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SECOND-ORDER STURM-LIOUVILLE BOUNDARY VALUE PROBLEM INVOLVING THE ONE-DIMENSIONAL p-LAPLACIAN

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ABSTRACT. In this paper, we prove the existence of at least three solutions for the Sturm-Liouville boundary value problem depending upon the parameter λ. Our main tool is a three critical points theorem given by Averna and Bonanno [**3**].

1. Introduction. In recent years, a great deal of work has been done in the study of the existence of multiple solutions of two-point boundary value problems, by which a number of physical and biological phenomena are described. For the background and results, we refer the reader to the monograph by Agarwal et al. and some recent contributions such as [**2, 6**–**9**].

Various fixed point theorems are applied to get interesting results, see for example, [**6**–**9**] and the references therein. Among them, Krasnosel'skii fixed point theorem, Leggett-Williams fixed point theorem, a five functionals fixed point theorem and fixed point theorems in cones are very frequently used.

In recent years, a three critical point theorem given by Ricceri [**10**] is also widely used and has been generalized by Averna and Bonanno [**3**]. Using the variational principle and the mountain pass theorem, Averna and Bonanno gave a definite interval, say $]1/\varphi_2(r), 1/\varphi_1(r)[$, in which λ lies, then $\Phi + \lambda\Psi$ has at least three critical points. Their result is as follows.

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Theorem 1.1 [**3**]. *Let X be a reflexive real Banach space, let $\Phi : X \to R$ be a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|x\|\to+\infty}(\Phi(x)+\lambda\Psi(x)) = +\infty$ *for all* $\lambda \in [0,+\infty[$;

(ii) *there is an $r \in R$ such that*

$$\inf_X \Phi < r$$

and

$$\varphi_1(r) < \varphi_2(r),$$

where

$$\varphi_1(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi}{r-\Phi(x)},$$

$$\varphi_2(r) := \inf_{x\in\Phi^{-1}(]-\infty,r[)} \sup_{y\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(x)-\Psi(y)}{\Phi(y)-\Phi(x)},$$

and $\overline{\Phi^{-1}(]-\infty,r[)}^w$ is the closure of $\Phi^{-1}(]-\infty,r[)$ in the weak topology.

Then, for each

$$\lambda \in \left]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right[,$$

the functional $\Phi+\lambda\Psi$ has at least three critical points in X.

This theorem has been applied to the Dirichlet and mixed boundary value problem, see [**3, 4, 5**]. The aim of this paper is to obtain further applications of Theorem 1.1 to the second order Sturm-Liouville boundary value problem

$$(1.1) \qquad \begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = \lambda f(t,x(t)) & t \in [a,b], \\ \alpha x'(a) - \beta x(a) = A, \quad \gamma x'(b) + \sigma x(b) = B, \end{cases}$$

where $p > 1$, $\Phi_p(x) := |x|^{p-2}x$, $\rho, s \in L^\infty[a,b]$ with $\operatorname{ess\,inf}_{[a,b]}\rho > 0$ and $\operatorname{ess\,inf}_{[a,b]} s > 0$, $\lambda \in]0,+\infty[$, A and B are constants, α, β, γ, $\sigma > 0$, f is an L^1-Carathéodory function and λ is a positive parameter.

With the p-Laplacian and the new type of boundary condition taken into consideration, difficulties, such as how to construct a suitable functional Φ and how to prove the equivalence between the critical points of $\Phi + \lambda\Psi$ and the solutions of BVP (1.1), have to be overcome. Under suitable hypotheses, we prove that the problem (1.1) has at least three solutions when λ lies in an explicitly determined open interval.

This paper is organized as follows. In Section 2, the variational approach is justified and the regularity of an appropriate functional involved is proved. In Section 3, existence results are given in Theorem 3.1 and Corollary 3.2. At the same time, we give a particular case (Theorem 3.3) of Theorem 3.1.

2. Preliminaries. To begin with, we introduce some notations. Here, and in the sequel, we assume that, $[a, b]$ is a compact real interval, X is the Sobolev space $W^{1,p}([a, b])$ equipped with the norm

$$\|x\| = \left(\int_a^b \rho(t)|x'(t)|^p + s(t)|x(t)|^p \, dt \right)^{1/p},$$

which is clearly equivalent to the usual one; F is the real function

$$F(t, \xi) = \int_0^\xi f(t, x) \, dx.$$

We denote $\|x\|_\infty := \sup_{x\in[a,b]} |x(t)|$ to be the norm in $C^0([a, b])$. Moreover, $\|x\|_1$ and $\|x\|_{L^\infty}$ stand for the norm in $L^1([a, b])$ and $L^\infty([a, b])$, respectively.

We say that x is a solution of BVP (1.1) if $x \in Z = \{x \in X : \rho\Phi_p(x')(\cdot) \in W^{1,\infty}([a, b])\}$ satisfies the boundary condition in BVP (1.1) and

$$-(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = \lambda f(t, x(t))$$

for almost every $t \in [a, b]$.

For each $x \in X$, put

(2.1)

$$\Phi(x) := \frac{\|x\|^p}{p} + \frac{\gamma\rho(b)}{\sigma p} \left| \frac{B - \sigma x(b)}{\gamma} \right|^p + \frac{\alpha\rho(a)}{\beta p} \left| \frac{A + \beta x(a)}{\alpha} \right|^p,$$

(2.2)

$$\Psi(x) := -\int_a^b F(t, x(t)) \, dt.$$

Clearly, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in X$ is the functional $\Phi'(x) \in X^*$, given by

(2.3)
$$\Phi'(x)(v) = \int_a^b (\rho(t)\Phi_p(x'(t))v'(t) + s(t)\Phi_p(x(t))v(t))\, dt \\ - \rho(b)\Phi_p\left(\frac{B - \sigma x(b)}{\gamma}\right)v(b) + \rho(a)\Phi_p\left(\frac{A + \beta x(a)}{\alpha}\right)v(a)$$

for every $v \in X$, and $\Phi' : X \to X^*$ is continuous. Moreover, taking into account that Φ is convex, from [**12**, Proposition 25.20 (i)], we obtain that Φ is a sequentially weakly lower semi-continuous functional.

It is easy to see that $\Psi : X \to R$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in X$ is the functional $\Psi'(x) \in X^*$ given by

$$\Psi'(x)(v) = -\int_a^b f(t, x(t))v(t)\, dt \tag{2.4}$$

for every $v \in X$.

Proposition 2.1. *$x \in X$ is a critical point of $\Phi + \lambda\Psi$ if and only if x is a solution of* BVP (1.1).

Proof. Let $x \in X$ be a critical point of the functional $\Phi + \lambda\Psi$; then, for any $v \in X$,

$$\langle (\Phi + \lambda\Psi)'(x), v\rangle = 0,$$

that is,

$$\int_a^b (\rho(t)\Phi_p(x'(t))v'(t) + s(t)\Phi_p(x(t))v(t))\, dt \\ - \rho(b)\Phi_p\left(\frac{B - \sigma x(b)}{\gamma}\right)v(b) + \rho(a)\Phi_p\left(\frac{A + \beta x(a)}{\alpha}\right)v(a) \\ - \lambda\int_a^b f(t, x(t))v(t)\, dt = 0.$$

Simple calculations show that

$$\begin{aligned}(2.5)\quad \int_a^b &[-(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) - \lambda f(t,x(t))]\, v(t)\, dt \\ &+ \rho(b)v(b)\left[\Phi_p(x'(b)) - \Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)\right] \\ &+ \rho(a)v(a)\left[\Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right) - \Phi_p(x'(a))\right] = 0\end{aligned}$$

for all $v \in X$ and hence for all $v \in C_0^\infty([a,b])$. Thus, by the fundamental lemma of variational method, x satisfies the equation in BVP (1.1) for almost every $t \in [a,b]$. Then (2.5) becomes

$$\begin{aligned}\rho(b)v(b)&\left[\Phi_p(x'(b)) - \Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)\right] \\ &+ \rho(a)v(a)\left[\Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right) - \Phi_p(x'(a))\right] = 0\end{aligned}$$

for all $v \in X$. We will show that x satisfies the boundary condition in BVP (1.1). If not, without loss of generality, we assume

$$\gamma x'(b) + \sigma x(b) > B,$$

which means that

$$\Phi_p(x'(b)) - \Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right) > 0.$$

Let $v(t) = t - a \in C^\infty([a,b]) \subset X$. Then

$$\begin{aligned}\rho(b)v(b)&\left[\Phi_p(x'(b)) - \Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)\right] \\ &+ \rho(a)v(a)\left[\Phi_p\left(\frac{A+\beta x(a)}{\alpha}\right) - \Phi_p(x'(a))\right] \\ &= \rho(b)(b-a)\left[\Phi_p(x'(b)) - \Phi_p\left(\frac{B-\sigma x(b)}{\gamma}\right)\right] > 0,\end{aligned}$$

a contradiction. So x is a solution of BVP (1.1).

If x is a solution of BVP (1.1), for any $v \in X$, multiplying $v(t)$ on the both sides of the equation in BVP (1.1), then integrating on $[a,b]$, in view of the boundary condition, it is easy to see that x satisfies $\langle (\Phi + \lambda\Psi)'(x),\, v\rangle = 0$. □

We need the following lemmas in the proof of Theorem 3.1.

Lemma 2.2. *If $x \in W^{1,p}([a,b])$ and there exists $r > 0$ such that $\Phi(x) < r$, then*

$$\|x\|_\infty \le \frac{|A|}{\beta} + \sqrt[p]{pr}\left[\left(\frac{\alpha}{\beta}\right)^{1/q}(\rho(a))^{-1/p} + (b-a)^{1/q}\left(\operatorname{ess}\inf_{[a,b]}\rho\right)^{-1/p}\right] := \Theta(r).$$

Proof. If $\Phi(x) < r$, then

$$\frac{\|x\|^p}{p} < r, \tag{2.6}$$

$$\frac{\alpha\rho(a)}{\beta p}\left|\frac{A+\beta x(a)}{\alpha}\right|^p < r \tag{2.7}$$

hold. By (2.6) and the mean value theorem we have

$$\begin{aligned} |x(t)| &\le |x(a)| + \int_a^b |x'(s)|\,ds \le |x(a)| + (b-a)^{1/q}\|x'\|_{L^p} \\ &\le |x(a)| + (b-a)^{1/q}\left(\operatorname{ess}\inf_{[a,b]}\rho\right)^{-1/p}\|x\| \\ &\le |x(a)| + (b-a)^{1/q}\left(\operatorname{ess}\inf_{[a,b]}\rho\right)^{-1/p}\sqrt[p]{pr}. \end{aligned} \tag{2.8}$$

By (2.7), we have

$$|x(a)| \le \frac{|A+\beta x(a)| + |A|}{\beta} \le \frac{\sqrt[p]{(pr\beta\alpha^{p-1})/\rho(a)} + |A|}{\beta}. \tag{2.9}$$

From (2.8) and (2.9), the result follows. □

Lemma 2.3. *$\Phi' : X \to X^*$ admits a continuous inverse on X^*.*

Proof. Firstly, for every $x \in X \setminus \{0\}$, it follows from (2.3) that

$$\lim_{\|x\|\to+\infty} \frac{\langle \Phi'(x), x\rangle}{\|x\|} = \lim_{\|x\|\to+\infty} \times \frac{\|x\|^p - \rho(b)\Phi_p\left(B - \sigma x(b)/\gamma\right)x(b) + \rho(a)\Phi_p\left(A + \beta x(a)/\alpha\right)x(a)}{\|x\|},$$

if $|x(a)|, |x(b)| < +\infty$; then

$$\lim_{\|x\|\to+\infty} \frac{-\rho(b)\Phi_p\left(B - \sigma x(b)/\gamma\right)x(b)}{\|x\|} = \lim_{\|x\|\to+\infty} \frac{\rho(b)\Phi_p\left(A + \beta x(a)/\alpha\right)x(a)}{\|x\|} = 0;$$

if $|x(a)|, |x(b)| \to +\infty$, then

$$-\rho(b)\Phi_p\left(\frac{B - \sigma x(b)}{\gamma}\right)x(b) > 0, \quad \rho(a)\Phi_p\left(\frac{A + \beta x(a)}{\alpha}\right)x(a) > 0.$$

So $\lim_{\|x\|\to+\infty} \langle \Phi'(x), x\rangle / \|x\| = +\infty$, that is, Φ' is coercive.

Moreover, given $u, v \in X$, it follows from the nondecreasing property of Φ_p that

$$\begin{aligned}
&\langle \Phi'(u) - \Phi'(v), u - v\rangle \\
&= \int_a^b [\rho(t)(\Phi_p(u'(t)) - \Phi_p(v'(t)))(u'(t) - v'(t)) \\
&\qquad\qquad + s(t)(\Phi_p(u(t)) - \Phi_p(v(t)))(u(t) - v(t))] \\
&\quad - \rho(b)\left(\Phi_p\left(\frac{B - \sigma u(b)}{\gamma}\right) - \Phi_p\left(\frac{B - \sigma v(b)}{\gamma}\right)\right)(u(b) - v(b)) \\
&\quad + \rho(a)\left(\Phi_p\left(\frac{A + \beta u(a)}{\alpha}\right) - \Phi_p\left(\frac{A + \beta v(a)}{\alpha}\right)\right)(u(a) - v(a)) \\
&\geq \int_a^b [\rho(t)(\Phi_p(u'(t)) - \Phi_p(v'(t)))(u'(t) - v'(t)) \\
&\qquad\qquad + s(t)(\Phi_p(u(t)) - \Phi_p(v(t)))(u(t) - v(t))]\, dt;
\end{aligned}$$

thus, by [**11**, (2.2)], there exist $c_p, d_p > 0$ such that

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle \tag{2.10}$$
$$\geq \begin{cases} c_p \int_a^b [\rho(t)|u'(t) - v'(t)|^p + s(t)|u(t) - v(t)|^p]\, dt, \\ \qquad\qquad \text{if} \quad p \geq 2; \\ d_p \int_a^b [|u'(t) - v'(t)|^2/(|u'(t)| + |v'(t)|)^{2-p} \\ +|u(t) - v(t)|^2/(|u(t)| + |v(t)|)^{2-p}]\, dt, \\ \qquad\qquad \text{if} \quad 1 < p < 2. \end{cases}$$

If $p \geq 2$, then it follows that

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle \geq c_p \|u - v\|^p,$$

so Φ' is uniformly monotone. By [**12**, Theorem 26.A (d)], we have that $(\Phi')^{-1}$ exists and is continuous on X^*.

If $1 < p < 2$, by the Holder inequality, we obtain

(2.11)
$$\begin{aligned}\int_a^b |u(t) - v(t)|^p\, dt &\leq \left(\int_a^b \frac{|u(t) - v(t)|^2}{(|u(t)| + |v(t)|)^{2-p}}\, dt\right)^{p/2} \\ &\quad \times \left(\int_a^b (|u(t)| + |v(t)|)^p\, dt\right)^{(2-p)/2} \\ &\leq \left(\int_a^b \frac{|u(t) - v(t)|^2}{(|u(t)| + |v(t)|)^{2-p}}\, dt\right)^{p/2} \\ &\quad \times 2^{(p-1)(2-p)/2}\left(\int_a^b |u(t)|^p + |v(t)|^p\, dt\right)^{(2-p)/2} \\ &\leq 2^{(p-1)(2-p)/2} M\left(\int_a^b \frac{|u(t) - v(t)|^2}{(|u(t)| + |v(t)|)^{2-p}}\, dt\right)^{p/2} \\ &\quad \times (\|u\| + \|v\|)^{((2-p)p)/2}.\end{aligned}$$

Similarly,

$$\int_a^b |u'(t) - v'(t)|^p\, dt \leq 2^{(p-1)(2-p)/2} M \tag{2.12}$$
$$\left(\int_a^b \frac{|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\, dt\right)^{p/2} (\|u\| + \|v\|)^{(2-p)p2}.$$

Thus from (2.10), (2.11) and (2.12), we have

$$
\begin{aligned}
(2.13)\quad & \langle \Phi'(u) - \Phi'(v), u - v \rangle \\
& \geq d_p \frac{\left(\int_a^b |u(t) - v(t)|^p \, dt\right)^{2/p} + \left(\int_a^b |u'(t) - v'(t)|^p \, dt\right)^{2/p}}{2^{(p-1)(2-p)/p} M^{2/p} (\|u\| + \|v\|)^{2-p}} \\
& \geq \frac{M' \left(\int_a^b |u(t) - v(t)|^p + |u'(t) - v'(t)|^p \, dt\right)^{2/p}}{(\|u\| + \|v\|)^{2-p}} \\
& \geq M'' \frac{\|u - v\|^2}{(\|u\| + \|v\|)^{2-p}};
\end{aligned}
$$

therefore, Φ' is strictly monotone. By [**12**, Theorem 26.A (d)] we obtain that $(\Phi')^{-1}$ exists and is bounded. Furthermore, given $g_1, g_2 \in X^*$, by (2.13) we have

$$
\begin{aligned}
\|(\Phi')^{-1}(g_1) - (\Phi')^{-1}(g_2)\| \leq \frac{1}{M''} \big(& \|(\Phi')^{-1}(g_1)\| \\
& + \|(\Phi')^{-1}(g_2)\|\big)^{2-p} \|g_1 - g_2\|_{X^*},
\end{aligned}
$$

so $(\Phi')^{-1}$ is Lipschitz continuous for $1 < p < 2$. Thus, we have showed $\Phi' : X \to X^*$ admits a continuous inverse on X^*. The proof is complete. $\square$

Lemma 2.4. *$\Psi' : X \to X^*$ is a continuous and compact operator.*

Proof. First we will show that Ψ' is strongly continuous on X. For this, let $u_n \rightharpoonup u$ as $n \to \infty$ on X; by [**12**] we have u_n converges uniformly to u on $[a, b]$ as $n \to \infty$. Since f is a L^1-Carathédory function, one has $f(t, u_n) \to f(t, u)$ as $n \to \infty$. So $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Thus we have showed that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by [**12**, Proposition 26.2]. Moreover, Ψ' is continuous since it is strongly continuous. The proof is complete. $\square$

3. Main results. In the following theorem, we will use the following notation:

$$\begin{aligned}
L(k) &:= \int_a^b \rho(t)dt \left[\left|\frac{A+\beta k}{\alpha - a\beta}\right|^p + \left|\frac{B-\sigma k}{\gamma + b\sigma}\right|^p \right.\\
&\quad \left. + \left|\frac{(2b+a)(B-\sigma k)}{(b-a)(\gamma+b\sigma)} - \frac{(2a+b)(A+\beta k)}{(b-a)(\alpha-a\beta)}\right|^p\right],\\
Q(k) &:= \int_a^b s(t)\,dt \left\{\max\left\{\left|\frac{A+\beta k}{\alpha-a\beta}a + k\right|^p, \left|\frac{A+\beta k}{\alpha - a\beta}\times\frac{2a+b}{3} + k\right|^p\right\}\right.\\
&\quad + \max\left\{\left|\frac{B-\sigma k}{\gamma+b\sigma}b + k\right|^p, \left|\frac{B-\sigma k}{\gamma+b\sigma}\times\frac{2b+a}{3}+k\right|^p\right\}\\
&\quad \left. + \max\left\{\left|\frac{B-\sigma k}{\gamma+b\sigma}\times\frac{2b+a}{3}+k\right|^p, \left|\frac{A+\beta k}{\alpha-a\beta}\times\frac{2a+b}{3}+k\right|^p\right\}\right\},\\
R(k) &:= \frac{\gamma\rho(b)}{\sigma}\left|\frac{B-\sigma k}{\gamma}\right|^p + \frac{\alpha\rho(a)}{\beta}\left|\frac{A+\beta k}{\alpha}\right|^p,\\
y_1(t) &= \frac{A+\beta k}{\alpha - a\beta}t + k,\\
y_2(t) &= \frac{1}{b-a}\left[\frac{(2b+a)(B-\sigma k)}{\gamma+b\sigma} - \frac{(2a+b)(A+\beta k)}{\alpha-a\beta}\right]t\\
&\quad - \frac{(2a+b)(2b+a)}{3(b-a)}\left[\frac{B-\sigma k}{\gamma+b\sigma} - \frac{A+\beta k}{\alpha - a\beta}\right] + k,\\
y_3(t) &= \frac{B-\sigma k}{\gamma+b\sigma}t + k,\\
\Gamma &:= \int_a^{(2a+b)/3} F(t, y_1(t))\,dt + \int_{(2a+b)/3}^{(2b+a)/3} F(t, y_2(t))\,dt\\
&\quad + \int_{(2b+a)/3}^b F(t, y_3(t))\,dt.
\end{aligned}$$

Theorem 3.1. *Assume that there exist three positive constants* $k, d, l > 0$ *with* $l < p$,

$$L(k) > pd > \frac{\rho(b)B^p}{\sigma\gamma^{p-1}} + \frac{\rho(a)A^p}{\beta\alpha^{p-1}},$$

and a positive function $\mu \in L^1([a,b])$ *such that:*

(H1)

$$\frac{\max_{|\xi|\le\Theta(d)}\int_a^b F(t,\xi)\,dt}{d-(\rho(b)B^p/\sigma p\gamma^{p-1})-(\rho(a)A^p/\beta p\alpha^{p-1})} < \frac{p}{L(k)+Q(k)+R(k)}\left[\Gamma-\max_{|\xi|\le\Theta(d)}\int_a^b F(t,\xi)\,dt\right];$$

(H2) $F(t,\xi)\le\mu(t)(1+|\xi|^l)$ *for almost every* $t\in[a,b]$ *and all* $\xi\in R$.

Then for each $\lambda\in\,]\lambda_1,\lambda_2[$, *the problem* (1.1) *has at least three solutions, where*

$$\lambda_1=\frac{L(k)+Q(k)+(\gamma\rho(b))/\sigma\,|(B-\sigma y(b))/\gamma|^p}{p\Gamma-p\max_{|\xi|\le\Theta(d)}\int_a^b F(t,\xi)\,dt} + \frac{(\alpha\rho(a))/\beta\,|(A+\beta y(a))/\alpha|^p}{p\Gamma-p\max_{|\xi|\le\Theta(d)}\int_a^b F(t,\xi)\,dt}$$

and

$$\lambda_2=\frac{d-[(\rho(b)B^p)/(\sigma p\gamma^{p-1})]-[(\rho(a)A^p)/(\beta p\alpha^{p-1})]}{\max_{|\xi|\le\Theta(d)}\int_a^b F(t,\xi)\,dt}.$$

Proof. From the previous section we have seen that $\Phi:X\to R$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative at the point $u\in X$ is the functional $\Phi'(u)\in X^*$, given by (2.3). $(\Phi')^{-1}$ exists and is continuous on X^*. $\Psi:X\to R$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative given by (2.4) is compact.

By Proposition 2.1, the solutions to Problem (1.1) are exactly the critical points of the functional $\Phi+\lambda\Psi$, so our aim is to apply Theorem 1.1 to Φ and Ψ.

For (i) in Theorem 1.1, by condition (H2) and (2.1) we have

$$\begin{aligned}\lim_{\|x\|\to+\infty}(\Phi(x)+\lambda\Psi(x)) &\geq \lim_{\|x\|\to+\infty}\Bigg(\frac{\|x\|^p}{p}+\frac{\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma x(b)}{\gamma}\right|^p \\ &\qquad -\lambda\int_a^b \mu(t)(1+|x(t)|^l)\,dt\Bigg) \\ &\geq \lim_{\|x\|\to+\infty}\Bigg(\frac{\|x\|^p}{p}+\frac{\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma x(b)}{\gamma}\right|^p \\ &\qquad -\lambda\|\mu\|_1\left\{1+\left[|x(b)|+\|x'\|_{L^p}(b-a)^{1/q}\right]^l\right\}\Bigg) \\ &= +\infty.\end{aligned}$$

So (i) is satisfied.

To prove (ii) in Theorem 1.1, first we claim that

$$\varphi_1(r) \leq \frac{\max_{|\xi|\leq\Theta(r)}\int_a^b F(t,\xi)\,dt}{[r-(\rho(b)B^p)/(\sigma p\gamma^{p-1})]-[(\rho(a)A^p)/(\beta p\alpha^{p-1})]} \tag{A1}$$

for each $r>0$ and

(A2) $\varphi_2(r)$

$$\geq p\frac{\int_a^b F(t,y(t))\,dt-\max_{|\xi|\leq\Theta(r)}\int_a^b F(t,\xi)\,dt}{\|y\|^p+(\gamma\rho(b))/\sigma\,|(B-\sigma y(b))/\gamma|^p+(\alpha\rho(a))/\beta\,|(A+\beta y(a))/\alpha|^p}$$

for each $r>0$ and every $y\in X$ such that

$$\Phi(y)\geq r \quad\text{and}\quad \int_a^b F(t,y(t))\,dt \geq \max_{|\xi|\leq\Theta(r)}\int_a^b F(t,\xi)\,dt. \tag{3.1}$$

In fact, for $r>0$, taking into account that $\overline{\Phi^{-1}(]-\infty,r[)}^{\omega} = \Phi^{-1}(]-\infty,r])$ and $x\equiv 0$ on $[a,b]$ obviously belongs to $\Phi^{-1}(]-\infty,r[)$ and that $\Psi(0)=0$, we have

$$\varphi_1(r)\leq\frac{\sup_{x\in\Phi^{-1}(]-\infty,r])}\int_a^b F(t,x(t))\,dt}{r-[(\rho(b)B^p)/(\sigma p\gamma^{p-1})]-[(\rho(a)A^p)/(\beta p\alpha^{p-1})]}.$$

Thus, since $x \in \Phi^{-1}(]-\infty, r])$, that is, $\Phi(x) \le r$, by Lemma 2.2, we have

$$\|x\|_\infty \le \Theta(r). \tag{3.2}$$

As a consequence,

$$\frac{\sup_{x\in\Phi^{-1}(]-\infty,r])}\int_a^b F(t,x(t))\,dt}{r-[(\rho(b)B^p)/(\sigma p\gamma^{p-1})]-[(\rho(a)A^p)/(\beta p\alpha^{p-1})]} \le \frac{[(\max_{|\xi|\le\Theta(r)}\int_a^b F(t,\xi)\,dt)}{(r-(\rho(b)B^p)/(\sigma p\gamma^{p-1})-[(\rho(a)A^p)/(\beta p\alpha^{p-1})])}.$$

So (A1) is proved.

Moreover, for each $r > 0$ and each $y \in X$ such that $\Phi(y) \ge r$, we have

$$\begin{aligned}\varphi_2(r) &\ge \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(x)-\Psi(y)}{\Phi(y)-\Phi(x)}\\ &= \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\int_a^b F(t,y(t))\,dt-\int_a^b F(t,x(t))\,dt}{\Phi(y)-\Phi(x)}.\end{aligned}$$

Since (3.2) holds for $x \in \Phi^{-1}(]-\infty, r])$, we obtain

$$\varphi_2(r) \ge \inf_{x\in\Phi^{-1}(]-\infty,r[)} \frac{\int_a^b F(t,y(t))\,dt-\max_{|\xi|\le\Theta(r)}\int_a^b F(t,\xi)\,dt}{\Phi(y)-\Phi(x)}$$

and, under further condition (3.1), we can write

$$\begin{aligned}\varphi_2(r) \ge{}& \frac{\int_a^b F(t,y(t))\,dt-\max_{|\xi|\le\Theta(r)}\int_a^b F(t,\xi)\,dt}{1/p\|y\|^p+(\gamma\rho(b))/(\sigma p)\,|(B-\sigma y(b))/\gamma|^p}\\ &+\frac{\int_a^b F(t,y(t))\,dt-\max_{|\xi|\le\Theta(r)}\int_a^b F(t,\xi)\,dt}{(\alpha\rho(a))/\beta p\,|(A+\beta y(a))/\alpha|^p}.\end{aligned}$$

So (A2) is proved.

Now, in order to prove (ii) in Theorem 1.1, taking into account (A1) and (A2), it suffices to find $r > 0$, $y \in X$ such that (3.1) and

$$\begin{aligned}&\frac{\max_{|\xi|\le\Theta(r)}\int_a^b F(t,\xi)\,dt}{r-(\rho(b)B^p/\sigma p\gamma^{p-1})-(\rho(a)A^p/\beta p\alpha^{p-1})}\\ &< p\frac{\int_a^b F(t,y(t))\,dt-\max_{|\xi|\le\Theta(r)}\int_a^b F(t,\xi)\,dt}{\|y\|^p+(\gamma\rho(b))/\sigma\,|(B-\sigma y(b))/\gamma|^p+(\alpha\rho(a))/\beta\,|(A+\beta y(a))/\alpha|^p}\end{aligned} \tag{3.3}$$

hold. To this end, we define

$$y(t) = \begin{cases} y_1(t) & t \in [a, (2a+b)/3]; \\ y_2(t) & t \in [(2a+b)/3, (2b+a)/3]; \\ y_3(t) & t \in [(2b+a)/3, b], \end{cases}$$

and $r := d$. Clearly $y \in X$, and

$$\begin{aligned} L(k) &= \int_a^b \rho(t)|y'(t)|^p \, dt < \|y\|^p \\ &< \int_a^b [\rho(t)|y'(t)|^p + r(t)\|y\|_\infty^p] \, dt \\ &= L(k) + Q(k). \end{aligned} \tag{3.4}$$

From (3.4) and $L(k) > pd$, we have $\Phi(y) \geq \|y\|^p/p > L(k)/p > d$. From condition (H1) and (3.4), it follows that (3.3) holds, which means (3.1) holds, too. □

For $\rho \equiv 1$, $s \equiv 1$, $A = B = 0$, $a = 0$, $b = 1$, we have the following corollary.

Corollary 3.2. *Assume that $g : R \to R$ is a positive continuous function and put $G(\xi) = \int_0^\xi g(s)\, ds$. Besides assume that there exist four positive constants $k, d, l, \mu > 0$ with $l < p$ such that $G(\xi) \geq 0$ for*

$$\xi \in \left[\frac{\gamma k}{\gamma + \sigma}, \frac{(3\gamma+\sigma)k}{3(\gamma+\sigma)}\right]$$

and that the following conditions hold:

(L1) $k^p\left[(\beta/\alpha)^p + (\sigma/(\gamma+\sigma))^p + ((2\sigma)/(\gamma+\sigma) + (\beta/\alpha))^p\right] > pd$;

(L2)

$$\begin{aligned} &\left[\frac{1}{d} + \frac{p}{k^p\left(\Lambda + [(\sigma^{p-1})/(\gamma^{p-1})] + [(\beta^{p-1})/(\alpha^{p-1})]\right)}\right] G(\Theta(d)) \\ &< \frac{p}{k^p\left(\Lambda + [(\sigma^{p-1})/(\gamma^{p-1})] + [(\beta^{p-1})/(\alpha^{p-1})]\right)} \left[\int_0^{1/3} G\left(\frac{\beta k t}{\alpha} + k\right) dt \right. \\ &\qquad \left. + \int_{1/3}^{2/3} G\left(-\left(\frac{2\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)kt + \frac{2}{3}\left(\frac{\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)k + k\right) dt\right]; \end{aligned}$$

(L3) $G(\xi) \le \mu(1+|\xi|^l)$ *for almost every* $t \in [0,1]$ *and all* $\xi \in R$. *Then for each* $\lambda \in \,]\lambda_1, \lambda_2[$, *the problem*

$$(3.5) \qquad \begin{cases} -(\Phi_p(x'(t)))' + \Phi_p(x(t)) = \lambda g(x(t)) & t \in [0,1], \\ \alpha x'(0) - \beta x(0) = 0, \ \gamma x'(1) + \sigma x(1) = 0, \end{cases}$$

has at least two nontrivial solutions, where

$$\lambda_1 = \frac{k^p(\Lambda + \sigma^{p-1}/\gamma^{p-1} + \beta^{p-1}/\alpha^{p-1})}{p(\int_0^{1/3} G((\beta kt/\alpha)+k)\,dt + \int_{1/3}^{2/3} G(-((2\sigma/\gamma+\sigma)+(\beta/\alpha))kt}$$
$$+ \frac{k^p(\Lambda + \sigma^{p-1}/\gamma^{p-1} + \beta^{p-1}/\alpha^{p-1})}{2/3(\sigma/(\gamma+\sigma)+(\beta/\alpha))k+k)\,dt - G(\Theta(d)))}$$

and

$$\lambda_2 = \frac{d}{G(\Theta(d))},$$
$$\Lambda = \left(\frac{\beta}{\alpha}\right)^p + \left(\frac{\sigma}{\gamma+\sigma}\right)^p + \left(\frac{2\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)^p + \left(\frac{\beta}{3\alpha}+1\right)^p + \left(\frac{3\gamma+\sigma}{3(\gamma+\sigma)}\right)^p + \max\left\{\left(\frac{3\gamma+\sigma}{3(\gamma+\sigma)}\right)^p, \left(\frac{\beta}{3\alpha}+1\right)^p\right\}.$$

Proof. Since $\rho \equiv 1$, $s \equiv 1$, $A = B = 0$, $a = 0$, $b = 1$, then

$$L(k) = \left(\frac{\beta}{\alpha}\right)^p + \left(\frac{\sigma}{\gamma+\sigma}\right)^p + \left(\frac{2\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)^p,$$
$$Q(k) = \left(\frac{\beta}{3\alpha}+1\right)^p + \left(\frac{3\gamma+\sigma}{3(\gamma+\sigma)}\right)^p + \max\left\{\left(\frac{3\gamma+\sigma}{3(\gamma+\sigma)}\right)^p, \left(\frac{\beta}{3\alpha}+1\right)^p\right\},$$
$$R(k) = \frac{\sigma^{p-1}}{\gamma^{p-1}}k^p + \frac{\beta^{p-1}}{\alpha^{p-1}}k^p,$$
$$y_1(t) = \frac{\beta k}{\alpha}t + k,$$

$$y_2(t) = -\left(\frac{2\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)kt + \frac{2}{3}\left(\frac{\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)k + k,$$

$$y_3(t) = -\frac{\sigma k}{\gamma+\sigma}t + k,$$

$$\begin{aligned}\Gamma = &\int_0^{1/3} G\left(\frac{\beta kt}{\alpha} + k\right) dt \\ &+ \int_{1/3}^{2/3} G\left(-\left(\frac{2\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)kt + \frac{2}{3}\left(\frac{\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)k + k\right) dt \\ &+ \int_{2/3}^{1} G\left(-\frac{\sigma kt}{\gamma+\sigma} + k\right) dt.\end{aligned}$$

So $\Lambda = L(k)+Q(k)$. Moreover, since $g : R \to R$ is a positive continuous function, G is nondecreasing on R, which means $\max_{|\xi|\leq\Theta(d)} G(\xi) = G(\Theta(d))$.

Clearly (L1) means $L(k) > pd$. Now we show that (L2) means (H1) in Theorem 3.1.

In fact, (L2) is equivalent to

(3.6)

$$\begin{aligned}\frac{1}{d}G(\Theta(d)) < \frac{p}{k^p\left(\Lambda + (\sigma^{p-1}/\gamma^{p-1}) + (\beta^{p-1}/\alpha^{p-1})\right)}\Bigg[&\int_0^{\frac{1}{3}} G\left(\frac{\beta kt}{\alpha} + k\right)dt \\ + \int_{1/3}^{2/3} G\Bigg(-&\left(\frac{2\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)kt \\ &+\frac{2}{3}\left(\frac{\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)k + k\Bigg)\ dt - G(\Theta(d))\Bigg].\end{aligned}$$

Noticing the assumption $G(\xi) \geq 0$ for $\xi \in [(\gamma k)/(\gamma+\sigma), [(3\gamma+\sigma)k/3(\gamma+\sigma)]]$, (3.6) means (H1) in Theorem 3.1.

It is clear that (L3) means (H2) in Theorem 3.1. Now, applying Theorem 3.1, the result holds. □

Example 3.1. The problem

(3.7)

$$\begin{cases} -(\Phi_3(x'))' + \Phi_3(x) = \lambda\left(e^{-xt}(20x^{19} - tx^{20}) + 2x\right) & t \in [0,1], \\ 2x'(0) - x(0) = 0, \quad 3x'(1) + x(1) = 0, \end{cases}$$

admits at least three nontrivial solutions for each $\lambda \in \left]1/(6^{16}, 1.7)\right[$. In fact, the function $F(t,x) = e^{-xt}x^{20} + x^2$ satisfies all assumptions of Theorem 3.1 by choosing, for instance, $d = 9$, $k = 6$.

As a consequence of Corollary 3.2, we obtain the following theorem.

Theorem 3.3. *Assume that $g : R \to R$ is a nonnegative continuous function satisfying*

$$\lim_{x \to 0^+} \frac{g(x)}{x^{p-1}} = 0,$$

and

$$\lim_{x \to +\infty} \frac{g(x)}{x^s} \in R$$

for some $s \in]0, p-1[$. Furthermore, assume there exists $k > 0$ such that $G(\xi) = \int_0^\xi g(s)\,ds > 0$ for

$$\xi \in \left[\frac{\gamma k}{\gamma + \sigma}, \frac{(\sigma + 3\gamma)k}{3(\gamma + \sigma)}\right].$$

Then, the problem (3.5) ($\lambda = 1$) *admits at least two nontrivial solutions.*

Proof. Fix $b > 0$, and put

$$\lambda > \frac{k^p\left(\Lambda + [(\gamma\sigma^{p-1})/((\gamma + b\sigma)^p)] + (\beta^{p-1}/\alpha)\right)}{p(\int_0^{1/3} G((\beta kt/\alpha) + k)\,dt + \int_{1/3}^{2/3} G(-([2\sigma/(\gamma+\sigma)] + (\beta/\alpha))kt} + \frac{k^p(\Lambda + [(\gamma\sigma^{p-1})/((\gamma + b\sigma)^p)] + (\beta^{p-1}/\alpha))}{2/3([\sigma/(\gamma+\sigma)] + (\beta/\alpha))k + k)\,dt)}.$$

Since $\lim_{x\to 0^+}(g(x)/x^{p-1}) = 0$, there exists $0 < d < k$ such that:

$$G(\Theta(d)) < \min\left\{\frac{pd}{k^p\left(\Lambda + [(\gamma\sigma^{p-1})/((\gamma+\sigma)^p)] + (\beta^p/\alpha^{p-1})\right)} \times \left[\int_0^{1/3} G\left(\frac{\beta kt}{\alpha} + k\right)dt + \int_{1/3}^{2/3} G\left(-\left(\frac{2\sigma}{\gamma+\sigma} + \frac{\beta}{\alpha}\right)kt\right.\right.\right.$$

$$+\frac{2}{3}\left(\frac{\sigma}{\gamma+\sigma}+\frac{\beta}{\alpha}\right)k+k\Big)\ dt\Big],\frac{d}{\lambda}\Big\},$$

and

$$\lambda > \frac{k^p\left(\Lambda+[(\gamma\sigma^{p-1})/((\gamma+\sigma)^p)]+(\beta^{p-1}/\alpha)\right)}{p[\int_0^{1/3}G((\beta kt/\alpha)+k)\,dt+\int_{1/3}^{2/3}G(-([(2\sigma)/(\gamma+\sigma)]+(\beta/\alpha))kt} + \frac{k^p\left(\Lambda+[(\gamma\sigma^{p-1})/((\gamma+\sigma)^p)]+(\beta^{p-1}/\alpha)\right)}{2/3([\sigma/(\gamma+\sigma)]+(\beta/\alpha))k+k)\,dt-G(\Theta(d))]}.$$

Since $\lim_{x\to+\infty}[(g(x))/(x^s)] \in R$, (L2) is satisfied. Hence, by Corollary 3.2, for each

$$\lambda > \frac{k^p\left(\Lambda+[(\gamma\sigma^{p-1})/((\gamma+\sigma)^p)]+(\beta^{p-1})/(\alpha)\right)}{p[\int_0^{1/3}G((\beta kt/\alpha)+k)\,dt+\int_{1/3}^{2/3}G(-([(2\sigma)/(\gamma+\sigma)]+(\beta/\alpha))kt} + \frac{k^p(\Lambda+[(\gamma\sigma^{p-1})/((\gamma+\sigma)^p)]+(\beta^{p-1})/(\alpha))}{2/3([\sigma/(\gamma+\sigma)]+(\beta/\alpha))k+k)\,dt]},$$

the problem (3.5) $(\lambda = 1)$ has at least two nontrivial solutions. □

Example 3.2. The problem

$$\begin{cases} -(\Phi_3(x'(t)))' + \Phi_3(x(t)) = g(x(t)), & t\in[0,1], \\ 2x'(0)-x(0)=0, \quad 3x'(1)+x(1)=0, \end{cases} \tag{3.8}$$

where

$$g(x) = \begin{cases} 4x^3 & x\le 1, \\ 2x+2 & x>1 \end{cases}$$

admits at least two nontrivial solutions.

In fact, the function

$$G(x) = \begin{cases} x^4 & x\le 1, \\ 1+x^2 & x>1 \end{cases}$$

satisfies all the assumptions of Theorem 3.3 by choosing $k = 3/4$.

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