$\Delta(\Psi_P)=\sum_{\hat{0}<x<\hat{1}}\Psi_{[\hat{0},x]}\otimes\Psi_{[x,\hat{1}]}.$$

From the above proposition and from coassociativity of the coproducts it follows that

$$\sum_{\hat{0}=x_0<x_1\cdots<x_k=\hat{1}} f_1(\Psi_{[x_0,x_1]})\cdots f_k(\Psi_{[x_{k-1},x_k]}) = \sum_{\Psi_P} f_1(\Psi_{P_{(1)}})\cdots f_k(\Psi_{P_{(k)}}) \tag{2}$$

for any linear maps $f_1,f_2,\ldots,f_k$ on the algebra $\mathbf{Q}\langle\mathbf{a},\mathbf{b}\rangle$.

Since $\Delta(\mathbf{a}+\mathbf{b})=2\cdot 1\otimes 1$ and $\Delta(\mathbf{ab}+\mathbf{ba})=\mathbf{c}\otimes 1+1\otimes\mathbf{c}$ it follows that $\mathbf{Q}\langle\mathbf{c},\mathbf{d}\rangle$ is closed under the coproduct Δ. Also, if $\mathcal{E}$ denotes the subspace of $\mathcal{P}$ spanned by all isomorphism types of Eulerian posets, it is easy to see that $\mathcal{E}$ is closed under the coproduct. The **cd**-index is a coalgebra homomorphism between $\mathcal{E}$ and $\mathbf{Q}\langle\mathbf{c},\mathbf{d}\rangle$.

In [**3**] the following lemma is stated

Lemma 2 [**3**]. *The linear map $\Psi:\mathcal{P}\mapsto\mathbf{Z}\langle\mathbf{a},\mathbf{b}\rangle$ is surjective.*

If P is the face lattice of a polytope V, then $P\times B_1$ is the face lattice of the pyramid over V. So, for a poset P we can define $\operatorname{Pyr}(P)=P\times B_1$.

Proposition 3 [**4**]. *Let P be a graded poset. Then*

$$\begin{aligned}\Psi_{P\times B_1}&=\Psi_P\cdot\mathbf{a}+\mathbf{b}\cdot\Psi_P+\sum_{\Psi_P}\Psi_{P_{(1)}}\cdot\mathbf{ab}\cdot\Psi_{P_{(2)}}\\&=\Psi_P\cdot\mathbf{b}+\mathbf{a}\cdot\Psi_P+\sum_{\Psi_P}\Psi_{P_{(1)}}\cdot\mathbf{ba}\cdot\Psi_{P_{(2)}}.\end{aligned}$$

The above proposition defines the linear map Pyr : $\mathbf{Q}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbf{Q}\langle \mathbf{a}, \mathbf{b}\rangle$ by

$$\begin{aligned} \operatorname{Pyr}(u) &= u \cdot \mathbf{a} + \mathbf{b} \cdot u + \sum_u u_{(1)} \cdot \mathbf{ab} \cdot u_{(2)} \\ &= u \cdot \mathbf{b} + \mathbf{a} \cdot u + \sum_u u_{(1)} \cdot \mathbf{ba} \cdot u_{(2)}, \end{aligned} \tag{3}$$

and for any polytope (poset) P we have that $\Psi_{\mathrm{Pyr}\, P} = \operatorname{Pyr}(\Psi_P)$. Also, in [**4**], it is proven that the map Pyr can be described by

$$\operatorname{Pyr}(u) = u \cdot (\mathbf{a} + \mathbf{b}) + G(u), \tag{4}$$

where the derivation G is defined on the generators of the algebra $\mathbf{Q}\langle \mathbf{a}, \mathbf{b}\rangle$ by $G(\mathbf{a}) = \mathbf{ba}$ and $G(\mathbf{b}) = \mathbf{ab}$. Since $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$ holds, the restriction of the map Pyr on $\mathbf{Q}\langle \mathbf{c}, \mathbf{d}\rangle$ is described with the formulae (4).

The **ab**-index also has a product structure. For two posets P and Q we have that $\Psi_{P*Q} = \Psi_P \cdot \Psi_Q$. If we denote with H_{n+1} the finite ladder poset of rank $n+1$, i.e., H_{n+1} is the star product of n copies of B_2, then $\Phi_{H_{n+1}} = \mathbf{c}^n$.

The *star involution* is defined on $\mathbf{Q}\langle \mathbf{a}, \mathbf{b}\rangle$ such that it reads **ab**-polynomials backwards. For $u = u_1 u_2 \cdots u_n$ we have $u^* = u_n u_{n-1} \cdots u_1$. If P^* denotes the dual poset of P, then we have that $\Psi_{P^*} = \Psi_P^*$.

The diamond product on posets is defined by

$$P \diamond Q = \big(P \setminus \{\hat{0}\}\big) \times \big(Q \setminus \{\hat{0}\}\big) \cup \{\hat{0}\}.$$

This product corresponds to the Cartesian product of polytopes, and therefore for a poset P we define $\operatorname{Prism}(P) = P \diamond B_2$. In [**4**] the linear map Prism : $\mathbf{Q}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbf{Q}\langle \mathbf{a}, \mathbf{b}\rangle$ is defined by

$$\operatorname{Prism}(u) = u \cdot (\mathbf{a} + \mathbf{b}) + \sum_u u_{(1)} \cdot (\mathbf{ab} + \mathbf{ba}) \cdot u_{(2)},$$

and it is proven that $\Psi_{\mathrm{Prism}\,(P)} = \operatorname{Prism}(\Psi_P)$ holds for any poset P.

2. The interval poset. The *interval poset* $I(P)$ of a graded poset P is the set of all closed intervals of P ordered by containment:

$$[x, y] \leq [x', y'] \text{ in } I(P) \text{ if and only if } x' \leq x \leq y \leq y' \text{ in } P.$$

We also adjoin the empty interval to $I(P)$ as the minimal element.

Lindström in [**6**] noted that for all $n \in \mathbf{N}$ the interval poset of the Boolean lattice B_n (the face lattice of an $(n-1)$-simplex) is the face lattice of an n-cube, i.e., $I(L(\Delta_{n-1})) \cong L(C_n)$. Also, in [**6**] he asked wether it is true for every polytope V that there exists a polytope W such that $I(L(V)) \cong L(W)$.

Proposition 4. (i) *For any poset P we have that $I(P) \cong I(P^*)$.*

(ii) *Intervals in the poset $I(P)$ have the following form*

$$[[x,y],[x',y']]_{I(P)} \cong [x',x]^* \times [y,y'], \quad \left[\hat{0},[x,y]\right]_{I(P)} \cong I([x,y]).$$

(iii) *Let P be a graded poset of rank n. Then $I(P)$ is a graded poset of rank $n+1$ and $r([x,y]) = r(y) - r(x) + 1$. Also, the f-vector of $I(P)$ is*

$$f_i(I(P)) = \sum_{j=0}^{n-i+1} f_{\{j,j+i-1\}}(P).$$

(iv) *For any poset P we have that*

$$I(P \times B_1) \cong I(P) \diamond B_2. \tag{5}$$

Proof. Obviously, statements (i)–(iii) follow directly from the definition of $I(P)$.

We define $F : I(P \times B_1) \to I(P) \diamond B_2$ by $F([(x,p),(y,q)]) = ([x,y],r)$, where

$$r = \begin{cases} \{1\} & \text{if } p = q = \varnothing, \\ \{2\} & \text{if } p = q = 1, \\ \{1,2\} & \text{if } p \neq q. \end{cases}$$

It is easy to verify that F is an isomorphism. $\square$

If V and W are polytopes such that $I(L(V)) \cong L(W)$, then (as a special case of (5)) we have that $I(L(\mathrm{Pyr}\,(V))) \cong L(\mathrm{Prism}\,(W))$.

Now, we wish to express the **ab**-index of $I(P)$ in terms of Ψ_P.

Proposition 5. *There exists a linear map $\mathcal{I} : \mathbf{Q}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbf{Q}\langle \mathbf{a}, \mathbf{b} \rangle$ such that for any graded poset P the identity $\Psi_{I(P)} = \mathcal{I}(\Psi_P)$ holds.*

Proof. We say that a chain

$$C = \left\{ \hat{0} = \varnothing < [x_0, y_0] < [x_1, y_1] < \cdots < [x_r, y_r] < \left[\hat{0}_P, \hat{1}_P\right] = \hat{1} \right\}$$

in $I(P)$ corresponds to a chain c in P if and only if the multi-chain

$$\hat{0}_P \le x_r \le x_{r-1} \le \cdots \le x_0 \le y_0 \le y_1 \le y_2 \le \cdots \le y_r \le \hat{1}_P$$

in P contains exactly those elements which appear in c. Note that for every chain C in $I(P)$ there exists the unique chain in P which corresponds with C. For a chain c in P we denote $S(c) = \{C \text{ chain in } I(P) : C \text{ corresponds with } c\}$. Obviously, $\{S(c) : c \text{ chain in } P\}$ is a partition of the set of all chains in $I(P)$. Therefore, we obtain that

$$\Psi_{I(P)} = \sum_{C \text{ chain in } I(P)} \mathrm{wt}\,(C) = \sum_{c \text{ chain in } P} \sum_{C \in S(c)} \mathrm{wt}\,(C).$$

Further, if two chains c and c' have the same contribution to Ψ_P, i.e., $\mathrm{wt}\,(c) = \mathrm{wt}\,(c')$, then there exists an obvious bijection between the sets $S(c)$ and $S(c')$ which preserves the weights of chains. So, with

$$\mathcal{I}(\mathrm{wt}\,(c)) = \sum_{C \in S(c)} \mathrm{wt}\,(C)$$

a linear map is defined which satisfies the statement of the proposition. □

Now, we describe recursive relations for $\mathcal{I}$. It is easy to see that $\mathcal{I}(1) = \mathbf{a} + \mathbf{b}$.

Theorem 6. *For any* $\mathbf{ab}$*-monomial u the following formulas hold*:

$$\mathcal{I}(u \cdot \mathbf{a}) = \mathcal{I}(u) \cdot \mathbf{a} + (\mathbf{ab} + \mathbf{ba}) \cdot u^* + \sum_u \mathcal{I}(u_{(2)}) \cdot \mathbf{ab} \cdot u^*_{(1)}, \tag{6}$$

$$\mathcal{I}(u \cdot \mathbf{b}) = \mathcal{I}(u) \cdot \mathbf{b} + (\mathbf{ab} + \mathbf{ba}) \cdot u^* + \sum_u \mathcal{I}(u_{(2)}) \cdot \mathbf{ba} \cdot u^*_{(1)}. \tag{7}$$

Proof. For an arbitrary poset P, we consider a new poset $\overline{P}$ obtained from P by adding a new element $\bar{1}$ such that $x < \bar{1}$ for all $x \in P$. Obviously, we have that $\Psi_{\overline{P}} = \Psi_{\mathbf{P}} \cdot \mathbf{a}$.

We divide all chains of $I(\overline{P})$ into four sets:

1. Chains in which intervals of $I(\overline{P})$ containing $\bar{1}$ do not appear. These are exactly the "old" chains of $I(P)$ which may end with $[\hat{0}, \hat{1}]$ (but not necessarily). From relation (1) it follows that the contribution of all such chains to $\Psi_{I(\overline{P})}$ is exactly $\Psi_{I(P)} \cdot \mathbf{a}$.

2. Chains in $I(\overline{P})$ which begin with $[\bar{1}, \bar{1}]$. All such chains may contain the interval $[\hat{1}, \bar{1}]$ (but not necessarily). From (ii) of Proposition 4, we obtain that the contribution of all such chains to $\Psi_{I(\overline{P})}$ is

$$\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \cdot \sum_{c \text{ chain in } P} \mathrm{wt}\,(c)^* + \mathbf{b} \cdot \mathbf{b} \cdot \sum_{c \text{ chain in } P} \mathrm{wt}\,(c)^* = \mathbf{ba} \cdot \Psi_P^*.$$

3. Chains which contain $[\hat{1}, \bar{1}]$ but do not contain $[\bar{1}, \bar{1}]$. These chains may (but again, may not) begin with $[\hat{1}, \hat{1}]$ and their contribution to $\Psi_{I(\overline{P})}$ is

$$\mathbf{b} \cdot \mathbf{b} \sum_{c \text{ chain in } P} \mathrm{wt}\,(c)^* + (\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \sum_{c \text{ chain in } P} \mathrm{wt}\,(c)^* = \mathbf{ab} \cdot \Psi_P^*.$$

4. Chains in which $\bar{1}$ appears for the first time in the interval $[x, \bar{1}]$ (for an $x \in P$, $x \neq \hat{1}, x \neq \hat{0}$). Such chains may contain $[x, \hat{1}]$ (but not necessarily).

From (ii) of Proposition 4 we have that $[\varnothing, [x, \hat{1}]]_{I(\overline{P})} \cong I([x, \hat{1}]_P)$ and $[[x, \bar{1}], [\hat{0}, \bar{1}]] \cong [\hat{0}, x]_P^*$. So, the contribution of all such chains to $\Psi_{I(\overline{P})}$ is exactly

$$\begin{aligned} &\sum_{x \in P \setminus \{\hat{0}, \hat{1}\}} \sum_{c \text{ chain in } I([x, \hat{1}])} \sum_{c' \text{ chain in } [0, x]} \mathrm{wt}\,(c) \cdot \mathbf{ab} \cdot \mathrm{wt}\,(c')^* \\ &\quad = \sum_{x \in P \setminus \{\hat{0}, \hat{1}\}} \mathcal{I}(\Psi_{[x, \hat{1}]}) \cdot \mathbf{ab} \cdot \Psi_{[x, \hat{0}]^*} = \sum_{\Psi_P} \mathcal{I}(\Psi_{P_{(2)}}) \cdot \mathbf{ab} \cdot \Psi_{P_{(1)}}^*. \end{aligned}$$

The last equation above follows from relation (2).

So, by adding the weights of all chains of $I(\overline{P})$ we obtain that (6) holds when u is the $\mathbf{ab}$-index of some poset P. From the linearity of $\mathcal{I}$ and Lemma 2 it follows that (6) holds for any $\mathbf{ab}$-monomial u.

In order to prove the formula (7), for an arbitrary poset P we shall consider the poset $P' = P * B_2$. Let us denote two coatoms of P' with $1'$ and $1''$, and the maximal element of P' with $\mathbf{1}$. Obviously,

$$\mathcal{I}(\Psi_{P'}) = \mathcal{I}(\Psi_P \cdot (\mathbf{a}+\mathbf{b})) = \mathcal{I}(\Psi_P \cdot \mathbf{a}) + \mathcal{I}(\Psi_P \cdot \mathbf{b}).$$

Chains in $I(P')$ in which $1''$ does not appear are in bijection with chains of $\overline{P}$, and their contributions to $\Psi_{I(P)}$ are exactly $\mathcal{I}(\Psi_P \cdot \mathbf{a})$. So, we have that $\mathcal{I}(\Psi_P \cdot \mathbf{b})$ is equal to the contribution of chains of $I(P')$ in which $1''$ appears. Again, we divide the set of all such chains of $I(P')$ into four parts:

1. Chains in which $1''$ appears only at the end (in the interval $[\hat{0}, 1'']$). These are exactly the old chains of $I(P)$ with added $[\hat{0}, 1'']$. Their contribution to $\Psi_{I(P')}$ is exactly $\mathcal{I}(\Psi_P) \cdot \mathbf{b}$.

2. Chains which contain $[1'', \mathbf{1}]$, but do not contain $[1'', 1'']$. All such chains may begin with the interval $[\mathbf{1}, \mathbf{1}]$ (but not necessarily). The contribution of all such chains to $\Psi_{I(P')}$ is $\mathbf{ab} \cdot \Psi_P^*$.

3. Chains in which $1''$ appears for the last time in an interval $[x, 1'']$ (for an $x \in P \setminus \{\hat{0}, \hat{1}\}$). Any such chain may contain $[x, \mathbf{1}]$ (but not necessarily). By using (ii) of Proposition 4 we obtain that the contribution of all such chains to $\Psi_{I(P')}$ is

$$\begin{aligned}\sum_{x \in P \setminus \{\hat{0},\hat{1}\}} \ \sum_{c \text{ chains in } I([x,\hat{1}])} \ \sum_{c' \text{ chains in } [\hat{0},x]} & wt(c) \cdot \mathbf{ba} \cdot wt(c')^* \\ = \sum_{x \in P \setminus \{\hat{0},\hat{1}\}} \mathcal{I}(\Psi_{[x,\hat{1}]}) \cdot \mathbf{ba} \cdot \Psi_{[x,\hat{0}]^*} &= \sum_{\Psi_P} \mathcal{I}(\Psi_{P_{(2)}}) \cdot \mathbf{ba} \cdot \Psi^*_{P_{(1)}}.\end{aligned}$$

4. Chains in $I(P')$ which contain $[1'', 1'']$ but do not contain $[x, 1'']$ (for all $x \in P$). All such chains may (but again, may not) contain $[1'', \mathbf{1}]$, and their contribution to $\Psi_{I(P')}$ is $\mathbf{ba} \cdot \Psi_P^*$.

By adding all the obtained contributions we can conclude that the formula (7) holds when u is the $\mathbf{ab}$-index of a poset. Using the same argument as before we obtain that (7) is true for any $\mathbf{ab}$-monomial u. □

From (i) and (iv) of Proposition 4 we obtain that

$$\mathcal{I}(\Psi_{P^*}) = \mathcal{I}(\Psi_P) \text{ and } \mathcal{I}(\mathrm{Pyr}\,(\Psi_P)) = \mathrm{Prism}\,(\mathcal{I}(\Psi_P))$$

holds for any graded poset P. Lemma 2 and the linearity of $\mathcal{I}$ provide that the above formulas hold for any **ab**-monomial u. Further, from the previous theorem, we obtain that the operator $\mathcal{I}$ commutes with the "bar" involution (which interchanges variables **a** and **b**), i.e., for any **ab**-monomial u we have that $\mathcal{I}(\bar{u}) = \overline{\mathcal{I}(u)}$.

From (ii) and (iii) of Proposition 4, it is easy to see that the operation $P \mapsto I(P)$ preserves the property of being Eulerian. Therefore, for an Eulerian poset P we have that $\Phi_{I(P)} = \mathcal{I}(\Phi_P)$.

The recursive relations for the operator $\mathcal{I}$ inside the algebra $\mathbf{Q}\langle \mathbf{c}, \mathbf{d}\rangle$ are described with the following

Corollary 7. *Let u be a* **cd***-monomial. Then*

$$\mathcal{I}(u \cdot \mathbf{c}) = \mathcal{I}(u) \cdot \mathbf{c} + 2\mathbf{d} \cdot u^* + \sum_u \mathcal{I}(u_{(2)}) \cdot \mathbf{d} \cdot u^*_{(1)},$$

$$\begin{aligned}\mathcal{I}(u \cdot \mathbf{d}) = {} & \mathcal{I}(u) \cdot \mathbf{d} + (\mathbf{dc} + \mathbf{cd}) \cdot u^* + \mathbf{d} \cdot u^* \cdot \mathbf{c} \\ & + \sum_u \Big(\mathcal{I}(u_{(2)}) \cdot \mathbf{d} \cdot \mathrm{Pyr}\,(u^*_{(1)}) + \mathbf{d} \cdot u^*_{(2)} \cdot \mathbf{d} \cdot u^*_{(1)} \Big).\end{aligned}$$

Proof. The first relation is easy to obtain by adding (6) and (7).

From (7) it follows that

$$\mathcal{I}(u \cdot \mathbf{ab}) = \mathcal{I}(u\mathbf{a}) \cdot \mathbf{b} + (\mathbf{ab} + \mathbf{ba}) \cdot \mathbf{a}u^* + \sum_{ua} \mathcal{I}((u\mathbf{a})_{(2)}) \cdot \mathbf{ba} \cdot (u\mathbf{a})^*_{(1)}.$$

As we have that $\Delta(u\mathbf{a}) = u \otimes 1 + \sum_u u_{(1)} \otimes u_{(2)}\mathbf{a}$, it follows that

$$\begin{aligned}
\sum_{ua} \mathcal{I}((u\mathbf{a})_{(2)}) \cdot \mathbf{ba} \cdot (u\mathbf{a})^*_{(1)} \\
&= (\mathbf{a}+\mathbf{b}) \cdot \mathbf{ba} \cdot u^* + \sum_u \mathcal{I}(u_{(2)}\mathbf{a}) \cdot \mathbf{ba} \cdot u^*_{(1)} \\
&= \text{applying relation (6)} \\
&\quad (\mathbf{a}+\mathbf{b}) \cdot \mathbf{ba} \cdot u^* + \sum_u \mathcal{I}(u_{(2)}) \cdot \mathbf{aba} \cdot u^*_{(1)} \\
&\quad + \sum_u (\mathbf{ab}+\mathbf{ba}) \cdot u^*_{(2)} \cdot \mathbf{ba} \cdot u^*_{(1)} \\
&\quad + \sum_u \mathcal{I}(u_{(3)}) \cdot \mathbf{ab} \cdot u^*_{(2)} \cdot \mathbf{ba} \cdot u^*_{(1)}.
\end{aligned}$$

Using the above relation and (6) we obtain that

$$\begin{aligned}
\mathcal{I}(u\mathbf{ab}) = {} & \mathcal{I}(u) \cdot \mathbf{ab} + (\mathbf{ab}+\mathbf{ba}) \cdot u^* \cdot \mathbf{b} \\
& + \sum_u \mathcal{I}(u_{(2)}) \cdot \mathbf{ab} \cdot u^*_{(1)} \cdot \mathbf{b} \\
& + (\mathbf{ab}+\mathbf{ba}) \cdot \mathbf{a} \cdot u^* + (\mathbf{a}+\mathbf{b}) \cdot \mathbf{ba} \cdot u^* \\
& + \sum_u \mathcal{I}(u_{(2)}) \cdot \mathbf{a} \cdot \mathbf{ba} \cdot u^*_{(1)} \\
& + \sum_u (\mathbf{ab}+\mathbf{ba}) \cdot u^*_{(2)} \cdot \mathbf{ba} \cdot u^*_{(1)} \\
& + \sum_u \mathcal{I}(u_{(3)}) \cdot \mathbf{ab} \cdot u^*_{(2)} \cdot \mathbf{ba} \cdot u^*_{(1)}.
\end{aligned} \tag{8}$$

Similarly, by applying (6) and (7), we obtain that

$$\begin{aligned}
\mathcal{I}(u\mathbf{ba}) = {} & \mathcal{I}(u) \cdot \mathbf{ba} + (\mathbf{ab}+\mathbf{ba}) \cdot u^* \cdot \mathbf{a} \\
& + \sum_u \mathcal{I}(u_{(2)}) \cdot \mathbf{ba} \cdot u^*_{(1)} \cdot \mathbf{a} \\
& + (\mathbf{ab}+\mathbf{ba}) \cdot \mathbf{b} \cdot u^* + (\mathbf{a}+\mathbf{b}) \cdot \mathbf{ab} \cdot u^* \\
& + \sum_u \mathcal{I}(u_{(2)}) \cdot \mathbf{b} \cdot \mathbf{ab} \cdot u^*_{(1)} + \sum_u (\mathbf{ab}+\mathbf{ba}) \cdot u^*_{(2)} \cdot \mathbf{ab} \cdot u^*_{(1)} \\
& + \sum_u \mathcal{I}(u_{(3)}) \cdot \mathbf{ba} \cdot u^*_{(2)} \cdot \mathbf{ab} \cdot u^*_{(1)}.
\end{aligned} \tag{9}$$

Now, by adding relations (8) and (9), and by using Proposition 1 and relation (3), we obtain the second formulae of the corollary. □

Hetyei in [**5**] introduced general Tchebyshev posets as follows:

For a locally finite poset Q, let $T(Q)$ denote the set of all ordered pairs $(x, y) \in Q \times Q$ satisfying $x < y$, and we define $(x_1, y_1) \leq (x_2, y_2)$ when $y_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$.

Let $\mathbf{P}^{\pm}$ denote the poset $-1 < 1 < -2,\ 2 < -3,\ 3 < \cdots < \cdots$. Pairs of elements separated with a comma are considered incomparable.

Definition 8. The Tchebyshev poset T_n is the interval $[(-1, 1), (-(n+1), -(n+2))]$ in $T(\mathbf{P}^{\pm})$.

One of the most interesting properties of the posets T_n is that the nth Tchebyshev polynomial arises from the **cd**-index upon evaluating Φ_{T_n} at $\mathbf{c} = x$, $\mathbf{d} = (x^2 - 1)/2$. For a **cd**-monomial w and a **cd**-polynomial Φ, let $[w]_\Phi$ denote the coefficient of w in Φ. In [**5**, Theorem 7.1], the following is proved

$$\left[\mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d}\cdots\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k_{r+1}}\right]_{\Phi_{T_{n+1}}} = 2^r(k_1+1)(k_2+1)\cdots(k_r+1). \tag{10}$$

Theorem 9. *The poset $I(H_{n+1})^*$ and the Tchebyshev poset T_{n+2} have the same* **cd***-index.*

Proof. From Corollary 7 we obtain

$$\Phi_{I(H_{n+1})} = \mathcal{I}(\mathbf{c}^n) = \Phi_{I(H_n)} \cdot \mathbf{c} + 2\mathbf{d}\mathbf{c}^n + 2\sum_{j=1}^{n-1} \Phi_{I(H_j)}\mathbf{d}\mathbf{c}^{n-1-j}. \tag{11}$$

Using induction, we can prove that the coefficient of the **cd**-monomial $w = \mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d}\cdots\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k_{r+1}}$ in $\Phi_{I(H_{n+1})}$ is

$$[w]_{\Phi_{I(H_{n+1})}} = 2^r(k_{r+1}+1)(k_r+1)\cdots(k_2+1). \tag{12}$$

When $k_{r+1} > 0$, from (11) it follows that

$$[w]_{\Phi_{I(H_{n+1})}} = \left[\mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d}\cdots\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k_{r+1}-1}\right]_{\Phi_{I(H_n)}} + 2\cdot[\mathbf{c}^{k_1}\mathbf{d}\mathbf{c}_2^{k}\mathbf{d}\cdots\mathbf{c}^{k_r}]_{\Phi_{I(H_{n-1-k_{r+1}})}}.$$

So, from the inductive assumption we have that

$$[w]_{\Phi_{I(H_{n+1})}} = 2^r k_{r+1}(k_r+1)\cdots(k_2+1) + 2\cdot 2^{r-1}(k_r+1)(k_{r-1}+1)\cdots(k_2+1)$$

and (12) holds. The proof of relation (12) is similar if $k_{r+1}=0$. Now, the statement of the theorem follows from (10). □

Although posets T_{n+1} and $I(H_n)^*$ have the same **cd**-index, for $n>2$ they are not combinatorially equivalent. Note that in T_{n+1} there is no element of rank 2 which covers $[-1,-2]$ and $[1,2]$ simultaneously, while any two elements of $I(H_n)$ of rank n have at least one element of rank $n-1$ which is covered by both of them.

Remark 10. Using the same ideas as in [**5**, Section 4], we can prove that the order complex of the unsigned Tchebyshev poset U_n (without the maximal element) and the order complex of the poset of intervals of the poset $1 < 2 < \cdots < (n+1)$ (without the minimal element) are the same triangulation of the n-simplex.

Let Γ be a geometric realization of the order complex of $\Delta(P \setminus \{\hat{0},\hat{1}\})$. The geometric realization of $\Delta(I(P)\setminus\{\hat{0}_{I(P)},\hat{1}_{I(P)}\})$ induces a triangulation of the suspension of Γ. As a consequence, it follows that the order complex $\Delta(I(H_{n-1})\setminus\{(\hat{0},\hat{1})\})$ gives us the same triangulation of the n dimensional cross-polytope as the order complex $\Delta(T_n\setminus\{\hat{0},\hat{1}\})$.

3. E_t-construction for posets. Paffenholz and Ziegler define in [**7**] the E_t-construction for posets as follows:

Let P be a graded poset of rank $d+1$ with a rank function r. For an integer $t\in\{0,1,\dots,d-1\}$, let $E_t(P)$ denote the set

$$\{(x,x): x\in P, r(x)=t+1\}\cup\{(y,z):\exists x\in P, r(x)=t+1, y<x<z\} \cup\{\varnothing\}$$

ordered by reversed inclusion. For example, $E_t(B_n)$ is combinatorially equivalent with the dual of the face lattice of the hypersimplex $\Delta_{n-1}(t+1)$. Note that $E_t(P)$ is also a graded poset of rank $d+1$. For any graded poset P we have that

$$E_{r(P)-1}(P) \cong P, \quad E_0(P) \cong P^*, \quad E_t(P) \cong E_{r(P)-t-1}(P^*). \tag{13}$$

Now, we are looking for linear maps $\mathcal{E}_t : \mathbf{Q}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbf{Q}\langle \mathbf{a}, \mathbf{b}\rangle$ such that $\Psi_{E_t(P)} = \mathcal{E}_t(\Psi_P)$ holds for any graded poset P. The existence of such maps can be proven in the same way as in Proposition 5.

From Lemma 2 and relation (13) we have that for all **ab**-monomials the following holds:

$$\mathcal{E}_{|u|-1}(u) = u,\ \mathcal{E}_0(u) = u^* \text{ and } \mathcal{E}_t(u) = \mathcal{E}_{|u|-t-1}(u^*), \tag{14}$$

where $|u|$ denotes the degree of monomial u. Using the same ideas as in the previous section we can prove the following theorem.

Theorem 11. *Let u be an* **ab***-monomial. Then, for all* $t = 1, 2, \dots, |u| - 2$:

$$\mathcal{E}_t(u \cdot \mathbf{a}) = \mathbf{a} \cdot \mathcal{E}_t(u) + \sum_{u, |u_{(1)}|<t} u_{(1)} \cdot \mathbf{ba} \cdot \mathcal{E}_{t-1-|u_{(1)}|}(u_{(2)}),$$
$$\mathcal{E}_t(u \cdot \mathbf{b}) = \mathbf{b} \cdot \mathcal{E}_t(u) + \sum_{u, |u_{(1)}|<t} u_{(1)} \cdot \mathbf{ab} \cdot \mathcal{E}_{t-1-|u_{(1)}|}(u_{(2)}).$$

Note that in the above sums only those summands of $\Delta(u)$ appear in which the degree of $u_{(1)}$ is lower than t.

If P is an Eulerian poset, then $E_t(P)$ is also Eulerian ([**7**, Theorem 1.4]). In that case, the computation of the **cd**-index of $E_t(P)$ is described by

Corollary 12. *Let u be a* **cd***-monomial. Then, for all* $t = 1, 2, \dots, |u| - 2$:

$$\mathcal{E}_t(u \cdot \mathbf{c}) = \mathbf{c} \cdot \mathcal{E}_t(u) + \sum_{u, |u_{(1)}|<t} u_{(1)} \cdot \mathbf{d} \cdot \mathcal{E}_{t-1-|u_{(1)}|}(u_{(2)}). \tag{15}$$

Also, for all $t = 1, 2, \dots, |u| - 3$, *we have that*

$$\mathcal{E}_t(u \cdot \mathbf{d}) = \mathbf{d} \cdot \mathcal{E}_t(u) + \sum_{u, |u_{(1)}| < t} \mathrm{Pyr}\,(u_{(1)}) \cdot \mathbf{d} \cdot \mathcal{E}_{t-1-|u_{(1)}|}(u_{(2)}).$$

From relation (14) it follows that $\mathcal{E}_{|u|-2}(u \cdot \mathbf{d}) = \mathcal{E}_1(\mathbf{d} \cdot u^*)$, which completes the recursive relations for operators $\mathcal{E}_t$ inside the coalgebra $\mathbf{Q}\langle \mathbf{c}, \mathbf{d} \rangle$.

The above formulas seem overly complicated, and we cannot express the coefficients of $\mathcal{E}_t(\mathbf{c}^n)$ (as we did in the proof of Theorem 9 for $\mathcal{I}(\mathbf{c}^n)$). But, if we define an operator $\mathcal{X} : \mathbf{Q}\langle \mathbf{c}, \mathbf{d} \rangle \to \mathbf{Q}\langle \mathbf{c}, \mathbf{d} \rangle$ with $\mathcal{X}(u) = \mathcal{E}_0(u) + \mathcal{E}_1(u) + \cdots + \mathcal{E}_{|u|-1}$, we have the following

Theorem 13. *Let* $w = \mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d} \cdots \mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k_{r+1}}$ *be a* **cd**-*monomial of degree* n. *Then the coefficient of* w *in* $\mathcal{X}(\mathbf{c}^n)$ *is*

$$2^r (k_1 + 1)(k_2 + 1) \cdots (k_r + 1) k_{r+1}.$$

Proof. We apply relation (15) and obtain

$$\mathcal{X}(\mathbf{c}^n) = \mathbf{c}^n + \mathbf{c} \cdot \mathcal{X}(\mathbf{c}^{n-1}) + 2 \cdot \sum_{j=0}^{n-3} \mathbf{c}^j \mathbf{d} \cdot \mathcal{X}(\mathbf{c}^{n-j-2}).$$

The rest of the proof is the same as in Theorem 9. □

For two posets P and Q of the same rank $n+1$ we define

$$P \circ Q = \left(P \setminus \left\{\hat{0}_P, \hat{1}_P\right\}\right) \cup \left(Q \setminus \left\{\hat{0}_Q, \hat{1}_Q\right\}\right) \cup \left\{\hat{0}, \hat{1}\right\}.$$

When P and Q are Eulerian with an odd rank $2k+1$, then $P \circ Q$ is Eulerian, and $\Phi_{P \circ Q} = \Phi_P + \Phi_Q - (\mathbf{c}^2 - 2\mathbf{d})^k$, see [**4**]. For $n = 2k+1$, we define the poset X_n as follows

$$X_n = E_0(H_n) + E_1(H_n) + \cdots + E_{n-1}(H_n).$$

Corollary 14. *Let n be an odd positive integer. Then the coefficient of the* $\mathbf{cd}$*-monomial* $w = \mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d}\cdots\mathbf{c}^{k_r}\mathbf{d}\mathbf{c}^{k_{r+1}}$ *of degree* $n-1$ *in* Φ_{X_n} *is*

$$[w] = \begin{cases} 2^r(k_1+1)(k_2+1)\cdots(k_r+1)k_{r+1} - (-2)^r(n-1) & \\ & \text{if all } k_i \text{ are even;} \\ 2^r(k_1+1)(k_2+1)\cdots(k_r+1)k_{r+1} & \\ & \text{otherwise.} \end{cases}$$

From [**7**, Theorem 2.1], we can conclude that the order complexes $\Delta(E_t(H_n))$ give us the subdivisions of the cross-polytope, which differs from the subdivisions described in [**5**, Section 4] and Remark 10.

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