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INHERENT COMPACTNESS OF UPPER CONTINUOUS SET VALUED MAPS

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ABSTRACT. Half of the paper, roughly, is devoted to an overview of research originally connected with the names of Vaĭnšteĭn, Choquet and Dolecki. 'VCD theorem' serves as a convenient code-name for a series of results in which 'compactness' of a set-valued map between topological spaces appears as a consequence of its 'continuity.' This gives a proper perspective for the other half, in which new 'VCD theorems' appear. Our stress is on showing an essential unity of methods that underlie results in this area of analysis/topology.

0. Names and notions. Let Y be a topological space, $\mathcal{B}$ a family of its subsets and $A \subset Y$. Following [**18**], we write $\mathcal{B} \rightsquigarrow A$ and say that $\mathcal{B}$ *aims* at A, if, for each neighborhood V of A, there exists a B in $\mathcal{B}$ such that $B \subset V$.

Let X be another topological space and $F : X \rightrightarrows Y$ a set-valued map. For a filter $\mathcal{U}$ on X, its image $F(\mathcal{U})$ is a filter base, and we keep the notation $F(\mathcal{U})$ for the generated filter. The map F is *upper continuous* at a point x_0 (*uc at* x_0), if $F(\mathcal{N}) \rightsquigarrow F(x_0)$, where $\mathcal{N} = \mathcal{N}(x_0)$ is the neighborhood filter of $x_0 \in X$. F is *upper continuous* (uc) if it is upper continuous at x for each $x \in X$.

Historically, the terminology concerning upper continuous maps varies greatly. Choquet [**9**] calls them *strongly upper semi-continuous*, Strother [**33**] *weakly continuous*, Ponomarev [**32**] *continuous*, Kuratowski [**24**], Michael [**28**] and others use the term *upper semi-continuous*. In recent literature upper semi-continuity became the accepted term. However, the original motivation for using this term, as reported, e.g., in Engelking [**19**, 1.7.17], seems a bit shaky. This feeling is compounded by the fact that an upper semi continuous set-valued map taking points as its values is *continuous*, making the terminology contradictory with the much older and universally accepted notion of

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upper semi-continuity in real analysis. Upon realizing that the quite natural terms of *upper* and *lower continuity* (instead of upper and lower semi-continuity) are 'free for use' (because they are not employed elsewhere), we decided they were worth a try in this survey. At any rate, successful or not, the reported terminological experiment is conducted in the first author's dissertation [**10**] whose part the present research represents.

The *external part* or *map* (of F at x_0) is the map $E(\,.\,) := F(\,.\,)\backslash F(x_0)$.

Although E should have subscripts referring to F and x_0, we simplify the notation because the map F and the point x_0 will be fixed throughout the paper.

Let $\mathcal{U}$ be a filter contained in $\mathcal{N}$. Then $E(\mathcal{U}) = \{F(U) \setminus F(x_0) : U \in \mathcal{U}\}$, and the generated filter is still denoted by $E(\mathcal{U})$. We call it the *external filter of F at x_0 relative to $\mathcal{U}$*. If $\mathcal{U} = \mathcal{N}$, we drop $\mathcal{U}$, and refer to it as the *external filter*. It may be degenerate, that is, it may contain the empty set. However, if it does, x_0 is not interesting from our point of view and is discarded from further considerations.

A set $K \subset Y$ is said to be a *$\mathcal{U}$-kernel of F at x_0* if $E(\mathcal{U}) \rightsquigarrow K$. If $\mathcal{U} = \mathcal{N}$, we drop $\mathcal{U}$ and speak about the *kernel of F at x_0*. If, moreover, $K \subset F(\mathcal{U}^\bullet)$, where $\mathcal{U}^\bullet = \cap\mathcal{U}$, we refer to K as a *Choquet $\mathcal{U}$-kernel* (and a Choquet kernel if $\mathcal{U} = \mathcal{N}$).

The *active $\mathcal{U}$-boundary* of F at x_0 is defined as the adherence of $E(\mathcal{U})$, that is,

$$\mathrm{Frac}_{\mathcal{U}} F(x_0) = \bigcap_{U \in \mathcal{U}} \overline{\{F(U) \setminus F(x_0)\}}.$$

Again, if $\mathcal{U} = \mathcal{N}(x_0)$, we write $\mathrm{Frac}\, F(x_0)$. $\mathrm{Frac}\, F(x_0)$ stands for the French 'frontière active.' Our general reference for things topological is [**23**].

1. Pioneer era. Here is how, in 1948, Choquet introduces upper semi-continuous maps in his paper [**9**, page 70].

> There exists, besides outer semicontinuity that we just studied, another type of semicontinuity which appears to be less interesting, and which we mention mainly because of its analogy with the lower semicontinuity to be examined later.

Then, after a few remarks concerning the definition, Choquet continues:

> One can stress the lack of interest of the upper semicontinuity by the following result which shows to what extent this type of semicontinuity is restrictive.
>
> **1.1** [**9**, Théorème 3]. *Let X, Y be metrizable and F uc at $x_0 \in X$. There exists a compact set $K \subset F(x_0)$ such that, for each neighborhood V of K in Y, there exists a neighborhood U of x_0 in X such that*
>
> $$F(U) \subset (V \cup F(x_0)).$$

One should therefore not be surprised that its author does not even bother to provide a proof to such a negative result!

As far as we know, the result was never used by others and was probably forgotten. This claim is corroborated by the fact that for a long time nobody noticed the close connection of Choquet's theorem to another, this time quite known, theorem in topology. A little earlier than Choquet, in December 1946, a Soviet mathematician Vaĭnšteĭn submitted a paper about continuous closed functions between metric spaces. Here is his theorem with its original proof.

1.2 [34, Theorem 1]. *Let X, Y be metric spaces and $f : Y \to X$ a function that is continuous closed and onto. Then, for each $x_0 \in X$, the boundary K of the fiber $f^{-1}(x_0)$ is compact.*

Proof. Let a sequence of points $y_n \in K = \operatorname{Fr} f^{-1}(x_0)$ be given. Define $U_n = f^{-1}[V(x_0, 1/n)]$, where $V(x_0, 1/n)$ is the ball of radius $1/n$ centered at x_0. As $y_n \in \operatorname{Fr} f^{-1}(x_0)$, there is a point $y_n' \in U_n$ such that $\rho(y_n, y_n') < 1/n$ and $x_n = f(y_n') \neq x_0$. Since $x_n \to x_0$ and $x_n \neq x_0$ for each $n \in \mathcal{N}$, we conclude by the closedness of f that the sequence y_n' has a cluster point $y_0 \in K$. Clearly, y_0 is also a cluster point of the sequence y_n.

The theorem of Vaĭnšteĭn received due attention (we only quote the paper [**29**] by Michael in which q-points were originally defined). Yet, in all this research, the Choquet theorem remained unknown.

Enters Dolecki. In the mid 1970s, he stressed the role of semi-continuity as a unifying theme for several problems in control theory and was led to define the active boundary $\mathrm{Frac}\, F(x_0)$, see [**12**] (wherein [**11**] is referred). This, most probably, is the set K that appears in the statement of Choquet. Here is Dolecki's theorem.

1.3 [**11**]. *Let $x_0 \in X$ be first countable, and let Y be metrizable. If F is* uc *at x_0, then* $\mathrm{Frac}\, F(x_0)$ *is compact.*

Proof. The proof given here is taken from [**17**] in which Dolecki and Rolewicz refer to [**11**]. The latter paper remained in the preprint form and was never published. Let $\{y_1, y_2, \dots\}$ be a countable subset of $\mathrm{Frac}\, F(x_0)$, and let $U_1 \supset U_2 \cdots$ be a fundamental sequence of neighborhoods of x_0. For each $n \in \mathbf{N}$, choose $z_n \notin F(x_0)$ contained in the $1/n$-ball centered at y_n. As $x_0 \notin F^{-1}(z_n)$ and U_n intersects $F^{-1}(z_n)$ for each n, we conclude that $F^{-1}(\{z_n\}_{n=1}^{\infty})$ is not closed. Hence, by upper continuity, $\{z_n : n \in \mathbf{N}\}$ is not closed. Therefore, $\{z_n\}$ and $\{y_n\}$ have a (common) cluster point y_0. $\square$

The fact, stated by Choquet, that in the above the inclusion $\mathrm{Frac}\, F(x_0) \subset F(x_0)$ takes place, is also shown in the just quoted paper, see [**17**, Lemma 2].

Here is the connection between the theorems above. A, not necessarily continuous, function f between topological spaces Y and X maps closed sets into closed sets if and only if the set-valued map $F = f^{-1} : X \to Y$ is upper continuous. Moreover, if f is continuous, then $\mathrm{Frac}\, F(x_0) = Frf^{-1}(x_0)$. In other words, the theorem of Vaĭnšteĭn concerns uc relations in $X \times Y$ that are graphs of continuous functions, while the result of Choquet and Dolecki is about arbitrary uc relations. We adopt the name 'Vaĭnšteĭn-Choquet-Dolecki theorems (VCD) for results of this type.

A subsequent paper, written with Lechicki and the first in which the Choquet contribution is accounted for, brings two improvements. Firstly, if X, Y are first countable, $\mathrm{Frac}\, F(x_0)$ is actually the smallest set which is a Choquet kernel of F at x_0, [**16**, Corollary 6.1(i)]. The second improvement, using today's terminology (to be explained below), [**16**, Theorem 7.5, 7.6] can be stated as follows.

1.4. Theorem. *Let X be first countable and Y Dieudonné complete. The external filter $E(\mathcal{N})$ and its adherence* $\operatorname{Frac} F(x_0)$ *are compact.*

2. Maturity era. We recall that a point $x_0 \in X$ is called *of countable character* if it admits a countable fundamental system of neighborhoods, i.e., $\mathcal{N}(x_0)$ is countably based. More generally, x_0 is a *q-point* if it admits a *q-sequence* $(Q_n)_{n=1}^{\infty}$, that is, a decreasing sequence of neighborhoods of x_0 having the following property: if $x_n \in Q_n$, $n = 1, 2 \ldots$, then the sequence (x_n) has a cluster point, say $x \in X$. A point of local countable compactness, though not necessarily of countable character, is also a q-point. A *q-space* is a space whose points are q-cluster points. For instance, Čech complete spaces are in the class. Michael [**29**] generalizes Vaĭnšteĭn's results about closed continuous functions by considering q-spaces instead of metric spaces.

We come to [**22**], which we would like to consider as the first paper of the 'maturity era' of the Vaĭnšteĭn-Choquet-Dolecki theorem, an era in which various authors seek generalizations of the theorem beyond its original setting. In [**22**], two, seemingly different, cases are considered. First is the case in which the point x_0 at the map F is studied and is *of countable character*. An essential new ingredient are Choquet kernels defined via cluster sets whose definitions use sequences and which are in general smaller than the active boundary. This line of reasoning was pushed towards its natural limits in [**25**].

2.1. Definition. The *countable adherence* of $\mathcal{B}$ is defined by

$$\operatorname{adh}_\omega \mathcal{B} = \{y \in Y : \text{there exists } (y_n) \geq \mathcal{B}, y_n \rightharpoonup y\}\,.$$

The semi-arrow or harpoon is used to denote that y is a cluster point of (y_n). We recall that a point y is a *cluster* point of the *sequence* (y_n) if, for every neighborhood V of y and for every $m \in \mathbf{N}$, there exists an $n > m$ such that $y_n \in V$.

2.2. Definition. The *sequential adherence* of $\mathcal{B}$ is defined by

$$\mathrm{adh}_\sigma \mathcal{B} = \{y \in Y : \text{there exists } (y_n) \geq \mathcal{B}, y_n \to y\}.$$

We note that the original definitions of the cluster sets $\mathrm{adh}_\omega \mathcal{B}$ and $\mathrm{adh}_\sigma \mathcal{B}$ were stated for the outer part of F at x_0 only and were slightly different in form. Here, as everywhere else in the present paper, a sequence is identified with the elementary filter [**2**] it generates and the notation $(y_n) \geq \mathcal{B}$ means that this elementary filter is finer than $\mathcal{B}$, compare Section 3 below.

Before stating the main result in this setting, we recall that a Hausdorff space X is said to be *angelic* if its relatively countably compact subsets are relatively compact and, moreover, if every such subset K in X has a sequentially determined closure. That is, if

$$x \in \overline{K} \setminus K \implies \text{there exists } (x_n) \subset K, \quad x_n \to x.$$

2.3 [**25**, Theorem 1]. *Let $x_0 \in X$ be of countable character, let Y be angelic and F* uc *at x_0. Then* $\mathrm{Frac}\, F(x_0) = \mathrm{adh}_\sigma E(\mathcal{N})$ *is compact. Moreover, it is the smallest Choquet kernel of F at x_0.*

The second case considered in [**22**] is that of a *q-point* x_0. The ideas of Michael's paper and the Dolecki-Lechicki paper are combined to provide a proof of the following theorem.

2.4. Theorem. *Let x_0 be a q-point of a regular space X. Let Y be regular and $F : X \rightrightarrows Y$ upper continuous. If $F(x_0)$ is closed in the $\mathcal{G}_\delta$-topology*[1] *of Y, then* $\mathrm{Frac}\, F(x_0) \subset F(x_0)$. *If Y is Dieudonné complete, then* $\mathrm{Frac}\, F(x_0)$ *is compact.*

The theorem has obvious flaws. For instance, it leaves unanswered the question whether $\mathrm{Frac}\, F(x_0)$ is a kernel of F at x_0. With hindsight, one can say that its proof and, more precisely, a specific technique of selection of a sequence of points providing a needed contradiction, was more important than the result itself.

Before we proceed with our discussion further, let us state and prove a simple lemma which is at the core of all the arguments leading to the VCD-type results. Recall that $x \in X$ is an *accumulation point* of a subset A of X if, for each neighborhood U of x, there exists a point $a \in A$, $a \neq x$, such that $a \in U$. Further, a subset D of X is *countably compact at* $A \subset X$ if every sequence of points of D has a cluster point in A; if $A = X$, we drop 'at X' and call it *relatively countably compact.* D is *countably compact* if it is countably compact at itself.

2.5. Basic lemma. *Let $(x_n) \subset X$ be a sequence having x_0 as its cluster point, and let $y_n \in F(x_n) \setminus F(x_0)$. Suppose F is upper continuous at x_0. Then the set $\{y_n : n \in \mathcal{N}\}$ has an accumulation point belonging to $F(x_0)$. In particular, if Y is a T_1-space, we can assert the existence of a cluster point of the sequence (y_n).*

Proof. Denote by C the closure of the set $\{y_n : n \in \mathbf{N}\}$, and suppose C is disjoint with $F(x_0)$. Then $V = Y \setminus C$ is an open set containing $F(x_0)$ so, by the upper continuity of F at x_0, for some $U \in \mathcal{N}(x_0)$, $F(U) \subset V$. Thus, for arbitrarily large indices $k \in \mathbf{N}$, we have $x_k \in U$. Yet, for the corresponding y_ks,

$$y_k \in F(x_k) \subset V \subset Y \setminus \{y_1, y_2, \dots\},$$

a contradiction. Hence, there is a $y \in C \cap F(x_0)$. In particular, $y \neq y_n$ for $n \in \mathbf{N}$. $\square$

Explicitly, in a less general form (with $x_n \to x_0$), the lemma seems to be isolated for the first time as Lemma 1 in [**25**]. In a form close to the one above, it appeared independently in [**6, 26**]. It is proven in [**26**] and, interestingly enough, quoted as known in [**6**] (an erroneous reference to [**1**] is given despite the fact that the proof needed is so simple). On the other hand, it seems that in one form or another it must have already been used by Choquet.

In [**26**], all the results of [**22, 25**] are improved in a unified way. Thus, the countable case is now completely covered by the q-point case making both [**22**] and [**25**] obsolete. Here is, for instance, one of the main results.

2.6. Theorem. *Let x_0 be a q-point of a regular space X, let Y be a Hausdorff space whose relatively countably compact subsets are relatively compact and let $F : X \rightrightarrows Y$ be upper continuous.*[2] *Then the active boundary* $\operatorname{Frac} F(x_0)$ *is the smallest compact kernel of F at x_0.*

Two ingredients play an essential role in its proof. On the one hand, the already-mentioned selection technique from [**22**] is improved and combined with the basic lemma. On the other hand, the techniques lifted from [**25**] are now applied to the Q-filter, i.e., the filter generated by the chosen q-sequence, rather than to the filter $\mathcal{N}$ of all the neighborhoods of x_0. This second development, as we recently realized, gives a link to an earlier and more abstract research concerning *compact filters*. A discussion of this link is the subject of Section 3.

Yet, to keep things in their proper order, we cannot finish the present section without mentioning a development which we treat as another remarkable offspring of the selection technique of [**22**]. The authors of [**6**], slightly generalizing a topological game invented by Bouziad in [**3**], 'axiomatize' the selection process into a specific game. This extends the class of domain spaces. Another crucial point of this approach is a 'transfer lemma' made possible by a clever use of the basic lemma and which asserts that a winning strategy in the game played on the filter $\mathcal{N}(x_0)$ can be transported via the map F on the external filter base $E(\mathcal{N})$. All of this will be discussed in Section 4.

3. Compactness. Although it was later found that compact filters were first introduced by Pettis [**31**], see also the survey article [**27**] where the origins of the notion are discussed, the fact remains that in the case of Dolecki and Lechicki the very notion was independently discovered after they had proven (see 1.4 above) that the filter $E(\mathcal{N})$ is compact (see [**14, 15, 30**]). Compactness of filters has been studied intensively ever since. We only introduce a few relevant notions needed in the present paper.

Let Y be a topological space, $\mathcal{A}$, $\mathcal{B}$ families of its subsets. We write $\mathcal{A}\#\mathcal{B}$ and say that $\mathcal{A}$ *meshes with* $\mathcal{B}$ if, for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$, $A \cap B \neq \varnothing$.

We fix $\mathcal{B}$ and assume that it is a filter base throughout this section.

The *adherence* of $\mathcal{B}$ is defined by

$$\operatorname{adh}\mathcal{B} = \bigcap\{\overline{B} : B \in \mathcal{B}\}$$

or, equivalently,

$$\operatorname{adh}\mathcal{B} = \{y \in Y : \text{there exists } \mathcal{F} \geq \mathcal{B}, \mathcal{F} \to y\}$$

where $\mathcal{F}$ is a filter on Y.

Let $\mathbf{D}$ denote a class of filters on Y. We say that $\mathcal{B}$ *is* $\mathbf{D}$*-compact at* $\mathcal{A}$, if

$$\mathcal{D} \in \mathbf{D}, \mathcal{D}\#\mathcal{B} \implies \operatorname{adh}\mathcal{D}\#\mathcal{A}.$$

If $\mathcal{A} = \{A\}$, we speak about compactness at A. If $A = Y$, we drop Y (provided no ambiguity about Y can occur). Our reference is [**27**] despite the change in terminology signaled in Endnote 3. We say that $\mathcal{B}'$ is a base of $\mathcal{B}$, or that $\mathcal{B}$ is *based* by $\mathcal{B}'$, if both $\mathcal{B}$ and $\mathcal{B}'$ generate the same filter.

Let $\mathcal{O}(\mathbf{D})$ be the class formed by filters from $\mathbf{D}$ which are *openly based*, that is, admit a generating base formed by open sets. With the above definitions and conventions repeated, we use the name of $\mathbf{D}$-*midcompactness* for objects that are $\mathcal{O}(\mathbf{D})$-compact.

Let $\aleph$ be a cardinal. We denote by $\mathbf{F}_\aleph$ the class of all filters that admit a base of cardinality (strictly) less than $\aleph$, by $\mathcal{O}(\mathbf{F}_\aleph)$ the class of all filters that admit a base of cardinality less than $\aleph$ which consists of open sets. $\mathcal{B}$ is said to be $\aleph$-*compact at* $\mathcal{A}$, if it is $\mathbf{F}_\aleph$-compact at $\mathcal{A}$. $\mathcal{B}$ is said to be $\aleph$-*midcompact at* $\mathcal{A}$, if it is $\mathcal{O}(\mathbf{F}_\aleph)$-compact at $\mathcal{A}$. The 'full' properties are obtained 'by dropping $\aleph$.' Thus, $\mathbf{F}$ is the class of all filters (on Y) and, for instance, $\mathcal{B}$ is *compact at* A, that is, at the family $\mathcal{A}$ composed of one set A, if it is $\mathbf{F}$-compact at A. This, of course, happens, if $\mathcal{B}$ is $\aleph$-compact at A for each $\aleph$.

Since we use strict inequality when dealing with cardinals, it is convenient and customary to refer to $\aleph_0$-compactness as *finite compactness*. Note that $\mathbf{F}_{\aleph_0}$ is the class of all principal filters. The notion of finite compactness, first considered in [**13**], is useful because

3.1. Fact. *$\mathcal{B}$ aims at A if and only if $\mathcal{B}$ is finitely compact at A.*

Hence, if $\mathcal{U}$ is any filter contained in $\mathcal{N}(x_0)$, we have

3.2. Fact. *A set K is a $\mathcal{U}$-kernel of F at x_0 if and only if $E(\mathcal{U})$ is finitely compact at K.*

Similarly, it is customary to refer to $\aleph_1$-compactness and its versions as *countable compactness*. Hence, for instance, $\mathcal{B}$ is *countably midcompact*, if any filter $\mathcal{F}$ having a countable base of open sets and meshing with $\mathcal{B}$, has a cluster point (in Y).

Let $\mathbf{D}$ again be a class of filters, and let $\mathcal{B}$ be $\mathbf{D}$-compact (with all possible variations of this notion thus far defined being allowed). We say that $\mathcal{B}$ is *nearly* $\mathbf{D}$-compact, in the sense considered, if the corresponding condition is satisfied for all filters $\mathcal{D} \in \mathbf{D}$ which are *finer* than $\mathcal{B}$. Thus, for instance, $\mathcal{B}$ is nearly $\aleph$-midcompact if for each $\mathcal{D} \in \mathcal{O}(\mathbf{F}_\aleph)$ such that $\mathcal{D} \geq \mathcal{B}$, $\mathcal{D}$ has a cluster point.

As already mentioned, we identify sequences with the elementary filters they generate. Thus, if (y_n) is a sequence in Y, then $(y_n) \geq \mathcal{B}$ means that the elementary filter $\mathcal{E} = \mathcal{E}\{(y_n)\}$ generated by (y_n) is finer than $\mathcal{B}$. Similarly, for $(y_n)\#\mathcal{B}$. Once this identification agrees, it is natural to consider (near) $\mathbf{D}$-compactness when $\mathbf{D}$ is the class of all sequences. Thus, $\mathcal{B}$ is (*nearly*) *sequence compact at A* if for each sequence (y_n) (finer than) meshing with $\mathcal{B}$, (y_n) has a cluster point in A. Although it is quite obvious that the near sequence compactness is equivalent to the more standard notion of near countable compactness, we have already encountered situations where the use of sequences seemed more natural. It should be clear that the *countable adherence* of $\mathcal{B}$ can also be written as

$$\mathrm{adh}_\omega \mathcal{B} = \{y \in Y : \text{there exists } \mathbf{F}_{\aleph_1} \ni \mathcal{D} \geq \mathcal{B}, \mathcal{D} \rightharpoondown y\},$$

where the harpoon is used to denote that y is a cluster point of $\mathcal{D}$.

Similarly, the *sequential* adherence of $\mathcal{B}$ can be written

$$\mathrm{adh}_\sigma \mathcal{B} = \{y \in Y : \text{there exists } \mathbf{F}_{\aleph_1} \ni \mathcal{D} \geq \mathcal{B}, \mathcal{D} \to y\}.$$

3.3. Theorem. *Let $\mathcal{B}$ be countably based and nearly countably compact. Its countable adherence* $\mathrm{adh}_\omega \mathcal{B}$ *is a countably closed set.*

Proof. Let $B_1 \supset B_2 \supset \cdots$ be a base of $\mathcal{B}$. Let (a_i) be a sequence of distinct points in $A = \mathrm{adh}_\omega \mathcal{B}$, and let a be its cluster point. For each

$i \in \mathbf{N}$, we find a sequence $(y_j^i : j \in \mathbf{N})$ finer than $\mathcal{B}$ such that $(y_j^i) \subset B_i$ has a_i as its cluster point.

Let (y_k) be the sequence obtained by reordering the double sequence (y_j^i) into a single sequence, taking together the elements for which $i+j$ has a common value and ordering these groups in increasing order of $i+j$. In what follows it will be convenient to refer to this reordering as the *Cauchy reordering*. The (Cauchy reordered) sequence (y_k) is finer than $\mathcal{E}$. Indeed, in a set $B \in \mathcal{B}$, we first find n so large that $B_n \subset B$. Hence, all the columns of the matrix which are indexed by the upper index i larger than n are contained in B. We can also find the lower index k so large that, for $i = 1, 2, \dots, n$, the tails are $(a_j^i)_{j=k}^{\infty} \subset B$. It is now obvious that we can find a tail of (y_k) so far that the corresponding triangle avoids the $k \times n$ rectangle and, consequently, the tail is contained in B. Hence $(y_k) \geq \mathcal{B}$ and, as a is a cluster point of (y_k), $a \in A$. This shows that A is countably closed. □

The following theorem is due to Cascales and Orihuela [**8**], see also [**7**].

3.4. Theorem. *The following are equivalent for $\mathcal{B}$ with a countable base.*

(i) *For each $(y_n) \geq \mathcal{B}$, the closure $\overline{\{y_n : n \in \mathbf{N}\}}$ is countably compact.*

(ii) *$\mathcal{B}$ is countably compact at* $\mathrm{adh}_\omega \mathcal{B}$ *which itself is countably compact.*

Proof. (i) $\Rightarrow$ (ii). With the notation of the previous proof, $\{a_i : i \in \mathbf{N}\} \subset \overline{\{y_n : n \in \mathbf{N}\}}$. Now, if (i) holds, the existence of a cluster point a used in that proof, is guaranteed. Hence, the fact that $a \in A$ shows the countable compactness of A. The fact that $\mathcal{B}$ is nearly countably compact is obvious. As $\mathcal{B}$ is countably based, it is countably refinable (see Section 5 where this notion is defined) and, therefore, countably compact at its countable adherence.

(ii) $\Rightarrow$ (i). Consider $(y_n) \geq \mathcal{B}$. Observe first that if (y_{n_k}) is a subsequence of (y_n), then $(y_{n_k}) \geq (y_n) \geq \mathcal{B}$. Hence, (y_{n_k}) has a cluster point. It follows that, if (y_n) is finer than $\mathcal{B}$, then $\{y_n : n \in \mathbf{N}\}$ is a relatively countably compact subset of Y. So suppose that $\{a_i : i \in \mathbf{N}\}$

is taken from $\overline{\{y_n : n \in \mathbf{N}\}} \setminus \{y_n : n \in \mathbf{N}\}$. Then $(a_i) \subset \mathrm{adh}_\omega \mathcal{B}$ and, by (ii), has a cluster point. This shows (i). □

The proof given here follows [**26**] rather than [**8**]. What is really interesting though, is the link between this result and the Vaĭnšteĭn-Choquet-Dolecki theorem. Cascales and Orihuela knew [**22**], but their applications do not go beyond points of countable character treated in [**25**]. On the other hand, a part of the arguments in [**26**] could have been skipped using the above theorem.

The close relationship of those papers is based on two facts. The first is just an observation that being a q-point means also that the filter $\mathcal{N}$ of all neighborhoods of x_0 admits a coarser *countably based* filter $\mathcal{Q}$ which is *countably compact* (at Y). The second fact is the possibility of transporting countable compactness from $\mathcal{Q}$ to $E(\mathcal{Q})$.

Here is the crucial lemma. When compared with [**26**], it is given in a somewhat improved form which will be needed in Section 5.

3.5. Transfer lemma. *Let X be regular. Let $\mathcal{R}$ be a nearly countably compact filter contained in $\mathcal{N}(x_0)$. Then $E(\mathcal{R})$ is nearly countably compact. Moreover, if $(y_n) \geq E(\mathcal{R})$, then the set $\{y_n : n \in \mathbf{N}\}$ is relatively countably compact.*

Proof. As we already noticed, the notion of near countable compactness is equivalent to that of near sequence compactness. Since our argument depends on the basic lemma, the proof will be done using the near sequence compactness.

Let $(y_n) \geq E(\mathcal{R})$. Pick $y_{n_1} = y_1$ and $x_1 \in X = L_1$ such that $y_1 \in F(x_1) \setminus F(x_0)$.

Choose a closed neighborhood L_2 of x_0 such that

$$L_2 \subset \{x \in X : F(x) \subset Y \setminus \{y_{n_1}\}\},$$

and find y_{n_2} so that $\{y_n\}_{n=n_2}^{\infty} \subset F(L_2) \setminus F(x_0)$.

Pick x_2 so that $y_{n_2} \in F(x_2) \setminus F(x_0)$, and choose a closed neighborhood L_3 of x_0 such that

$$L_3 \subset \{x \in X : F(x) \subset Y \setminus \{y_{n_1}, y_{n_2}\}\}.$$

Find y_{n_3} so that $\{y_n\}_{n=n_3}^{\infty} \subset F(L_3) \setminus F(x_0)$. Pick x_3 so that $y_{n_3} \in F(x_3) \setminus F(x_0) \dots$. Continue.

For the inductively defined sequence (y_{n_k}), we have $(y_{n_k}) \geq (y_n) \geq E(\mathcal{R})$. Further, (x_k) is the sequence in the domain space such that

$$y_{n_k} \in F(x_k) \setminus F(x_0),$$

which implies that $(x_k) \geq \mathcal{R}$. By assumption on $\mathcal{R}$, (x_k) has a cluster point, say ξ. By the choice of (L_n), for each $n \in \mathbf{N}$,

$$\xi \in L_{n+1} \subset \{x : F(x) \cap \{y_1, y_2, \dots, y_n\} = \varnothing\},$$

and so

$$F(\xi) \cap \{y_1, y_2, \dots\} = \varnothing.$$

Hence, $y_n \in F(x_n) \setminus F(\xi)$ and, by our selection process, (y_n) is a sequence of distinct points. By the basic Lemma 2.5, the sequence (y_n) must have a cluster point in $F(\xi)$.

In order to see the 'moreover' statement, it suffices to note that any subsequence of a sequence finer than $E(\mathcal{R})$ is still finer than $E(\mathcal{R})$. □

We now give 'the ultimate proof' of the Vaĭnšteĭn-Choquet-Dolecki theorem. We apply it to Theorem 2.6 but treat it rather as a 'canonical scheme of proving' this type of result. It will be used later to give other refinements of our leitmotive theorem.

Proof of Theorem 2.6. Let $\mathcal{Q}$ be a q-filter contained in $\mathcal{N}(x_0)$. Then its image filter $E(\mathcal{Q})$ by the external part E of F at x_0 is, by the transfer lemma, nearly countably compact. As $E(\mathcal{Q})$ is countably based, Theorem 3.4 of Cascales and Orihuela can be applied. By our condition on the space Y combined with the last statement of the transfer lemma, condition (i) of Theorem 2.5 is obviously satisfied. Hence, by Theorem 2.5 (ii), $\operatorname{adh} E(Q)$ is countably compact. Applying the condition on the space Y for the second time, we conclude that $\operatorname{adh} E(\mathcal{Q})$ is compact. Further, $E(\mathcal{Q})$, being countably based, is countably compact which, combined with the compactness of $\operatorname{adh} E(\mathcal{O})$, implies that $E(\mathcal{Q})$ is compact. Hence, the finer filter $E(\mathcal{N})$ is compact. Its nonempty adherence $\operatorname{Frac} F(x_0)$ being a closed subset of $\operatorname{adh} E(\mathcal{Q})$, is also compact. The minimality property of the kernel Frac $F(x_0)$ is a known general fact about compact filters.

4. Favorable points. We start by describing the topological game considered in [**6**]. This is, in essence, the game of Bouziad [**3**] (who played it on filters of neighborhoods only). Because no other game is considered here, in the sequel the word *game* will denote this particular game.

Let $\mathcal{B}$ be a family of nonempty subsets of a topological space X. The *game on* $\mathcal{B}$ between players α and β is played as follows. α starts by choosing a set $B_1 \in \mathcal{B}$. Then β chooses a point $x_1 \in B_1$ and α answers by choosing a set $B_2 \in \mathcal{B}$. They continue. This produces a sequence $p \equiv ((B_n, x_n) : n \in \mathbf{N})$ called a *play* of the game on $\mathcal{B}$. α *wins* whenever the sequence (x_n) has a cluster point. Otherwise α loses and β wins. A *strategy* s *for player* α is a 'recipe' that prescribes its move in every possible situation. Thus, $s \equiv (s_0, s_1, s_2 \dots)$ is a sequence of $\mathcal{B}$-valued functions such that $s_0(\varnothing) = B_1$, $s_1(x_1) = B_2, \dots s_n(x_1, x_2, \dots, x_n) = B_{n+1}, \dots$. In particular, we see that the domain of s_n is the set of all finite sequences $(x_1, x_2, \dots, x_n)$ of length n satisfying the condition $x_{i+1} \in s_i(x_1, x_2, \dots, x_i)$ for all $0 < i < n$. A strategy s is said to be *winning on* $\mathcal{B}$ if it guarantees that α wins in each play, i.e., that each outcome $(x_n)_{n=1}^{\infty}$ of a play according to s has a cluster point. We call the family $\mathcal{B}$ *favorable* (for our game here) if a winning strategy exists for player α on $\mathcal{B}$.

We are interested in favorable families and, in particular, *favorable filters*. More precisely, a pair $(\mathcal{B}, s)$ is favorable whenever s is a winning strategy for player α, and $\mathcal{B}$ is favorable if such a winning strategy s exists on $\mathcal{B}$.

Remark. There are situations in which it is more convenient to use the *monotone game*. Player α is supposed to choose sets in a *decreasing* way, i.e., the set in the next move must be contained in the set chosen in the previous move. Thus, the function s_n in a strategy s is now defined on a $2n$-tuple $(B_1, x_1, B_2, x_2, \dots, B_n, x_n)$. It is easy to see that if a filter $\mathcal{F}$ is favorable for the game, then its winning strategy s can be redefined into a winning strategy s' in the monotone game on $\mathcal{F}$.

A point $x_0 \in X$ is called *favorable* if $\mathcal{N}(x_0)$ is favorable; a space X is *favorable* if all points of X are. The q-points are favorable, and favorable spaces are stable under closed subspaces and formation of products.

A favorable pair $(\mathcal{B}, t)$ *is finer than* a favorable pair $(\mathcal{A}, s)$ if $\mathcal{B}$ is finer than $\mathcal{A}$ and the range of t is contained in the range of s, i.e., each play according to t is a play according to s.

4.1 [**6**, Proposition 2.2]. *If a filter $(\mathcal{R}, s)$ is favorable, and $\mathcal{S}$ is a filter finer than $\mathcal{R}$, then $\mathcal{S}$ is favorable. That is, a winning strategy t exists on $\mathcal{S}$, such that $(\mathcal{S}, t) \geq (\mathcal{R}, s)$.*

The next proposition should be thought of as the 'transfer lemma' for the game. Its proof and its role are somewhat similar to those of Lemma 3.5.

4.2 [**6**, Proposition 3.2]. *Let X and Y be regular spaces and $x_0 \in X$ favorable. If $F : X \rightrightarrows Y$ is upper continuous, then $E(\mathcal{U})$ is favorable.*

Once the propositions are established, the approach of Cao, Moors and Reilly is quite simple: they declare a space Y as *satisfying the condition* (∂) provided any favorable filter $\mathcal{H}$ in Y has a cluster point. In view of Proposition 4.1, this also means that every finer filter has a cluster point, i.e., that $\mathcal{H}$ is compact. In other words, Y is the space in which the Vaĭnšteĭn-Choquet-Dolecki theorem holds.

Of course, one is interested in what (∂)-spaces are. That they are stable under closed subspaces and formation of products is rather clear. Also, if a filter base $\mathcal{B}$ is favorable, then it must be totally bounded. Indeed, if $\mathcal{B}$ is not totally bounded, then the player β has a winning strategy. Namely, there exists a vicinity V and a play in which no matter what α does, the second player forces it to produce a V-discrete sequence of moves. At this point, by the argument already used by Dolecki and Lechicki [**16**, Theorem 7.5] which, in turn, was an adaptation of a classical proof of Bourbaki [**2**, Chapter 2, Section 4, Theorem 3], one concludes that Dieudonné complete spaces are in the class. Another class of (∂)-spaces, according to [**6**], is provided by $C(K)$ spaces equipped with the topology of pointwise convergence. We refer an interested reader to study the original paper and concentrate, in the next section, on an alternative generalization of the Vaĭnšteĭn-Choquet-Dolecki theorem. As we shall see, our approach exploits the transfer lemma proven in Section 3.

Besides [**6**], two other papers by Cao et al. concern the Vaĭnšteĭn-Choquet-Dolecki theorem. Reference [**5**] repeats the results proven in [**25**]. In [**4**], Cao feels compelled to reproduce, almost verbatim, the historical information given in [**22**], yet [**25**] is not mentioned. The substance of [**4**] is a complicated proof that stratifiable spaces satisfy the condition ($\Game$). However, every stratifiable space is paracompact [**21**, Theorem 5.7] and every paracompact space is Dieudonné complete [**23**, Chapter 6, Exercises], so the result does not say anything new.

5. The r-point case. Again, we fix a filter base $\mathcal{B}$ of subsets of Y throughout this section. We say that $\mathcal{B}$ is $\mathbf{D}/\mathbf{J}$-*refinable* if for each filter $\mathcal{D} \in \mathbf{D}$ meshing with $\mathcal{B}$ there exists a filter $\mathcal{J} \in \mathbf{J}$ such that $\mathcal{J}$ is finer than both $\mathcal{D}$ and $\mathcal{B}$. $\mathcal{B}$ is $\mathbf{D}/\mathbf{J}$-*midrefinable* if it is $\mathcal{O}(\mathbf{D})/\mathbf{J}$-refinable. Therefore, we will call $\mathcal{B}$ *countably midrefinable* if for each $\mathbf{D} \in \mathcal{O}(\mathbf{F}_{\aleph_1})$ meshing with $\mathcal{B}$ there exists a countably based filter $\mathcal{J}$ which is finer than $\mathcal{D}$ and finer than $\mathcal{B}$. We note that, see e.g., [**13**], $\mathbf{F}_{\aleph_1}/\mathbf{F}_{\aleph_1}$-refinable filters or filter bases are also called *strongly Fréchet*.

Our reference concerning uniform spaces is [**19**]. Let Y be a uniform space with the filter $\mathcal{V}$ of entourages of the diagonal. $\mathcal{B}$ is said to be *totally bounded* if for each $V \in \mathcal{V}$ there exists a finite set K and $B \in \mathcal{B}$ such that $B \subset V(K)$. As far as we know, totally bounded filters appear first in [**20**]. The next theorem should be compared with Theorem 3.4.

5.1. Theorem. *The following conditions are equivalent for a countably midrefinable base $\mathcal{B}$ in a Hausdorff uniform space Y.*

(i) *For each $(y_n) \geq \mathcal{B}$, the set $\{y_n : n \in \mathbf{N}\}$ is totally bounded.*

(ii) *$\mathcal{B}$ is totally bounded.*

Proof. (ii) $\Rightarrow$ (i). The sequence (y_n), being finer than $\mathcal{B}$, must be totally bounded. (i) follows easily.

(i) $\Rightarrow$ (ii). Consider the completion $\widehat{Y}$ of Y and

$$\hat{A} = \left\{ y \in \widehat{Y} : \text{there exists } Y \supset (y_n) \geq \mathcal{B}, y_n \rightharpoondown \hat{y} \right\}.$$

We stress that $\hat{A}$ is a subset of the completion, but sequences (y_n) are taken out of Y. It will be convenient to use the entourages of the diagonal that are closed in the product $\widehat{Y} \times \widehat{Y}$.

We first show that $\hat{A}$ is a totally bounded subset of $\widehat{Y}$. If $\hat{A}$ is not totally bounded, we can find a $\widehat{V}$-discrete infinite sequence (a_n) in $\hat{A}$. Then, we pick $\widehat{U} \in \widehat{\mathcal{V}}$ such that $4\widehat{U} \subset \widehat{V}$. Setting $\widehat{W_n} = \bigcup_{i=n}^{\infty} \hat{U}(a_i)$, $n \in \mathbf{N}$, we define a filter base $(\widehat{W_n})$ which is meshing with $\mathcal{B}$. But by $\mathcal{B}$ being composed of sets contained in Y, this also means that the base (W_n), where $W_n = \widehat{W}_n \cap Y$, is also meshing with $\mathcal{B}$. As $\mathcal{B}$ is countably midrefinable, we find a finer countably based filter and then pick out of it a sequence $(y_n) \subset Y$ such that (y_n) is finer than both $\mathcal{B}$ and (W_n). By passing to a subsequence if needed, and in view of the definition of W_ns, we can make sure that (y_n) is U-discrete. This is a contradiction with the fact that (y_n) must be totally bounded by (i). □

Claim. *Given $\widehat{V} \in \widehat{\mathcal{V}}$, there exists a finite set $\widehat{K} \subset \hat{A}$ and $B \in \mathcal{B}$ such that $B \subset \widehat{V}(\widehat{K})$.*

As $\hat{A}$ is totally bounded, we can find $\widehat{K}$ such that $\hat{A}$ is contained in the interior (relative to $\widehat{Y}$) of $\widehat{V}(\widehat{K})$. There is a $B \in \mathcal{B}$ contained in $\widehat{V}(\widehat{K})$. Indeed, if not, consider the open set $\widehat{O} = \widehat{Y} \setminus \widehat{V}(\widehat{K})$. It meshes with $\mathcal{B}$. Its trace O on Y is still open and still meshes with $\mathcal{B}$ (because $\mathcal{B}$ is contained in Y). By the refinability property of $\mathcal{B}$ there is a sequence (y_n) in O which is finer than $\mathcal{B}$. Thus, by (i), (y_n) is totally bounded and has a cluster point, say η in $\widehat{Y}$. Then η, being in the closure of $\widehat{O}$ relative to $\widehat{Y}$, is disjoint with $\widehat{A}$. But it is also in $\widehat{A}$ by the very definition of $\widehat{A}$, a contradiction.

Now, as $B \subset \widehat{V}(\widehat{K})$ and Y is dense in $\widehat{Y}$, we can find a finite set $K \subset Y$ so close to $\widehat{K}$ that $B \subset 2\widehat{V}(K)$. Since $B \subset Y$, this means that $B \subset 2V(K)$. This shows (ii). □

The following corollary generalizes [**13**, Theorem 3.7].

5.2. Corollary. *Let $\mathcal{B}$ be a countably midrefinable and nearly countably compact filter base in a Dieudonné complete space Y. Then $\mathcal{B}$ is compact* (*at* $\operatorname{adh}\mathcal{B}$). *In particular, $\mathcal{B}$ aims at its adherence which is compact.*

Proof. We may assume that Y is a complete uniform space. If $(y_n) \geq \mathcal{B}$, then it must be relatively countably compact and so, in particular, condition (i) of the previous theorem is satisfied. Hence $\mathcal{B}$ is totally bounded and, as Y possesses a base of closed entourages of the diagonal, $\overline{\mathcal{B}}$ is also totally bounded. Hence, any ultrafilter, say $\mathcal{U}$, finer than $\overline{\mathcal{B}}$ must be totally bounded. It follows that $\mathcal{U}$ is Cauchy (this is exactly the crux of the already-mentioned argument of Bourbaki [**2**]) and therefore convergent. This shows that $\overline{\mathcal{B}}$ is compact at its intersection $A = \operatorname{adh}\mathcal{B}$.

In particular, $\mathcal{B} \rightsquigarrow A$. Now, take a filter $\mathcal{F}$ meshing with A. Then $\mathcal{F}$ meshes with $\overline{\mathcal{B}}$, and therefore has a cluster point in A. Hence, A is compact. $\square$

Theorem 5.1 is close in spirit to Theorem 3.4. We want to combine it together with the method of the proof we already used for Theorem 2.6. We need just one more 'transfer lemma'.

5.3. Lemma. *Let $\mathcal{R}$ be a countably midrefinable filter on X. Then $E(\mathcal{R})$ is countably midrefinable.*

Proof. Let $\mathcal{G} \in \mathcal{O}(\mathbf{F}_{\aleph_1})$ be meshing with $E(\mathcal{R})$. Let $G_1 \supset G_2 \cdots$ be an open base of $\mathcal{G}$. Then $F^{\urcorner}(G_n) = \{H \subset X : F(H) \subset G_n\}$ is a sequence of open decreasing sets and it is clear that the filter they generate meshes with $\mathcal{R}$. By the refinability assumption on $\mathcal{R}$, we can find a countably based filter $\mathcal{H}$ finer than both $(F^{\urcorner}(G_n))$ and $\mathcal{R}$. Then $E(\mathcal{H})$ is as needed. $\square$

We say that x_0 is an *r-point* if its neighborhood filter $\mathcal{N}(x_0)$ contains a filter $\mathcal{R}$ which is countably midrefinable and nearly countably compact.

5.4. Theorem. *Let X and Y be regular spaces. Suppose x_0 is an r-point in X, Y is topologically complete and F is* uc. *Then $E(\mathcal{R})$, and therefore $E(\mathcal{N})$, is compact.*

Proof. By Lemma 5.3, if $\mathcal{R}$ is countably midrefinable, then so is $E(\mathcal{R})$. By Theorem 3.5, $E(\mathcal{R})$ is nearly countably compact. Apply Corollary 5.2. □

Remarks 1) In a regular space, a filter $\mathcal{F}$ is compact if and only if so is its closure filter $\overline{\mathcal{F}}$ if and only if $\mathcal{F}$ aims at its adherence which is compact. Moreover, its adherence is the smallest closed set at which the filter aims, see [**27, 35**]. For this reason, one may treat the compactness of $E(\mathcal{R})$, and therefore $E(\mathcal{N})$, as an ultimate property revealing the nature of upper continuity of set-valued maps. We gave Corollary 5.2 with a proof, because its proof is simpler than the just quoted general fact.

2) As usual, one may ask what are spaces, call them *VCD-spaces*, in which countably midrefinable nearly countably compact filters are compact? As usual, it is easy to see that VCD-spaces are productive and stable with respect to closed subspaces. The following two questions do not seem quite trivial. Are the spaces $C_p(K)$ (continuous functions on a compact K, with pointwise convergence) VCD-spaces? Are metacompact spaces in the class?

The main technical tool to achieve our goal is not the active boundary of F at x_0, $\operatorname{Frac} F(x_0)$, but rather the cluster set $\operatorname{adh}_\omega E(\mathcal{R})$. In the special case of a q-point, i.e., when $\mathcal{R}$ is countably based, a series of conditions on the space Y are considered in [**26**] under which the set $E(\mathcal{R})$ admits a sequential representation. This allows, among other things, to determine whether the cluster sets considered are Choquet kernels in the absence of G_δ-closedeness. Here is a result of similar flavor for a general r-point. Recall that a Hausdorff space Y is called *σ-angelic* if it is angelic with respect to countable subsets. That is, if C is a relatively countably compact subset of Y, then C is also relatively sequentially compact. Moreover, if η is a cluster point of a sequence $(y_n) \subset C$, then there exists a subsequence (y_{n_k}) of (y_n) which converges to η.

5.5. Proposition. *Let x_0 be an r-point in a regular space X and $F \rightrightarrows Y$ a* uc *map. If Y is σ-angelic, then* $\mathrm{adh}_\omega E(\mathcal{R}) = \mathrm{adh}_\sigma E(\mathcal{R}) \subset F(\mathrm{adh}_\omega \mathcal{R})$.

Proof. In the proof of the transfer Lemma 3.5 we have found sequences $x_k \rightharpoondown \xi$ and $y_{n_k} \rightharpoondown \eta$ such that $(x_k) \geq \mathcal{R}$ and $y_{n_k} \in F(x_k) \setminus F(\xi)$. Applying the fact that (y_{n_k}) is relatively countably compact and the fact that Y is σ-angelic, we may assume that $y_{n_k} \to \eta$. Then, by the basic lemma, $\eta \in F(\xi)$. □

Finally, the condition of $\mathcal{G}_\delta$-closedness (introduced in [**22**] for that very purpose) can be used in order to assure that the active boundary is a subset of $F(x_0)$.

5.6. Theorem. *Let x_0 be an r-point in a regular space X, Y regular, and $F \rightrightarrows Y$ a* uc *map. If the value $F(x_0)$ is closed in the G_δ-topology of Y, then the active $\mathcal{R}$-boundary is a Choquet $\mathcal{R}$-kernel of F at x_0. In particular,* $\mathrm{Frac}\, F(x_0)$ *is a Choquet kernel of F at x_0.*

Proof. As the filter $E(\mathcal{R})$ is nearly countably compact and also countably midrefinable, it must be countably midcompact. Although Corollary 2.5 is stated in [**26**] in the countably compact case, its proof shows that it applies in the midcompactoid case as well, showing that $E(\mathcal{R}) \subset F(x_0)$. The theorem follows. □

ENDNOTES

1. That is, the topology generated by declaring the G_δ-sets open.
2. That is, uc everywhere (or at least in a neighborhood of x_0).
3. In the quoted paper the term 'compactoid filter' instead of 'compact filter' is used.

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