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ON A-CONVEX NORMS ON COMMUTATIVE ALGEBRAS

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ABSTRACT. We study such norms on commutative algebras for which the multiplication is separately continuous. By comparing a given norm $\|\ \|$ to its operator semi-norm $\|\ \|_{\mathrm{op}}$, we get two constants $m(\|\ \|)$ (the modulus of m-convexity) and $r(\|\ \|)$ (the modulus of regularity). We study how these constants are connected to the m-convexity and to the A-convexity of $\|\ \|$. In particular, we give a concept of an irregular norm and study some properties of such norms. Further, we will give a generalization of the famous theorem of Gelfand, which states that a complete A-convex norm $\|\ \|$ is always equivalent to some m-convex norm $|\ |$, and if the algebra has a unit element e, this norm can be chosen so that $|e| = 1$.

1. Introduction. In this paper, A will denote a commutative algebra over the field $\mathbf{C}$ of complex numbers. If A has a unit element, it will be denoted by e. Let $\|\ \|$ be a usual linear-space norm on A. The topology on A defined by $\|\ \|$ will be denoted by $T(\|\ \|)$. It is said that the multiplication on A is separately continuous with respect to the norm $\|\ \|$, if the mapping $(x, y) \mapsto xy$ from $A \times A$ into A is continuous with respect to one component, when the other one is fixed ($A \times A$ is provided with the usual product topology induced by $T(\|\ \|)$). Moreover, the multiplication on A is said to be jointly continuous with respect to the norm $\|\ \|$, if the mapping $(x, y) \mapsto xy$ from $A \times A$ into A is continuous with respect to both components at the same time. The norm $\|\ \|$ is said to be absorbingly convex (shortly A-convex) on A, if for each x in A there exists a constant $M_x \geq 0$ (depending on x) such that

$$\|xy\| \leq M_x \|y\| \quad \text{for all } y \text{ in } A.$$

Moreover, the norm $\|\ \|$ is said to be submultiplicative or multiplicatively convex (shortly m-convex) on A, if

$$\|xy\| \leq \|x\| \|y\| \quad \text{for all } x \text{ and } y \text{ in } A.$$

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Clearly, every m-convex norm is A-convex, but the converse is not true in general. It is easy to verify that the multiplication on A is separately continuous with respect to the norm $\| \;\|$ if and only if $\| \;\|$ is A-convex on A. Further, the multiplication on A is jointly continuous with respect to the norm $\| \;\|$ if and only if there exists a constant $M \geq 0$ such that $\|xy\| \leq M\|x\|\|y\|$ for all x and y in A. If $M > 0$, then $M\| \;\|$ is an m-convex norm on A, and if $M = 0$, then the multiplication on A is trivial. Hence, in this case $\| \;\|$ is m-convex on A. So the multiplication on A is jointly continuous with respect to the norm $\| \;\|$ if and only if $\| \;\|$ is equivalent to some m-convex norm on A.

Let now $\| \;\|$ be an A-convex norm on A. The operator semi-norm $\| \;\|_{\mathrm{op}}$ of $\| \;\|$ is defined on A by

$$\|x\|_{\mathrm{op}} = \sup_{\|y\| \leq 1} \|xy\|, \quad x, y \in A. \tag{1.1}$$

Obviously $\| \;\|_{\mathrm{op}}$ is an m-convex norm or semi-norm on A. If $\|x\|_{\mathrm{op}} = 0$ for all $x \in A$, then the multiplication on A is trivial. Further, every such $x \in A$ for which $\|x\|_{\mathrm{op}} = 0$, is an annihilator of A. Hence, $\| \;\|_{\mathrm{op}}$ is a norm on A if and only if A does not contain any nonzero annihilators. It is easy to see that

$$\|xy\| \leq \|x\|_{\mathrm{op}}\|y\| \quad \text{for all } x \text{ and } y \text{ in } A. \tag{1.2}$$

Note also that $(k\| \;\|)_{\mathrm{op}} = \| \;\|_{\mathrm{op}}$ for every positive real k.

In this paper, we study A- and m-convexity of norms on commutative algebras. We are also interested in how the presence or the absence of a unit element impacts the multiplicative structure of such algebras. As will be shown, the operator semi-norm $\| \;\|_{\mathrm{op}}$ of a given norm $\| \;\|$ plays an important role in the study of these properties. In particular, the case where $\| \;\|$ and $\| \;\|_{\mathrm{op}}$ are not equivalent is interesting, since the operator norm has a strong influence on many properties of $(A, \| \;\|)$. In the rest of the paper, $\| \;\|$ will always denote a linear-space norm on A, which is at least A-convex. When we study a commutative algebra A equipped with $\| \;\|$, we will say that $(A, \| \;\|)$ is a normed algebra. If the norm $\| \;\|$ is m-convex and complete on A, then $(A, \| \;\|)$ is said to be a Banach algebra.

2. On m-convexity of norms. We first state an important result concerning A- and m-convexity of norms. Namely, according to the well-known theorem of Gelfand (see [**11**, Theorem 1]), we have the following:

Theorem 2.1. *Let A be a commutative algebra and $\| \ \|$ a complete norm on A, making the multiplication separately continuous. Then there exists an m-convex norm $| \ |$ on A that is equivalent to $\| \ \|$, and if A has a unit element e, the norm $| \ |$ can be chosen so that $|e| = 1$.*

As a matter of fact, in the case when A has a unit element, we can just take $| \ | = \| \ \|_{\mathrm{op}}$. On the basis of Theorem 2.1, complete and A-convex norms can be considered as m-convex norms. Further, in the case when A is unital, it can also be assumed that the norm of the unit element is one.

For nonunital algebras, the proof of Theorem 2.1 has been presented in the literature usually in the following way: Let $A_e = A \times \mathbf{C}$ be the unitization of A, and let $\| \ \|_e$ be an extension (the so-called l_1-extension) of $\| \ \|$ onto A_e, defined by

$$\|(x, \alpha)\|_e = \|x\| + |\alpha|, \quad (x, \alpha) \in A_e. \tag{2.1}$$

Now A can be considered as a subalgebra of $B(A_e)$ = the algebra of all $\| \ \|_e$-bounded linear operators on A_e, and the equivalent norm to $\| \ \|$ is the operator norm of $\| \ \|_e$ on A_e restricted to A, i.e.,

$$|x| = (\|(x, 0)\|_e)_{\mathrm{op}\,(e)} = \sup_{\|y\| + |\beta| \le 1} \|xy + \beta x\|.$$

Here $\mathrm{op}\,(e)$ denotes the operator norm on A_e. It must be noted that this norm includes an extra scalar component coming outside the algebra A, which makes the use of $(\|(\cdot, 0)\|_e)_{\mathrm{op}\,(e)}$ more or less complicated (see for example [**15**, Theorem 1.3.1], [**20**, Theorem 2.4 and Corollary 2.5], [**19**, pages 3–5] and [**18**, Proposition 1.1.9]). We will later show that the norm $(\|(\cdot, 0)\|_e)_{\mathrm{op}\,(e)}$ can be replaced by another equivalent norm, in which only the elements of A are used (see Example 4.1). Another method for proving Theorem 2.1 is the following: First it is shown that for the complete norm $\| \ \|$, separate continuity of the multiplication implies that the multiplication is in fact jointly continuous. Then the norm $\| \ \|$ is replaced by a new norm $C\| \ \|$ (C a positive constant), which is m-convex. For this technique, see [**12**, Theorem 0.3.4 and

Proposition 1.2.5]. We will later take a closer look at this method, but in a slightly different way. The weakness of this technique is that the property $|e| = 1$, or its extension to the nonunital case, cannot be taken into account so easily. However, we shall later introduce a technique, where the condition $|e| = 1$ can be generalized to nonunital algebras.

Let $\|\ \|$ be an A-convex norm on A. In a way, the operator semi-norm $\|\ \|_{\mathrm{op}}$ gives frames for the m-convexity of $\|\ \|$. We now consider this. The next result follows directly from (1.1) and (1.2).

Lemma 2.2. *Let A be a commutative algebra with an A-convex norm $\|\ \|$. Then $\|\ \|$ is m-convex on A if and only if $\|x\|_{\mathrm{op}} \leq \|x\|$ for every $x \in A$.*

Thus, if we want to study the multiplicative structure of $\|\ \|$, it is useful to compare $\|\ \|$ to its operator semi-norm $\|\ \|_{\mathrm{op}}$. We now define a constant, which is important in studying the relationship between $\|\ \|$ and $\|\ \|_{\mathrm{op}}$. So, let

$$m(\|\ \|) = \sup_{\|y\| \leq 1} \|y\|_{\mathrm{op}}, \quad y \in A.$$

We say that $m(\|\ \|)$ is the modulus of m-convexity of $\|\ \|$ on A. From the definition of $m(\|\ \|)$ it follows that, for every $x \in A$, we have $\|x\|_{\mathrm{op}} \leq m(\|\ \|)\|x\|$. Hence, by Lemma 2.2, $\|\ \|$ is m-convex on A if and only if $m(\|\ \|) \leq 1$. On the other hand, it is easy to verify that if $m(\|\ \|)$ is finite, then $m(k\|\ \|) = (1/k)m(\|\ \|)$ for every $k > 0$. Thus, $\|\ \|$ can be considered as an m-convex norm on A whenever $m(\|\ \|)$ is finite (if $m(\|\ \|) \neq 0$, then $m(\|\ \|)\|\ \|$ is an equivalent m-convex norm on A, and if $m(\|\ \|) = 0$, then $\|\ \|$ is m-convex on A). So we get the following result.

Theorem 2.3. *Let A be a commutative algebra and $\|\ \|$ a norm on A, making the multiplication separately continuous. If $m(\|\ \|)$ is finite, then $\|\ \|$ is equivalent to some m-convex norm on A.*

Note that the modulus of m-convexity is in fact the optimal value of those constants $k > 0$, making the norm $k\|\ \|$ m-convex on A. That is, if $k \geq m(\|\ \|)$, then $k\|\ \|$ is m-convex on A, and if $k < m(\|\ \|)$, then $k\|\ \|$

is not m-convex on A. The following result shows that Theorem 2.3 gives us a slight generalization of Theorem 2.1 (Gelfand).

Theorem 2.4. *Let A be a commutative algebra and $\| \ \|$ a complete norm on A, making the multiplication separately continuous. Then $m(\| \ \|)$ is finite.*

Proof. For a given $y \in A$, let T_y be the left multiplication operator on A, defined by $T_y(x) = xy$, $x \in A$. It is easy to verify that T_y is a $\| \ \|$-bounded linear operator on A. Moreover, for every $x \in A$, we have $\|T_y(x)\| = \|xy\| \le \|x\|_{\mathrm{op}}\|y\|$. Thus, for each fixed $x \in A$, the set $\{T_y(x) \mid \|y\| \le 1\}$ is $\| \ \|$-bounded on A. Hence, by the theorem of Uniform Boundedness, $m(\| \ \|) = \sup_{\|y\| \le 1} \|y\|_{\mathrm{op}} = \sup_{\|y\| \le 1} \|T_y\|$ is finite. □

As there exist noncomplete norms for which the modulus of m-convexity is finite, we see that it is indeed a more general condition than the completeness of $\| \ \|$, to ensure $\| \ \|$ to be equivalent to some m-convex norm on A.

We will next study the structure of normed algebras with infinite modulus of m-convexity. Note that a norm $\| \ \|$ cannot be complete on A if $m(\| \ \|) = \infty$. Moreover, such a norm cannot be equivalent to any m-convex norm on A, and therefore the finiteness of $m(\| \ \|)$ is also a necessary condition for $\| \ \|$ to be equivalent to some m-convex norm on A. We say that a norm $\| \ \|$ with $m(\| \ \|) = \infty$ is properly A-convex.

Example 2.5. Let $A = C([a,b])$ be the algebra of all continuous complex-valued functions defined on a closed interval $[a,b]$ and equipped with pointwise algebraic operations. Further, let v be a continuous function from $[a,b]$ into $\mathbf{R}$ for which $v(a) = v(b) = 0$ and $v(t) > 0$, if $t \ne a, b$. Let $\| \ \|_{l_1}$ and $\| \ \|_v$ be incomplete A-convex norms on A, defined by

$$\|x\|_{l_1} = \int_a^b |x(t)| \, dt \text{ and } \|x\|_v = \sup_{t \in [a,b]} v(t)|x(t)|, \quad x \in A.$$

It is easy to see that $(\| \ \|_{l_1})_{\mathrm{op}} = (\| \ \|_v)_{\mathrm{op}} = \| \ \|_\infty$ and further, that $m(\| \ \|_{l_1}) = m(\| \ \|_v) = \infty$. Note that the completions of both

$(A, \| \|_{l_1})$ and $(A, \| \|_v)$ are not normed algebras. We will next show that this is a characteristic feature of every normed algebra $(A, \| \|)$ with $m(\| \|) = \infty$.

For a given norm $\| \|$ on A, we denote by A_c the completion of A with respect to $\| \|$ and by $\| \|_c$ the extension of $\| \|$ onto A_c.

Theorem 2.6. *Let A be a commutative algebra and $\| \|$ an incomplete A-convex norm on A. Then $(A_c, \| \|_c)$ is a normed algebra if and only if $m(\| \|)$ is finite.*

Proof. If $m(\| \|)$ is finite, then $\| \|$ is either m-convex on A or equivalent to some m-convex norm on A. Hence, it follows from the general theory of normed algebras that $(A_c, \| \|_c)$ is a normed algebra. Suppose next that $(A_c, \| \|_c)$ is a normed algebra. Since $\| \|_c$ is complete on A_c, we have $m(\| \|_c) < \infty$. On the other hand, it is easy to verify that $m(\| \|) \leq m(\| \|_c)$, and so $m(\| \|)$ is finite. □

For further properties on completions of topological algebras, see [**16**].

When we used the constant $m(\| \|)$ in proving Theorem 2.3, we did not need any information about the condition $\|e\| = 1$. So we can deduce that we must know something more about the norm in order to be able to extend this condition also to a nonunital case. We shall consider this in the next section.

3. On regularity of norms. In order that we could extend the condition $\|e\| = 1$ from unital to nonunital algebras, we need the following constant. So, for a given A-convex norm $\| \|$ on A, set

$$r(\| \|) = \sup_{\|y\|_{\mathrm{op}} \leq 1} \|y\|, \quad y \in A.$$

We will say that $r(\| \|)$ is the modulus of regularity of $\| \|$ on A (this notion will be clarified later). From the definition of $r(\| \|)$ it follows that for every $x \in A$, we have $\|x\| \leq r(\| \|)\|x\|_{\mathrm{op}}$. Moreover, for every $k > 0$, we have $r(k\| \|) = kr(\| \|)$. The importance of the modulus of regularity comes from the fact that by using it, we can divide all A-convex norms into three classes. The classification we shall next represent has a strong impact on many properties of normed algebras. These include the Gelfand representation (see [**2**–**5**]), approximation properties of function algebras (see [**6, 13**]) and multipliers of Banach algebras (see [**14**]).

Let $\| \ \|$ be an A-convex norm on A. We will say that the norm $\| \ \|$ is regular on A if $\| \ \| = \| \ \|_{\mathrm{op}}$, weakly regular on A if $r(\| \ \|)$ is finite, and further, irregular on A if $r(\| \ \|) = \infty$. Clearly every regular norm is m-convex and weakly regular. Moreover, the operator semi-norm of a weakly regular norm is always a norm. The following shows that in a unital algebra, every A-convex norm is automatically weakly regular.

Theorem 3.1. *Let A be a commutative algebra with a unit element e. If $\| \ \|$ is an A-convex norm on A, then $r(\| \ \|) = \|e\|$.*

Proof. It is easy to verify that $\|e\|_{\mathrm{op}} = 1$. Thus, $\|e\| \leq r(\| \ \|)$. On the other hand, for every $x \in A$, we have $\|x\| = \|ex\| \leq \|e\| \|x\|_{\mathrm{op}}$. Hence, $r(\| \ \|) \leq \|e\|$, and so $r(\| \ \|) = \|e\|$. $\square$

From the above one can also see that in a unital algebra, the operator norm of a given norm is always regular. This motivates us to give the following generalization of Theorem 2.1.

Theorem 3.2. *Let A be a commutative algebra with a complete norm $\| \ \|$, making the multiplication separately continuous. Then there exists an equivalent m-convex norm $| \ |$ on A. Further, if $\| \ \|$ is weakly regular and $\| \ \|_{\mathrm{op}}$ is regular on A, then the norm $| \ |$ can be chosen so that $r(| \ |) = 1$.*

Proof. Take $| \ | = \| \ \|_{\mathrm{op}}$. Then $| \ |$ is an m-convex norm on A for which $r(| \ |) = 1$. Further, by Theorem 2.4, both $m(\| \ \|)$ and $r(\| \ \|)$ are finite. Hence, by the inequalities $\| \ \|_{\mathrm{op}} \leq m(\| \ \|) \| \ \|$ and $\| \ \| \leq r(\| \ \|) \| \ \|_{\mathrm{op}}$, $\| \ \|$ and $\| \ \|_{\mathrm{op}}$ are equivalent norms on A. $\square$

As there exist nonunital algebras with norms satisfying the conditions of Theorem 3.2, we see that Theorem 3.2 generalizes Theorem 2.1 to nonunital algebras. Further, the condition $r(| \ |) = 1$ corresponds to the condition $|e| = 1$, generalizing it to nonunital algebras. It is worth noting that the regularity of the norm $\| \|_{\text{op}}$ is needed to guarantee the condition $r(| |) = 1$ to be valid on A. We shall later consider the condition $(\| \|_{\text{op}})_{\text{op}} = \| \|_{\text{op}}$ in greater detail. We finally note that for an irregular norm the condition $\|e\| = 1$ cannot be generalized to nonunital algebras. That is, if $\| \|$ is irregular on A, then there does not exist an equivalent m-convex norm $| |$ on A for which $r(| |) = 1$ (or even finite) would hold.

It follows from the properties above that weakly regular normed algebras behave in one sense (topologically) like normed algebras with a unit. On the other hand, irregular normed algebras have in many senses a different kind of structure. We consider this next. We start with the following result, which gives a class of irregular norms. The proof is trivial.

Theorem 3.3. *Let A be a commutative algebra. If A has a nonzero annihilator, then every A-convex norm on A is irregular.*

We will now give a less trivial example of an algebra with an irregular norm. Note that a norm can be complete even though its modulus of regularity would be infinite.

Example 3.4. For $1 < p < \infty$, consider the Banach space $(L_p(G), \| \|_p)$ of equivalence classes of complex-valued functions on an infinite compact topological abelian group G whose pth powers are absolutely integrable with respect to a Haar measure λ on G which is normalized so that $\lambda(G) = 1$, and the norm $\| \|_p$ is defined on $L_p(G)$ by

$$\|f\|_p = \left(\int_G |f(t)|^p d\lambda(t) \right)^{1/p}.$$

It is well-known that $(L_p(G), \| \|_p)$ is a commutative Banach algebra with respect to convolution as multiplication. Further, it follows from the inequality $\|f * g\|_p \le \|f\|_1 \|g\|_p$, $f, g \in L_p(G)$, that $(\|f\|_p)_{\text{op}} \le \|f\|_1$ for every $f \in L_p(G)$. Hence, $\| \|_p$ is an irregular norm on $L_p(G)$.

We saw in Theorem 3.1 that the existence of a unit element forces every A-convex norm to be at least weakly regular. From this condition one can deduce that if $(A, \| \ \|)$ has an approximate identity, then its boundedness should be somehow connected with the value of the modulus of regularity. We consider this next. First we recall the definition of an approximate identity.

Let $\| \ \|$ be an A-convex norm on A. A net $(e_\alpha)_{\alpha\in\Omega}$ of A for which $\|xe_\alpha - x\| \to 0$ for every $x \in A$, is said to be an approximate identity of $(A, \| \ \|)$. If the set $\{\|e_\alpha\| \mid \alpha \in \Omega\}$ is bounded, then it is said that $(e_\alpha)_{\alpha\in\Omega}$ is a bounded approximate identity of $(A, \| \ \|)$ and further, if $\|e_\alpha\| \leq 1$ for every $\alpha \in \Omega$, then $(e_\alpha)_{\alpha\in\Omega}$ is said to be a minimal approximate identity of $(A, \| \ \|)$. Note that if $(A, \| \ \|)$ has an approximate identity, then $\| \ \|_{\mathrm{op}}$ is a norm on A:

Theorem 3.5. *Let A be a commutative algebra with an irregular norm $\| \ \|$. If $(A, \| \ \|)$ has an approximate identity $(e_\alpha)_{\alpha\in\Omega}$, then it is unbounded with respect to $\| \ \|$.*

Proof. Let $x \in A$ and $\alpha \in \Omega$ be arbitrary. Then $\|x\| \leq \|xe_\alpha - x\| + \|xe_\alpha\| \leq \|xe_\alpha - x\| + \|x\|_{\mathrm{op}}\|e_\alpha\|$, and from this it easily follows that the $\| \ \|$-boundedness of $(e_\alpha)_{\alpha\in\Omega}$ would imply $r(\| \ \|)$ to be finite. Hence, $(e_\alpha)_{\alpha\in\Omega}$ is unbounded with respect to $\| \ \|$. □

In the theory of normed algebras, it is often assumed that approximate identities are bounded. However, in doing so, irregular norms are excluded even though there are classical normed and Banach algebras whose norms are only irregular. On the other hand, it must be noted that, although approximate identities are unbounded with respect to irregular norms, the situation can be different with respect to corresponding operator norms.

Theorem 3.6. *Let $(A, \| \ \|)$ be a commutative Banach algebra with a sequential approximate identity $(e_n)_{n=1}^{\infty}$ (bounded or unbounded). Then $(e_n)_{n=1}^{\infty}$ is a bounded approximate identity of $(A, \| \ \|_{\mathrm{op}})$.*

For the proof of the theorem, see [**14**]. We do not know whether Theorem 3.6 is valid for every approximate identity. However, it is clear that the problem for the general case comes from the cardinality of the index set Ω. In the next example, $\| \|_\infty$ denotes the usual supremum norm.

Example 3.7. For $1 \leq p < \infty$, consider the Banach space $(l_p(\mathbf{Z}_+), \| \|_p)$ of all sequences $x = (x_n)_{n=1}^\infty$, where the x_n are complex numbers satisfying the condition $\sum_{n=1}^\infty |x_n|^p < \infty$, and the norm $\| \|_p$ is defined on $l_p(\mathbf{Z}_+)$ by

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

It is easy to verify that $(l_p(\mathbf{Z}_+), \| \|_p)$ is a commutative Banach algebra with respect to coordinatewise multiplication. Moreover, it follows from the inequality $\|xy\|_p \leq \|x\|_\infty \|y\|_p$, $x, y \in l_p(\mathbf{Z}_+)$, that $(\|x\|_p)_{\mathrm{op}} \leq \|x\|_\infty$ for all $x \in l_p(\mathbf{Z}_+)$. Hence, $\| \|_p$ is an irregular norm on $l_p(\mathbf{Z}_+)$. Note that $(l_p(\mathbf{Z}_+), \| \|_p)$ has a sequential approximate identity. By Theorems 3.5 and 3.6, it is unbounded with respect to $\| \|_p$ and bounded with respect to $(\| \|_p)_{\mathrm{op}}$.

Note also that the unbounded approximate identity of the algebra $(L_p(G), \| \|_p)$ of Example 3.4 is bounded with respect to the corresponding operator norm (see [**15**, pages 110–111]).

It is interesting to note that the operator norm is a very useful tool in describing properties of normed algebras, which only contain unbounded approximate identities. In particular, this is the case when the approximate identities are bounded with respect to the corresponding operator norms. Examples of such methods are found in [**5**] (Gelfand representation), [**13**] (approximation properties of function algebras) and [**14**] (multipliers).

We now return to study the condition $(\| \|_{\mathrm{op}})_{\mathrm{op}} = \| \|_{\mathrm{op}}$, which played an important part in generalizing the theorem of Gelfand to nonunital algebras. It must be noted that, even for an irregular norm, the corresponding operator norm can be regular.

Theorem 3.8. *Let A be a commutative algebra with an A-convex norm $\| \|$. If $(e_\alpha)_{\alpha\in\Omega}$ is a minimal approximate identity of $(A, \| \|_{\mathrm{op}})$, then the equality $(\| \|_{\mathrm{op}})_{\mathrm{op}} = \| \|_{\mathrm{op}}$ is valid on A.*

Proof. Let $x \in A$ be given. Then for every $\alpha \in \Omega$, we have $\|xe_\alpha\|_{\mathrm{op}} \le (\|x\|_{\mathrm{op}})_{\mathrm{op}}$. Since $\lim_\alpha \|xe_\alpha\|_{\mathrm{op}} = \|x\|_{\mathrm{op}}$, we have $\|x\|_{\mathrm{op}} \le (\|x\|_{\mathrm{op}})_{\mathrm{op}}$. On the other hand, as $\|\ \|_{\mathrm{op}}$ is m-convex on A, the converse inequality is automatically valid on A, and so we have $(\|\ \|_{\mathrm{op}})_{\mathrm{op}} = \|\ \|_{\mathrm{op}}$. □

Note that from Theorems 3.6 and 3.8 we get the following corollary.

Corollary 3.9. *Let $(A, \|\ \|)$ be a commutative Banach algebra with a sequential approximate identity $(e_n)_{n=1}^{\infty}$. Then $(\|\ \|_{\mathrm{op}})_{\mathrm{op}}$ and $\|\ \|_{\mathrm{op}}$ are equivalent on A.*

Of course it is possible that the condition $(\|\ \|_{\mathrm{op}})_{\mathrm{op}} = \|\ \|_{\mathrm{op}}$ is valid also for normed algebras without any approximate identities. For example, any function algebra equipped with the supremum norm satisfies this condition. On the other hand, $(\|\ \|_{\mathrm{op}})_{\mathrm{op}} = \|\ \|_{\mathrm{op}}$ is not a general property of normed algebras. For example, take a commutative algebra A in which there exists an element x such that $xA \neq \{0\}$ and $xyz = 0$ for all y and z in A. Then for any A-convex norm $\|\ \|$ on A, we have $(\|x\|_{\mathrm{op}})_{\mathrm{op}} = 0$, but $\|x\|_{\mathrm{op}} > 0$.

At the end of this section, we shall study irregular norms on algebras of continuous functions. In the following example, X is a noncompact, locally compact Hausdorff space. We denote by $C_b(X)$ the set of all continuous and bounded complex-valued functions on X and by $C_0(X)$ the set of all continuous complex-valued functions on X, which vanish at infinity, i.e., for each $f \in C_0(X)$ and $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that $|f(t)| < \varepsilon$ for all $t \in X \setminus K_\varepsilon$. With respect to pointwise algebraic operations, $C_b(X)$ and $C_0(X)$ are algebras and, further, $C_0(X)$ is a subalgebra of $C_b(X)$. It is well known that, under the usual supremum norm $\|\ \|_\infty$, $C_b(X)$ and $C_0(X)$ are Banach algebras. We now introduce two interesting function algebras whose norms are irregular (note that the supremum norm is always regular norm). The importance of these algebras follows from the fact that they are not B^*-algebras, but still their structures (for example, the ideal structure) are very similar to the structures of the well-known B^*-algebras $(C_0(X), \|\ \|_\infty)$ and $(C_b(X), \|\ \|_\infty)$.

Example 3.10. Let v be an upper semi-continuous real-valued function on X for which $\inf_{t\in X} v(t) > 0$. Unless it is otherwise stated, we also assume that there exists a net (t_α) in X such that $v(t_\alpha) \to \infty$ when $t_\alpha \to t_\infty$ (by t_∞ we mean the point at infinity of X). Hence, v is unbounded on X. We now define two subsets of $C(X)$ (= the set of all continuous complex-valued functions on X) by

$$C_0^v(X) = \{f \in C(X) \mid vf \text{ vanishes at infinity}\}$$

and

$$C_b^v(X) = \{f \in C(X) \mid vf \text{ is bounded on } X\}.$$

Obviously $C_0^v(X)$ and $C_b^v(X)$ are algebras with respect to pointwise algebraic operations. Further, $C_0^v(X)$ is a proper subalgebra of $C_0(X)$, $C_b^v(X)$ is a proper subalgebra of $C_b(X)$ and $C_0^v(X)$ is a subalgebra of $C_b^v(X)$. If v is chosen so that $v(t_\alpha) \to \infty$ for all nets (t_α) in X for which $t_\alpha \to t_\infty$, then it is easy to see that also $C_b^v(X)$ is a subalgebra of $C_0(X)$. Note that if we assume v to be bounded on X, then we clearly have $C_0^v(X) = C_0(X)$ and $C_b^v(X) = C_b(X)$. So $C_0(X)$ (correspondingly $C_b(X)$) is a special case of $C_0^v(X)$ (correspondingly of $C_b^v(X)$). We next study the topological structure of these algebras.

It is easy to see that, with respect to the supremum norm, $C_0^v(X)$ and $C_b^v(X)$ are not complete, and therefore it is not the natural norm for them. On the other hand, if we equip $C_0^v(X)$ and $C_b^v(X)$ with a weighted norm $\| \|_v$, defined by

$$\|f\|_v = \sup_{t \in X} v(t)|f(t)|,$$

then $\| \|_v$ is a complete A-convex norm on $C_0^v(X)$ and on $C_b^v(X)$ (for the proof of $C_0^v(X)$, see [**6**, Theorem 3.2], and for $C_b^v(X)$ the proof is similar). Thus, $(C_0^v(X), \| \|_v)$ and $(C_b^v(X), \| \|_v)$ can be considered as Banach algebras. Note that if v is assumed to be bounded on X, then $\| \|_v$ and $\| \|_\infty$ are equivalent norms, and therefore Banach algebras $(C_0(X), \| \|_\infty)$ and $(C_b(X), \| \|_\infty)$ are special cases of Banach algebras $(C_0^v(X), \| \|_v)$ and $(C_b^v(X), \| \|_v)$. On the other hand, in general $\| \|_v$ is properly stronger than the supremum norm. Further, then $\| \|_v$ is irregular on $C_0^v(X)$ and on $C_b^v(X)$. This follows from the equality $(\| \|_v)_{\mathrm{op}} = \| \|_\infty$, which can be shown to be valid on both $C_0^v(X)$ and $C_b^v(X)$.

It is interesting to note that the structure of the Banach algebra $(C_0^v(X), \| \|_v)$ is very similar to the structure of the Banach algebra $(C_0(X), \| \|_\infty)$. For example, $(C_0^v(X), \| \|_v)$ satisfies the Stone-Weierstrass property, i.e., each of its points separating symmetric subalgebra, which is bounded away from zero, is $\| \|_v$-dense in $C_0^v(X)$. Further, every closed ideal I of $(C_0^v(X), \| \|_v)$ is of the form $I = \{f \in C_0^v(X) \mid f(t) = 0 \text{ for all } t \in E\}$, where E is some closed subset of X. Also, $(C_0^v(X), \| \|_v)$ has an approximate identity (unbounded by Theorem 3.5). On the other hand, the structure of the Banach algebra $(C_b^v(X), \| \|_v)$ is very complicated and, moreover, from some parts it differs a lot from the structure of the Banach algebra $(C_b(X), \| \|_\infty)$. For example, $(C_b^v(X), \| \|_v)$ does not have an approximate identity in general. It is also worth noting that $(C_b^v(X), \| \|_v)$ does not have the Stone-Weierstrass property. Further, the ideal structure of $(C_b^v(X), \| \|_v)$ is very complicated, and even the description of all of its closed maximal ideals seems to be a difficult problem. For a more detailed study on algebras $C_0^v(X)$ and $C_b^v(X)$, see [**6, 13**].

Note also that again the approximate identity of $(C_0^v(X), \| \|_v)$ is bounded with respect to the operator norm $\| \|_\infty$.

Some of the results of this paper can be extended in a natural way to a noncommutative case and also to p-normed algebras with $0 < p < 1$. For example, in a noncommutative case the definition of A-convexity could be represented in the following way: A norm $\| \|$ on a noncommutative algebra A is A-convex if for each $x \in A$ there exists a constant $M(x) \geq 0$ such that $\max\{\|xy\|, \|yx\|\} \leq M(x)\|y\|$ for all $y \in A$. Moreover, the operator semi-norm of x is then the infimum of these constants. For the properties of these types of norms or semi-norms, see [**1, 10, 18**]. We focused our attention only upon a commutative case, since our applications of this theory were restricted to commutative algebras.

4. Norms on unitization of algebras. We mentioned earlier that Theorem 2.1 can be proved by extending a nonunital normed algebra $(A, \| \|)$ to a unital normed algebra $(A_e, \|(\cdot, \cdot)\|_e)$, defined in (2.1). More generally, in the theory of normed algebras, this unitization method has been used very often to simplify proofs concerning nonunital algebras. However, as the structure of irregular normed algebras differs a lot from the structure of unital normed algebras, one can deduce that the unitization method may not be the best possible way to study algebras, which are equipped with irregular norms. Indeed, as $(A_e, \|(\cdot,\cdot)\|_e)$ is always a regular normed algebra, we see that the unitization method "hides" the irregularity of the original norm. Therefore, it is easier to study irregular normed algebras without unitization (see also [**6, 8**]). We will now look in greater detail at how regularity, weak regularity or irregularity impacts on the extension of the norm $\| \ \|$ (or the norm topology $T(\| \ \|)$) onto the unitization A_e. We assume here that normed algebras in question are nonunital and complete. This is due to the fact that otherwise it may happen that completions of these algebras are unital, and this can cause problems when studying unitizations on nonunital algebras (see, for example, [**7**, Remark 5]).

Suppose first that $\| \ \|$ is a regular norm on A. Then it has two extreme extensions onto the unitization A_e. The maximal extension is the l_1-extension $\|(\cdot,\cdot)\|_e$, and the minimal extension is the so-called operator extension $\|(\cdot,\cdot)\|_{\mathrm{op}}$, defined by

$$\|(x,\alpha)\|_{\mathrm{op}} = \sup_{\|y\|\leq 1} \|xy+\alpha y\|, \ (x,\alpha) \in A_e.$$

By [**7**, Corollary 2], we have

$$\|(x,\alpha)\|_{\mathrm{op}} \leq \|(x,\alpha)\|_e \leq 3\|(x,\alpha)\|_{\mathrm{op}}, \ (x,\alpha) \in A_e.$$

Moreover, 3 is the best (minimal) constant for the upper bound. For this result, see also [**18**, Proposition 1.1.13] and [**9**].

Suppose next that $\| \ \|$ is a weakly regular norm on A. Again, $\|(\cdot,\cdot)\|_e$ is an extension of the norm $\| \ \|$. On the other hand, $\|(\cdot,\cdot)\|_{\mathrm{op}}$ need not be an extension of $\| \ \|$. However, also in this case the topologies $T(\|(\cdot,\cdot)\|_e)$ and $T(\|(\cdot,\cdot)\|_{\mathrm{op}})$ on A_e are equivalent and, further, they are extensions of the norm topology $T(\| \ \|)$ onto A_e. Therefore, for weakly regular norms, it is better to study extensions of topologies instead of extensions of norms.

Suppose finally that $\| \ \|$ is an irregular norm on A. The main problem in studying extensions of irregular norms, is that $\|(\cdot,\cdot)\|_{\mathrm{op}}$ is no longer an extension of the norm $\| \ \|$ and $T(\|(\cdot,\cdot)\|_{\mathrm{op}})$ is not an extension of the norm topology $T(\| \ \|)$. Further, even though $\|(\cdot,\cdot)\|_e$ is still an extension of $\| \ \|$, the l_1-extension can cause problems when studying $(A, \| \ \|)$ by means of $(A_e, \|(\cdot,\cdot)\|_e)$. For example, as it was shown in [**6, 13**], the structure of $(A_e, \|(\cdot,\cdot)\|_e)$ can be much more complicated compared to the structure of $(A, \| \ \|)$.

We now give an example on how Theorem 2.1, and other theorems connected with it in this paper, can be proved by using just the elements of A. In the following example, irregularity of a norm $\| \ \|$ implies irregularity of the new norm.

Example 4.1. Let A be a commutative algebra with or without a unit. Given an A-convex norm $\| \ \|$ on A (complete or noncomplete), we define a norm $\| \ \|_M$ on A by

$$\|x\|_M = \max\{\|x\|, \|x\|_{\mathrm{op}}\}, \quad x \in A.$$

Obviously $\| \ \|_M$ is an m-convex norm on A and $\| \ \|$ is m-convex on A if and only if $\| \ \|_M = \| \ \|$. We next give some general properties for $\| \ \|_M$. Let $x \in A$ be arbitrary. Then $\|x\| \leq \|x\|_M \leq \max\{1, m(\| \ \|)\}\|x\|$ and $\|x\|_{\mathrm{op}} \leq \|x\|_M \leq \max\{1, r(\| \ \|)\}\|x\|_{\mathrm{op}}$. From this we see that if $m(\| \ \|)$ is finite, then $\| \ \|$ is equivalent to $\| \ \|_M$. Further, if $r(\| \ \|)$ is finite, then $\| \ \|_{\mathrm{op}}$ is equivalent to $\| \ \|_M$ and finally, if both constants $m(\| \ \|)$ and $r(\| \ \|)$ are finite, then all three norms $\| \ \|$, $\| \ \|_{\mathrm{op}}$ and $\| \ \|_M$ are mutually equivalent. It is interesting to note that in some cases $\| \ \|_M$ is the minimal m-convex norm on A, which majorizes the original norm $\| \ \|$. For this, see [**1, 17**]. The norm $\| \ \|_M$ can now be used in proving Theorem 2.1. Namely, the norm $(\|(\cdot, 0)\|_e)_{\mathrm{op}\,(e)}$ introduced in Section 2, is equivalent to $\| \ \|_M$. This follows from the inequalities

$$\big\|\|_M \leq (\|(\cdot, 0)\|_e)_{\mathrm{op}\,(e)} \leq \|\big\| + \|\|_{\mathrm{op}} \leq 2\|\|_M .$$

By using $\| \ \|_M$ instead of $(\|(\cdot, 0)\|_e)_{\mathrm{op}\,(e)}$, we can avoid extra scalar components of the latter norm. The norm $(\|(\cdot, 0)\|_e)_{\mathrm{op}\,(e)}$ has been used in the literature quite often. See, for example, [**12**, Proof of Proposition 1.1.9].

REFERENCES

1. M. Abel and J. Arhippainen, *Locally m-pseudoconvex topologies on locally A-pseudoconvex algebras*, Czech. Math. J. **54** (2004), 675–680.

2. J. Arhippainen, *On functional representation of locally A-convex algebras*, Rocky Mountain J. Math. **30**, (2000), 777–794.

3. ———, *On functional representation of uniformly A-convex algebras*, Math. J. **46** (2004), 509–515.

4. ———, *On Gelfand representation of topological algebras*, Topol. Algebras Appl., Contemp. Math. **341** (2004), 25–30.

5. J. Arhippainen, *On functional representation of normed algebras*, J. Math. Anal. Appl. **329** (2007), 790–797.

6. J. Arhippainen and J. Kauppi, *Generalization of the B^*-algebra* $(C_0(X), \| \|_\infty)$, Math. Nachr. **282** (2009), 7–15.

7. J. Arhippainen and V. Müller, *Norms on unitizations of Banach algebras revisited*, Acta Math. Hungar. **114** (2007), 201–204.

8. A.K. Gaur and Z.V. Kovarik, *Norms, states and numerical ranges on direct sums*, Analysis **11** (1991), 155–164.

9. ———, *Norms on unitizations of Banach algebras*, Proc. Amer. Math. Soc. **117** (1993), 111–113.

10. A. El Kinani and M. Oudadess, *Classes of locally p-convex algebras*, Acta Univer. Ouluensis **3** (2003), 58–69.

11. I. Gelfand, D. Raikov and G. Shilov, *Commutative normed rings*, Chelsea Publishing Company, Bronx, New York, 1964.

12. A.Ya. Helemskii, *Banach and locally convex algebras*, Clarendon Press, Oxford, 1993.

13. J. Kauppi, *Approximation properties for subalgebras of* $C_0(X)$, Contemp. Math. **435** (2007), 245–252.

14. ———, *Irregularity on normed and function algebras*, Ph.D., University of Oulu, 2008.

15. R. Larsen, *Banach algebras*, *An introduction*, Pure Appl. Math. **24**, Marcel Dekker, Inc., New York, 1973.

16. A. Mallios, *Topological algebras*: *Selected topics*, Elsevier Science Publishers, New York, 1986.

17. L. Oubbi, *Locally A-convex algebras revisited*, Topological Algebras Appl., Contemp. Math. **427** (2007), 319–329.

18. T.W. Palmer, *Banach algebras and general theory of *-algebras* (Volume I), Cambridge University Press, Cambridge, 1994.

19. C.E. Rickart, *General theory of Banach algebras*, D. Van Nostrand Co., Inc., Princeton, 1960.

20. W. Zelazko, *Banach algebras*, Elsevier Publishing Co. and Polish Sci. Publishers, Warsaw, 1973.

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