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AN OVERVIEW OF GRAPHS ASSOCIATED WITH CHARACTER DEGREES AND CONJUGACY CLASS SIZES IN FINITE GROUPS

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1. Introduction. Let G be a finite group. One of the key tools to studying G is the set of irreducible characters of G. The characters are constant on conjugacy classes, and in fact the conjugacy classes can be viewed as being "dual" to the characters. The value of each character at the identity is the degree of the character, and we call the set $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ the (irreducible) character degrees of G. The corresponding value for the conjugacy class is its size, and we call the set $\mathrm{cs}(G) = \{|C| \mid C \in \mathrm{class}(G)\}$ the class sizes of G. These are both finite sets of positive integers that include 1.

In this field of study, there are two main questions that arise. Which sets of positive integers can occur as either $\mathrm{cd}(G)$ or $\mathrm{cs}(G)$ for some group G, and if there is some set X so that $X = \mathrm{cd}(G)$ or $X = \mathrm{cs}(G)$ for a group G, what can be said about the structure of G? To aid in the study of these questions, we will attach several graphs to $\mathrm{cd}(G)$ and $\mathrm{cs}(G)$. Again the questions that arise are: which graphs can occur in these situations, and if some graph does occur, what can be said about the associated groups?

This will be an expository paper in which we outline the major results about these graphs. In this paper we will survey many of the known results, and we will provide references to the literature for their proofs.

There are two main graphs considered in this paper. In Section 3, we define these graphs for general sets of positive integers, and we outline the connections between these graphs for a given set of integers. In Sections 4, 5, 6 and 7, we look at the properties of the graphs attached to $\mathrm{cd}(G)$. We look at graphs associated with $\mathrm{cs}(G)$ in Section 8.

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Sections 9 and 10 look at subgraphs and generalizations of the graphs considered for $\mathrm{cd}(G)$ and $\mathrm{cs}(G)$.

2. An overview. The genesis for this paper occurred several years ago when Isaacs was preparing [**29**]. At that time, he asked us for a summary of the results known about these graphs. Many times before and since, we have been asked for a good place to find an expository account of the results associated with these graphs. We usually referred people to [**24**], but that paper was published in 1991 and, due to the large number of results that have appeared since then, that account is now out of date. A good introduction to the basic results can be found in [**25**]. A more extensive introduction to the subject is in [**55**], and it included most of the results known for the character degree graphs at that time. Unfortunately, it too has become out of date. Thus, we feel that an expository account of the results associated with these graphs is appropriate.

The character degree portion of this paper is based on a series of lectures that we gave in the algebra seminar at Kent State University. At that time, it was suggested that it would be useful to include graphs associated with conjugacy classes. We decided to follow this advice. We were not as familiar with the literature on the graphs associated with conjugacy classes, and we were surprised with the extent of it.

Recall the two questions posed in the introduction. What graphs can occur in each context? If a given graph is associated with G, what can be said regarding the structure of G? Most of the known results address the first question. In particular, we will find that in all but one of the graphs we study, the number of connected components will be at most two or three. (There is one graph where there is no bound on the number of connected components.) For the most part, the only other restriction that will be placed on the graphs will be a bound on the diameter. In a couple of cases, we will find further restrictions on which graphs can occur, but even in those cases we do not have a full characterization of which graphs occur. As to the second question, most of the results look at the structure of G when the associated graph is disconnected.

3. Graphs associated with sets of integers. For the moment, we are going to set the groups aside and just focus on the graphs associated with sets of positive integers. The results in this section are considered part of the folklore of the subject. Since the proofs of these results are easy and have not been well distributed, we include them here.

If $\mathcal{G}$ is a graph, we let $n(\mathcal{G})$ be the number of connected components of $\mathcal{G}$. If x and y are vertices lying in the same connected component of $\mathcal{G}$, then the distance between x and y, written $d\,(x, y)$, is the number of edges in a path between x and y with the fewest number of edges. If x and y lie in different connected components of $\mathcal{G}$, we do not define the distance between them. The diameter of $\mathcal{G}$ is the maximum distance between vertices in the same connected component, and is denoted by diam $(\mathcal{G})$. Note this means that the diameter of $\mathcal{G}$ is the largest diameter among the connected components. (We should mention this is perhaps not the standard graph theoretic definition of diameter. In that case, the diameter of a disconnected graph would be infinite. This definition is consistent with the usage in the literature on these graphs.)

If x is a positive integer, we use $\pi(x)$ to denote the set of prime divisors of x. For our purposes, X will be a set of positive integers. We define $\rho(X)$ to be the set of primes dividing integers in X, i.e., $\rho(X) = \cup\pi(x)$ where x runs through the elements of X. The *prime vertex graph* for X is denoted by $\Delta(X)$. The vertex set for $\Delta(X)$ is $\rho(X)$, and there is an edge between $p, q \in \rho(X)$ if pq divides x for some integer $x \in X$. The *common divisor graph* for X is denoted by $\Gamma(X)$. The vertex set for $\Gamma(X)$ is $X^* = X \setminus \{1\}$, and there is an edge between $a, b \in X^*$ if a and b have a nontrivial common divisor. (Note, we are not necessarily assuming $1 \in X$. If 1 is not in X, then $X^* = X$.)

If Y is a subset of X, it is easy to see that $\Delta(Y)$ is a subgraph of $\Delta(X)$ and $\Gamma(Y)$ is a subgraph of $\Gamma(X)$. If W is a set of positive integers with the property that, for every element $w \in W$, there is some integer $x \in X$ so that $w \mid x$, then one can show that $\Delta(W)$ is a subgraph of $\Delta(X)$. On the other hand, $\Gamma(W)$ need not be a subgraph of $\Gamma(X)$. It is easy to believe that $\Delta(X)$ and $\Gamma(X)$ are intimately connected. In the following lemma, we present one connection that has proven to be important. Since it is possible for primes to occur in X, we will distinguish distance in $\Gamma(X)$ from distance in $\Delta(X)$ by a subscript. (To demonstrate the possible problem, consider the following. If $X = \{2, 3, 6\}$, then $d_{\Gamma(X)}(2, 3) = 2$ and $d_{\Delta(X)}(2, 3) = 1$.)

Lemma 3.1. *Let X be a set of positive integers. Suppose $a, b \in X^*$. Fix primes p and q so that $p \mid a$ and $q \mid b$. Then a and b lie in the same connected component of $\Gamma(X)$ if and only if p and q lie in the same connected component of $\Delta(X)$. Furthermore, if this occurs, then $|d_{\Gamma(X)}(a, b) - d_{\Delta(X)}(p, q)| \leq 1$.*

Proof. Suppose a and b lie in the same connected component of $\Gamma(X)$. We choose the integers $a = a_0, a_1, \ldots, a_n = b \in X^*$ so that $a = a_0 - a_1 - \cdots - a_n = b$ is a path with the fewest number of edges between a and b in $\Gamma(X)$, so $d_{\Gamma(X)}(a, b) = n$. For each $i = 1, \ldots, n$, we can find a prime $p_i \in \rho(X)$ so that $p_i \mid a_{i-1}$ and $p_i \mid a_i$. It follows that $p - p_1 - \cdots - p_n - q$ is a path of length $n + 1$ in $\Delta(X)$, and $d_{\Delta(X)}(p, q) \leq n + 1 = d_{\Gamma(X)}(a, b) + 1$.

Conversely, suppose that p and q lie in the same connected component of $\Delta(X)$. We find primes $p = q_0, q_1, \ldots, q_l = q \in \rho(X)$ with $p = q_0 - q_1 - \cdots - q_l = q$ is a path of shortest length between p and q in $\Delta(X)$. It follows that $d_{\Delta(X)}(p, q) = l$. For $i = 1, \ldots, l$, we can find $b_i \in X^*$ so that $q_{i-1}q_i \mid b_i$, and this implies $a - b_1 - \cdots - b_l - b$ is a path of length $l + 1$ in $\Gamma(X)$, and we see that $d_{\Gamma(X)}(a, b) \leq l + 1 = d_{\Delta(X)}(p, q) + 1$. Therefore, $|d_{\Gamma(X)}(a, b) - d_{\Delta(X)}(p, q)| \leq 1$. □

Corollary 3.2. *Let X be a set of positive integers so that X^* is not empty. Then we have the equation $n(\Delta(X)) = n(\Gamma(X))$. Furthermore, $|\mathrm{diam}(\Delta(X)) - \mathrm{diam}(\Gamma(X))| \leq 1$.*

Proof. Notice that in Lemma 3.1, we actually proved that there was a correspondence between the connected components of $\Gamma(X)$ and the connected components of $\Delta(X)$. In particular, a subset A of X^* that is maximal with prime set $\rho(A)$ corresponds to a connected component of $\Gamma(X)$ if and only if $\rho(A)$ corresponds to a connected component of $\Delta(X)$. Furthermore, the diameters of these connected components differ by at most 1. □

Let $X = \{4, 6, 9\}$. It is easy to see that $\Gamma(X)$ has diameter 2 while $\Delta(X)$ has diameter 1. On the other hand, if $Y = \{6, 15\}$, then $\Gamma(Y)$ has diameter 1 and $\Delta(Y)$ has diameter 2. Thus, it is possible for the graphs to have different diameters and for either graph to have the larger diameter. If we let $Z = \{4, 6\}$, then both $\Gamma(Z)$ and $\Delta(Z)$ have diameter 1 which shows that the two graphs may also have the same diameter.

Next, we will present examples to show that any graph can appear in either context. This shows that when we look at sets of integers coming from groups, the restrictions on the graphs occurring are group properties and not artifacts arising from the construction of the graphs.

Lemma 3.3. *Let $\mathcal{G}$ be any graph. Then there exist sets of positive integers X and Y so that $\Delta(X) \cong \mathcal{G}$ and $\Gamma(Y) \cong \mathcal{G}$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the distinct vertices, and let $e_1, e_2, \ldots, e_m$ be the distinct edges of $\mathcal{G}$. Take $p_1, p_2, \ldots, p_n$ to be distinct primes. We can view each edge e_i as an unordered pair $\{v_{a_i}, v_{b_i}\}$ where a_i and b_i are integers between 1 and n. For $i = 1, \ldots, m$, we set $x_i = p_{a_i} p_{b_i}$, and $X = \{x_i \mid i = 1, \ldots, m\} \cup \{p_1, p_2, \ldots, p_n\}$. It is not difficult to see that $\Delta(X) \cong \mathcal{G}$. Let $q_1, q_2, \ldots, q_m$ be distinct primes that are all different from $p_1, p_2, \ldots, p_n$. We define $c(i,j)$ to be 1 if e_j is incident to v_i and 0 otherwise. For $i = 1, \ldots, n$, we define $y_i = p_i \prod_{j=1}^{m} q_j^{c(i,j)}$, and $Y = \{y_i \mid i = 1, \ldots, n\}$. It is not difficult to see that $\Gamma(Y) \cong \mathcal{G}$. □

4. Which graphs occur as $\Delta(G)$? Let G be a group. We define $\rho(G) = \rho(\mathrm{cd}(G))$, $\Delta(G) = \Delta(\mathrm{cd}(G))$ and $\Gamma(G) = \Gamma(\mathrm{cd}(G))$. It is not difficult to see that, if G is any group and A is an abelian group, then both $\Delta(G) = \Delta(G \times A)$ and $\Gamma(G) = \Gamma(G \times A)$ since $\mathrm{cd}(G) = \mathrm{cd}(G \times A)$. Thus, once one has one example of a group that gives rise to a graph $\mathcal{G}$, then one has a family of examples that gives rise to $\mathcal{G}$.

If N is a normal subgroup of G, we know that $\mathrm{cd}(G/N)$ is a subset of $\mathrm{cd}(G)$, so $\Delta(G/N)$ is a subgraph of $\Delta(G)$ and $\Gamma(G/N)$ is a subgraph of $\Gamma(G)$. The set $\mathrm{cd}(N)$ has the property that every degree $a \in \mathrm{cd}(N)$ divides some degree $b \in \mathrm{cd}(G)$. It follows that $\Delta(N)$ is a subgraph of $\Delta(G)$. On the other hand, it is not clear what relationships exist between $\Gamma(G)$ and $\Gamma(N)$. Because of this fact, the prime vertex graph $\Delta(G)$ is more amenable to inductive arguments than the common divisor graph $\Gamma(G)$. This may explain why most of the results in the literature are about $\Delta(G)$. Furthermore, much more is known about

$\Delta(G)$ when G is solvable. In this section, we discuss which graphs occur as $\Delta(G)$ when G is solvable.

Perhaps the most sweeping result regarding which graphs arise as the prime graph, $\Delta(G)$, when G is solvable, is due to Pálfy [**67**]. This theorem should be viewed as eliminating many graphs as possibilities for $\Delta(G)$ when G is solvable. Interestingly, the first published proof of this result was in [**55**].

Theorem 4.1 (Pálfy [**67**]). *Let G be a solvable group, and let π be a subset of $\rho(G)$. If $|\pi| \geq 3$, then there exist primes $p, q \in \pi$ and a degree $a \in cd(G)$ so that pq divides a. In other words, any three primes in $\rho(G)$ must have an edge in $\Delta(G)$ that is incident to two of those primes.*

We will say that *a graph satisfies Pálfy's condition* if it has the property that every three vertices have an edge incident to two of those vertices. It is easy to see that a disconnected graph that satisfies Pálfy's condition will have two connected components, and each connected component will be a complete graph. Also, it is not difficult to see that (connected) graphs satisfying Pálfy's condition have diameter at most 3. We state these results as a corollary. In fact, it was known when G is solvable that $n(\Delta(G)) \leq 2$ and $\mathrm{diam}\,(\Delta(G)) \leq 3$ before Pálfy's theorem. (See [**52**, Proposition 2] and [**54**, Corollary 4.5]. It is surprising to note that Manz proved the number of connected components was at most 2 before the prime vertex graph $\Delta(G)$ had been defined.) In Theorem 4.4 of [**54**], they proved a weaker version of Theorem 4.1. In particular, they proved that if G was solvable and π was a subset of $\rho(G)$ with $|\pi| \geq 4$, then there exist primes $p, q \in \pi$ and a degree $a \in \mathrm{cd}\,(G)$ so that $pq \mid a$. This led to the immediate corollary that if G is solvable and $\Delta(G)$ is disconnected, then at least one connected component of $\Delta(G)$ is a complete graph and the other connected component has diameter at most 2, see [**54**, Corollary 4.5]. As far as we can tell, Pálfy was the first to prove that if $\Delta(G)$ is disconnected when G is solvable, then both connected components are complete graphs.

Corollary 4.2. *Let G be a solvable group.*

1. (Manz [**52**]). *Then $n(\Delta(G)) \leq 2$.*

2. (Manz, Willems and Wolf [**54**]). *If $\Delta(G)$ is connected, then* $\operatorname{diam}(\Delta(G)) \leq 3$.

3. *If $\Delta(G)$ is disconnected, then each connected component is a complete graph.*

As we will see, the situation when $\Delta(G)$ is disconnected has been studied in detail. The next result was first mentioned in [**24**]. When we read this result, it was such a surprise that we did not believe it was true. It was not clear in [**24**] who had discovered this result, so we wrote Huppert to ask. He stated the result was due to Pálfy who had never published it. We had managed to prove the result ourselves, and we were within a day of submitting for publication a paper containing our proof when we received a preprint of [**68**]. In that paper, we found:

Theorem 4.3 (Pálfy [**68**]). *Suppose the prime vertex graph of* $\operatorname{cd}(G)$ *for a solvable group G has two connected components, and let the cardinalities of the vertex sets of the two components be n and N with $n \leq N$. Then $N \geq 2^n - 1$.*

This shows that many graphs satisfying Pálfy's condition do not occur as $\Delta(G)$ for some solvable group G. For example, the four-vertex graph in Figure 1 does not occur as $\Delta(G)$ for a solvable group G.

FIGURE 1. Not allowed by Theorem 4.3.

Question 4.4. Let n and N be integers greater than 1 so that $N \geq 2^n - 1$. Does there exist a solvable group G so that $\Delta(G)$ has two connected components with cardinalities n and N?

We do not know any pairs (n, N) as in the question where there fails to be an example of solvable group G with a disconnected prime vertex graph with connected components of sizes n and N. For every positive integer N, there is a solvable group G so that $\Delta(G)$ has two connected components, one an isolated vertex and the other having N vertices. We illustrate this with the following example. Let p be an odd prime. We can find an integer a so that $p^a - 1$ has at least N prime divisors. (Use the Zsigmondy prime theorem.) Take b to be a divisor of $p^a - 1$ with exactly N distinct prime divisors. Define E to be an extra-special group of order p^{2a+1} and exponent p. Now, E has an automorphism σ of order b that centralizes the center of E, and set G as the semi-direct product $E\langle\sigma\rangle$. It is relatively easy to prove that $\mathrm{cd}(G) = \{1, b, p^a\}$, see [**66**], and so the connected components of $\Delta(G)$ have vertices $\{p\}$ and $\pi(b)$.

When $n \geq 2$, the answer to the question will probably rely on number theory rather than character theory or group theory. From [**36**], we see that if G is a solvable group whose prime vertex graph on $\mathrm{cd}(G)$ has two connected components, both components with at least two vertices, then the connected components of $\Delta(G)$ are $\pi(b)$ and $\pi(m)$ where b and m are coprime, composite integers. Furthermore, there is a prime power q where $(q^m - 1)/(q - 1)$ divides b and b divides $q(q^m - 1)$. If we have integers b and m that satisfy the above condition, then we use the construction in [**28**] to find a solvable group G whose prime vertex graph on $\mathrm{cd}(G)$ has the two connected components: $\pi(b)$ and $\pi(m)$. This reduces the question of the existence of a solvable group whose graph has two connected components having sizes n and N to a question of the existence of a prime power q and integers m and b so that the following conditions are satisfied: b and m are coprime, m has n distinct prime divisors, b has N distinct prime divisors, $(q^m - 1)/(q - 1)$ divides b and b divides $q(q^m - 1)$.

We have seen that we have a reasonable idea of which graphs occur as disconnected prime vertex graphs for $\mathrm{cd}(G)$ given G is solvable. When $\Delta(G)$ is connected and G is solvable, the picture is murkier. At the time [**54**] was published saying that $\mathrm{diam}(\Delta(G)) \leq 3$ for a solvable group G, no solvable group G was known to have a prime vertex graph for $\mathrm{cd}\, G$ whose diameter was 3. In fact, it was conjectured that $\mathrm{diam}(\Delta(G)) \leq 2$ when G is solvable. To further this belief Zhang proved:

Theorem 4.5 (Zhang [**75**]). *If G is a solvable group, then $\Delta(G)$ is not the graph in Figure* 2.

FIGURE 2. Not allowed by Theorem 4.5.

It is easy to see that the graph in Figure 2 is the only graph with 4 vertices and diameter 3. We also showed in [**41**] that no graph with 5 vertices and diameter 3 occurs as $\Delta(G)$ with G solvable.

Theorem 4.6 [**41**]. *If G is a solvable group and $|\rho(G)| = 5$, then* $\operatorname{diam}(\Delta(G)) \leq 2$.

On the other hand, at the time Theorem 4.6 was proved, we also found an example that proved the conjecture is false.

Theorem 4.7 [**40**]. *There is a solvable group G where $\Delta(G)$ is the graph in Figure* 3.

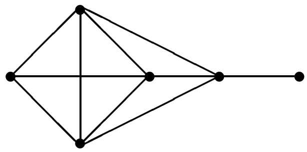

FIGURE 3. The graph of Theorem 4.7.

The group referred to in Theorem 4.7 is quite complicated. (For example, it has order $2^{45} \cdot 3 \cdot 5 \cdot 7 \cdot 31 \cdot 151$.) Thus, we will not present it here. Only one group is constructed in [**40**] and, as mentioned earlier, other examples can be found by taking direct products with abelian groups. Lately, our graduate student, Carrie Dugan, has replicated the construction in [**40**] to find other solvable groups whose prime vertex graphs for $\operatorname{cd}(G)$ have diameter 3.

We now ask which graphs with diameter 3 occur as $\Delta(G)$ for some solvable group G. Suppose $\mathcal{G}$ is a connected graph of diameter 3 that satisfies Pálfy's condition. It is not difficult to see that the vertices of $\mathcal{G}$ can be partitioned into 4 nonempty sets ρ_1, ρ_2, ρ_3, and ρ_4, so that no vertex in ρ_1 is adjacent to any vertex in $\rho_3 \cup \rho_4$ and no vertex in ρ_4 is adjacent to any vertex in $\rho_1 \cup \rho_2$. Also, every vertex in ρ_2 is adjacent to some vertex in ρ_3 and every vertex in ρ_3 is adjacent to something in ρ_2. Applying Pálfy's condition, we see that $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4$ determine complete subgraphs of $\mathcal{G}$. Finally, we label the sets so that $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$. (For the full details on this partition see Observation 1 of [**41**].)

The evidence we have seen suggests that if G is solvable and $\Delta(G)$ has diameter 3, then there is a normal subgroup N so that $\rho(G/N) = \rho(G)$ and $\Delta(G/N)$ is disconnected where the connected components match up with the sets $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4$ from the last paragraph. In fact, the groups that arise as G/N in the examples we have seen suggest the following conjecture, which should be compared with Theorem 4.3.

Conjecture 4.8. *Let G be a solvable group with* $\operatorname{diam}(\Delta(G)) = 3$. *Label the vertices of $\Delta(G)$ as above. If $n = |\rho_1 \cup \rho_2|$, then $|\rho_3 \cup \rho_4| \geq 2^n$.*

Since the ρ_i's are nonempty, $|\rho_1 \cup \rho_2| \geq 2$. If this conjecture is true, then it would say that $|\rho_3 \cup \rho_4| \geq 4$ when $|\rho_1 \cup \rho_2| = 2$. In particular, it would imply that there are no solvable groups whose degree graphs have diameter 3 and fewer than 6 vertices, which is the content of Theorems 4.5 and 4.6. It also would imply that if $\Delta(G)$ has diameter 3 and 6 vertices, then $|\rho_1 \cup \rho_2| = 2$ and $|\rho_3 \cup \rho_4| = 4$, which is proved in [**41**]. On the other hand, we have no feeling on the relationships between $|\rho_1|$ and $|\rho_2|$ and between $|\rho_3|$ and $|\rho_4|$. In Figure 3, $|\rho_1| = |\rho_2| = |\rho_4| = 1$ and $|\rho_3| = 3$. We ask whether there is a solvable group G where $\operatorname{diam}(\Delta(G)) = 3$ and $|\rho_3| = |\rho_4| = 2$ or $|\rho_3| = 1$ and $|\rho_4| = 3$.

Finally, we consider graphs with few vertices. In [**24**], Huppert lists all the graphs with 4 or fewer vertices that occur as $\Delta(G)$ for some solvable group G. In fact, except for the graphs shown in Figures 1 and 2, every graph with 4 or fewer vertices that satisfies Pálfy's condition occurs as $\Delta(G)$ for some solvable group G. We saw in Theorem 4.6 that the 5 vertex graphs with diameter 3 do not occur as $\Delta(G)$ for any solvable group G. In [**44**], we show that two other graphs with 5 vertices that satisfy Pálfy's condition do not occur as $\Delta(G)$ for any solvable group G. Also, for all but one of the remaining graphs having 5 vertices that satisfies Pálfy's condition, we find a solvable group with an isomorphic prime vertex graph. There is one graph with 5 vertices where we do not know of a solvable group G so that $\Delta(G)$ is isomorphic to this graph, but also, we are unable to prove that no such group exists. This graph is shown in Figure 4. At this time, it is unknown which graphs with 6 vertices occur as $\Delta(G)$ when G is solvable.

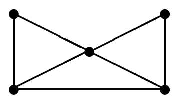

FIGURE 4. Unknown graph.

5. The structure of G given $\Delta(G)$. In the previous section, we considered which graphs occur as $\Delta(G)$ when G is solvable. In this section, we look at what can be said about the structure of G from knowing the structure of $\Delta(G)$. Perhaps the most basic result in this direction is the fact that if G has a nonabelian p-group quotient for some prime p, then p will be adjacent to every other prime in $\Delta(G)$. Restating this fact, if G is solvable and $\Delta(G)$ has no vertex adjacent to all the other vertices, then all nilpotent quotients of G are abelian.

We next look at the structure of a group in terms of a prime p where p is not adjacent in $\Delta(G)$ to some other prime in $\rho(G)$. We will show that the structure of G will be limited in terms of p. We define the following series of characteristic subgroups:

$$P_0 = \mathbf{O}_{p'}(G),\ P_1 = \mathbf{O}_{p',p,p'}(G),\ P_2 = \mathbf{O}_{p',p,p',p,p'}(G),\ \dots .$$

We define the *p-length* of G, written $l_p(G)$, to be the smallest integer l so that $P_l = G$. If G is solvable, $l_p(G)$ is guaranteed to exist. In its full generality, the following result was proved by Zhang in [**76**]. A weaker version of this theorem had been proved earlier by Pense in [**69**].

Theorem 5.1 (Zhang [**76**]). *Let G be a solvable group. Suppose that $p, q \in \rho(G)$ are not adjacent in $\Delta(G)$. Then $l_p(G) \leq 2$ and $l_q(G) \leq 2$. Furthermore, $l_p(G) + l_q(G) = 4$ if and only if $pq = 6$ and the normal closure of a Sylow 2-subgroup of G contains a quotient that is isomorphic to the semi-direct product of $\mathrm{GL}_2(3)$ acting faithfully on $Z_3 \times Z_3$.*

Now, we switch from thinking about the structure of G in terms of a single prime to the whole structure of G. Recall that the Fitting subgroup $\mathbf{F}(G)$ is the unique largest normal nilpotent subgroup of G. We can define a series of normal subgroups $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq G$ by $F_0 = 1$ and $F_{i+1}/F_i = \mathbf{F}(G/F_i)$. A group G is solvable if and only if $G = F_i$ for some integer i. The *Fitting height* of G is the smallest integer i so that $G = F_i$. The following result appears in [**55**].

Theorem 5.2 (Manz and Wolf [**55**]). *Let G be a solvable group where $\Delta(G)$ is disconnected. Then G has Fitting height at most 4, and $G/\mathbf{F}(G)$ has derived length at most 4.*

In fact, more can be said. The following result was nearly proved by Pálfy in [**68**] and Zhang in [**78**]. This form of the result was proved in [**36**].

Theorem 5.3 [**36**]. *Let G be a solvable group where $\Delta(G)$ is disconnected. Then the following are true*:

1. *If G has Fitting height 4, then $G/\mathbf{Z}(G)$ is isomorphic to the semi-direct product of $\mathrm{GL}_2(3)$ acting on $Z_3 \times Z_3$ and $\mathrm{cd}(G) = \{1, 2, 3, 4, 8, 16\}$. (Consequently, if $\rho(G) \neq \{2, 3\}$, then G has Fitting height at most 3.)*

2. *If both of the connected components of $\Delta(G)$ have at least 2 vertices, then G has Fitting height 3.*

3. *If one connected component of $\Delta(G)$ has $n \geq 2$ vertices and the other has $2^n - 1$ vertices, then G has derived length 3.*

In [**36**] we proved a stronger result. We defined six families of groups, and we proved that G is solvable with $\Delta(G)$ disconnected if and only if G is in one of the six families. In other words, we have found a classification for the solvable groups with $\Delta(G)$ disconnected.

Let $\mathcal{G}$ be a graph that occurs as $\Delta(G)$ for some solvable group G. We say that $\mathcal{G}$ has *bounded Fitting height* if there is a bound on the Fitting height of any solvable group H where $\Delta(H) \cong \mathcal{G}$. We know that if $\mathcal{G}$ is disconnected, then $\mathcal{G}$ has bounded Fitting height. On the other hand, if $\mathcal{G}$ is the graph with two vertices and an edge, then for any integer n it is possible to find a solvable group H with Fitting height at least n so that $\Delta(H) \cong \mathcal{G}$. Thus, $\mathcal{G}$ does not have bounded Fitting height. In [**35**], we were able to characterize which graphs have bounded Fitting height.

Theorem 5.4 [**35**]. *Let $\mathcal{G}$ be a graph with n vertices that occurs as $\Delta(G)$ for some solvable group G. Then $\mathcal{G}$ has bounded Fitting height if and only if $\mathcal{G}$ has at most one vertex of degree $n-1$.*

If $\mathcal{G}$ has bounded Fitting height, the bound that is found in [**35**] is linear in the number of vertices of $\mathcal{G}$. On the other hand, when $\mathcal{G}$ is disconnected, the bound is 4. In fact, we do not know of any graph with bounded Fitting height that occurs as $\Delta(G)$ for a solvable group G where the bound is bigger than 4. This leads to the following conjecture.

Conjecture 5.5. *Let G be a solvable group where $\Delta(G)$ is a graph with bounded Fitting height. Then G has Fitting height at most* 4.

To provide further evidence in support of this conjecture, we looked at some other graphs with bounded Fitting height, and in [**37**], we showed the conjecture holds for these graphs. Specifically, we proved the following theorem.

Theorem 5.6 [**37**]. *Let G be a solvable group where either:*

1. $\rho(G) = \pi_1 \cup \pi_2 \cup \{p\}$ *is a disjoint union where each of the π_i's is not empty, and no prime in π_1 is adjacent in $\Delta(G)$ to any prime in π_2, or*

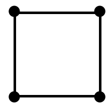

FIGURE 5. Graph of Theorem 5.6.2.

2. $\Delta(G)$ *is the graph with four vertices where every vertex has degree* 2.

Then G has Fitting height at most 4.

The proof of Theorem 5.4 is strongly based on Theorem 5.1. In fact, Moretó has been able to prove the existence of a universal bound. Specifically, he has proved the following.

Theorem 5.7 (Moretó [**63**]). *Let G be a solvable group with bounded Fitting height graph. Then the Fitting height of G is at most* 31. *Furthermore, if* $|G|$ *is odd, then the Fitting height of G is at most* 7.

To prove that the universal bound is 4, a different idea will likely be required. At this time, a reasonable next step would be to look at other families of graphs that arise as prime vertex graphs with bounded Fitting height, and prove that 4 is a universal bound for them. One set of candidates for study that we would consider next is the set of graphs with diameter 3.

Thus far, we have asked what is the largest Fitting height that can arise for a solvable group G where $\Delta(G) \cong \mathcal{G}$, given the graph $\mathcal{G}$. One can also ask what is the smallest Fitting height that a solvable group G can have when $\Delta(G) \cong \mathcal{G}$. When G is nonabelian and nilpotent, i.e., has Fitting height 1, we know that $\Delta(G)$ is a complete graph. Therefore, if $\Delta(G)$ is not a complete graph, then G must have Fitting height at least 2. In [**42**], we characterize $\Delta(G)$ when G is a solvable group of Fitting height 2.

Theorem 5.8 [**42**]. *Let* $\mathcal{G}$ *be a graph with* n *vertices. There exists a solvable group G of Fitting height* 2 *with* $\Delta(G) \cong \mathcal{G}$ *if and only if the vertices of* $\mathcal{G}$ *with degree less than* $n-1$ *can be partitioned into two sub-*

sets, each of which induces a complete subgraph of $\mathcal{G}$ and one of which contains only vertices of degree $n-2$.

We conclude this section with another question suggested by Theorem 5.3. Thus far, we have considered the relationship between Fitting height and the prime vertex graph, but conclusion (3) of Theorem 5.3 suggests that there might be a connection between $\Delta(G)$ and the derived length of G. We can say that a graph $\mathcal{G}$ which occurs as $\Delta(G)$ for some solvable group G has *bounded derived length* if there is an upper bound on the derived length of H when $\Delta(H) \cong \mathcal{G}$. Obviously, any such graph that does not have bounded Fitting height will not have bounded derived length. Notice that the graph consisting of a single vertex has bounded Fitting height but not bounded derived length since this is the graph $\Delta(P)$ when P is any nonabelian p-group for a prime p. Thus, there exist graphs with bounded Fitting height that do not have bounded derived length. In fact, the only graphs we know which have bounded derived length are the ones found in conclusion (3) of Theorem 5.3. It would be interesting to find other graphs with bounded derived lengths. Is it possible to characterize which graphs have bounded derived lengths?

6. $\Delta(G)$ when G is not solvable. In the previous two sections we considered the relationship between the prime vertex graph $\Delta(G)$ and the group G when G is solvable. In this section, we wish to consider this relationship when G is not solvable. These graphs have received much less attention when G is not solvable, but recent research has begun to better our understanding.

When looking at nonsolvable groups, the basic building blocks are the nonabelian simple groups, so it makes sense to understand the graphs that arise when G is a nonabelian simple group. We use the classification to list these graphs. Since the character tables of the sporadic simple groups are all in the Atlas [**19**], we can read off the graphs of the sporadic simple groups from there.

Theorem 6.1. *If G is a sporadic simple group other than M_{11}, J_1 or M_{23}, then $\Delta(G)$ is a complete graph. The prime vertex graphs of M_{11} and M_{23} are connected with diameter* 2, *and the prime vertex graph of J_1 is connected of diameter* 3.

The next family of groups to consider are the alternating groups. Looking at the character degree sets for Alt (5) and Alt (6), we see that $\Delta(G)$ is disconnected when G is either of these groups. It is well known that Alt (5) $\cong$ PSL (2, 4) and Alt (6) $\cong$ PSL (2, 9), and we shall show that these groups can be handled with the groups of the form PSL $(2, q)$ where $q \geq 4$ is a prime power. Also, $\Delta(\mathrm{Alt}\,(8))$ is graph of diameter 2. Looking at cd (G) when $G \cong$ Alt (7), we see that the graph $\Delta(\mathrm{Alt}\,(7))$ is a complete graph. By computing particular degrees for a large number of examples in [**4**], Alvis and Barry present compelling evidence that $\Delta(\mathrm{Alt}(n))$ is a complete graph when $n \geq 9$. Recently, Barry and Ward have proved this conjecture [**5**]. In fact, a stronger result is proved.

Theorem 6.2 (Barry and Ward [**5**]). *For every integer $n \geq 15$, there is a degree $a \in \mathrm{cd}\,(\mathrm{Alt}\,(n))$ such that a is divisible by every prime in $\rho(\mathrm{Alt}\,(n))$.*

For $9 \leq n \leq 14$, one can check via the actual character degrees that $\Delta(\mathrm{Alt}\,(n))$ is complete.

The remaining simple groups are the groups of Lie type. The graphs for these groups have been determined by White. The exceptional groups of Lie type are handled in [**72**]. He proved that if G is an exceptional simple group of Lie type except the Suzuki groups, then $\Delta(G)$ is a complete graph. If G is a Suzuki group, say $G \cong {}^2B_2(q^2)$ where $q^2 = 2^{2m+1}$ and $m \geq 1$, then $\rho(G)$ can be partitioned as $\{2\} \cup \pi(q^2 - 1) \cup \pi(q^4 + 1)$. The subgraph of $\Delta(G)$ on $\rho(G) - \{2\}$ is complete and 2 is adjacent in $\Delta(G)$ to precisely the primes in $\pi(q^2 - 1)$, see [**72**, Theorem 3.3].

Finally, we consider the classical groups of Lie type. The character table for PSL $(2, q)$ is well-known. From the table, we see $\Delta(\mathrm{PSL}\,(2, 2^n))$ with $n \geq 2$ has three connected components $\{2\}$, $\pi(2^n - 1)$ and $\pi(2^n + 1)$. When p is an odd prime and $p^n > 5$, then $\Delta(\mathrm{PSL}\,(2, p^n))$ has two connected components $\{p\}$ and $\pi(p^{2n} - 1)$. It might appear that we have omitted $\Delta(\mathrm{PSL}\,(2, 5))$, but in fact, we have not. It is known that PSL $(2, 4) \cong$ PSL$(2, 5)$, and so, this graph is handled as $\Delta(\mathrm{PSL}\,(2, 4))$. If G is any other classical Lie-type nonabelian simple group, then $\Delta(G)$ is connected of diameter at most 2. In fact, $\Delta(G)$ is a complete graph except when G is one of PSL $(3, 4)$, PSL $(3, q)$ where $\pi(q - 1) \not\subseteq \{2, 3\}$, PSL $(4, 2)$, or PSU $(3, q)$ where $\pi(q + 1) \not\subseteq \{2, 3\}$. (See [**47, 73, 74**].)

In summary, we have the following results regarding nonabelian simple groups of Lie type.

Theorem 6.3 (White [**47, 72, 73, 74**]). *Let G be a nonabelian simple group of Lie type.*

1. *If G is not isomorphic to* $\mathrm{PSL}\,(2, p^n)$ *for some prime p and integer n with $p^n \geq 4$, then $\Delta(G)$ is connected.*

2. *The graphs $\Delta(\mathrm{PSL}\,(2, p^n))$ have two connected components where p is an odd prime and n is an integer so that $p^n > 5$.*

3. *The graphs $\Delta(\mathrm{PSL}\,(2, 2^n))$ have three connected components where $n \geq 2$ is an integer.*

4. *If G is isomorphic to* $\mathrm{Sz}(2^n)$ *where $n \geq 3$ is odd,* $\mathrm{PSL}\,(3, q)$ *where $q \geq 3$ is a prime power with $q = 4$ or $\pi(q-1) \not\subseteq \{2, 3\}$,* $\mathrm{PSL}\,(4, 2)$, *or* $\mathrm{PSU}\,(3, q)$ *where $q \geq 3$ is a prime power with $\pi(q+1) \not\subseteq \{2, 3\}$, then $\Delta(G)$ is connected of diameter 2.*

5. *If G is any other simple group of Lie type, then $\Delta(G)$ is a complete graph.*

Looking at general nonsolvable groups, we have the following. Most of these results used results regarding the graphs of the nonabelian simple groups. We first look at the disconnected case.

Theorem 6.4. *Let G be a nonsolvable group. Then the following are true*:

1. (Manz, Staszewski and Willems [**53**]). $n(\Delta(G)) \leq 3$.

2. (Lewis and White [**47**]). *$n(\Delta(G)) = 3$ if and only if $G = S \times A$ where $S \cong$* $\mathrm{PSL}\,(2, 2^n)$ *for some integer $n \geq 2$ and A is an abelian group. The connected components are $\{2\}$, $\pi(2^n - 1)$, and $\pi(2^n + 1)$, and each component has diameter at most 1.*

3. (Lewis and White [**47, 48**]). *If $n(\Delta(G)) = 2$, then G has normal subgroups $N \subseteq K$ so that $K/N \cong$* $\mathrm{PSL}\,(2, p^n)$ *where p is a prime and n is an integer so that $p^n \geq 4$. Furthermore, G/K is abelian and N is either abelian or metabelian. In fact, the connected components of $\Delta(G)$ are $\{p\}$ and $\pi(p^{2n} - 1) \cup \pi(|G\!:\!CK|)$ where $C/N = \mathbf{C}_{G/N}(K/N)$. This second component has diameter at most 2.*

In [**47**], we prove a stronger condition than the one in Theorem 6.4 (3). In fact, we are able to prove a necessary and sufficient condition for G to be nonsolvable with $\Delta(G)$ having two connected components. Using the notation of the theorem, we proved that K is one of the following: $\mathrm{PSL}\,(2, p^n)$, $\mathrm{SL}\,(2, p^n)$, or there is a subgroup L in K that is normal in G so that $K/L \cong \mathrm{SL}\,(2, p^n)$, L is elementary abelian group of order p^{2n}, and K/L acts transitively on the nonprincipal characters in $\mathrm{Irr}\,(L)$. Also, it is necessary that p does not divide $|G : CK|$ and C/N is central in G/N. When $p = 2$ or $p^n = 5$, then either $N > 1$ or $CK < G$. This last condition is necessary to ensure that $\Delta(G)$ has two connected components instead of three. Finally, because $\mathrm{PSL}\,(2, 4) \cong \mathrm{PSL}\,(2, 5)$, there is some ambiguity in the value p in the theorem. We discuss in [**47**] how to handle this ambiguity.

Notice that, unlike the solvable case, $\Delta(G)$ can have three connected components, and when G is not solvable and $\Delta(G)$ is disconnected, one of the connected components is an isolated vertex. Recall that when G is solvable and $\Delta(G)$ has two connected components, then each component is a complete graph. If G is not solvable and $\Delta(G)$ has two connected components, it is possible that the other connected component has diameter 2, see $\Delta(\mathrm{PSL}\,(2, 11))$, for example. We have recently shown that the upper bound on the diameter of $\Delta(G)$ when G is not solvable is the same as the upper bound on the diameter of $\Delta(G)$ when G is solvable.

Theorem 6.5 (Lewis and White [**48, 49**]). *Let G be a nonsolvable group where $\Delta(G)$ is connected. Then* $\mathrm{diam}(\Delta(G)) \leq 3$.

7. The graph $\Gamma(G)$. We now consider the common divisor graph $\Gamma(G)$. By Corollary 3.2, we know from the results regarding $\Delta(G)$ that $\Gamma(G)$ has at most two connected components when G is solvable and $\Gamma(G)$ has at most three connected components when G is not solvable. The various classification results for when $\Delta(G)$ is disconnected will apply when $\Gamma(G)$ is disconnected. Furthermore, when G is solvable and $\Gamma(G)$ is disconnected, the Fitting height of G will be at most 4. Using Corollary 3.2, and the various results on $\Delta(G)$, we obtain upper bounds on the diameter of $\Gamma(G)$. In fact, we can usually do better than the bounds obtained this way.

Theorem 7.1. *Suppose that* $\Gamma(G)$ *is disconnected.*

1. *If* G *is solvable, then there is a prime dividing all the degrees corresponding to the vertices of one connected component of* $\Gamma(G)$*, and the other connected component has diameter at most* 2*.*

2. *If* $n(\Gamma(G)) = 3$*, then* G *is not solvable and each component is an isolated vertex.*

3. *If* G *is nonsolvable and* $n(\Gamma(G)) = 2$*, then one connected component is an isolated vertex, and the other component has diameter at most* 2*.*

Note that conclusion (2) of the theorem is an immediate consequence of Theorem 6.4 (2). Conclusion (3) has never appeared in the literature, but it can be deduced by a careful reading of the proofs of Theorem 6.3 of [**47**] and Lemma 3.2 of [**48**]. To see that the second component can actually have diameter 2, let G be the semi-direct product of $\mathrm{SL}(2,8)$ acting on an elementary abelian group of order 64 in the natural fashion. It is not difficult to see that $\mathrm{cd}(G) = \{1,7,8,9,63\}$ and the connected components of $\Gamma(G)$ are $\{8\}$ and $\{7,9,63\}$, which has diameter 2.

Conclusion (1) can easily be deduced from the classification in [**36**]. (It was known to be true before that paper.) Also, one can find a solvable group G where the common divisor graph $\Gamma(G)$ is disconnected and one of the connected components actually has diameter 2. In fact, there is a solvable group G with $\mathrm{cd}(G) = \{1,2,5,20,1023\}$.

Let m and n be any two positive integers. In [**57**], McVey has shown that there is a solvable group G where $\Gamma(G)$ has two connected components that are complete graphs of sizes m and n. Notice this contrasts with the situation for $\Delta(G)$. Also, it does not answer the question of which graphs arise where one of the two components is not a complete graph. We do not know which graphs do occur for $\Gamma(G)$ when G is solvable, $\Gamma(G)$ is disconnected, and one of the connected components is not a complete graph.

From Corollary 3.2 and the results regarding $\Delta(G)$, we see that $\Gamma(G)$ has diameter at most 4 when G is solvable and at most 5 when G is not solvable. McVey has been able to improve these bounds. The solvable case is handled in [**58**] and the nonsolvable case is handled in [**59**]. McVey proved the following.

Theorem 7.2 (McVey [**58, 59**]). *Let G be any group. Then* $\operatorname{diam}(\Gamma(G)) \leq 3$.

It is not difficult to find solvable and nonsolvable groups G whose diameter is 3. Let H be any group with $\operatorname{cd}(H) = \{1, 7\}$. (Notice that H is necessarily solvable.) For a solvable example, take $G = \operatorname{Sym}(4) \times H$. In this case, $\operatorname{cd}(G) = \{1, 2, 3, 7, 14, 21\}$ and $\Gamma(G)$ has diameter 3. For a nonsolvable example, take $K = \operatorname{Alt}(5) \times H$, and observe that $\operatorname{cd}(K) = \{1, 3, 4, 5, 7, 21, 28, 35\}$, so that $\Gamma(K)$ has diameter 3.

McVey's results give an upper bound for the diameter of $\Gamma(G)$. As for a lower bound, we see that if G is a p-group for some prime p, then $\Gamma(G)$ is a complete graph. Furthermore, in [**71**], Turull has a construction which shows that for every p-group P of order p^n there is a solvable p'-group Q of Fitting height n that P acts on with $\mathbf{C}_Q P = 1$. If G is the resulting semi-direct product, then p divides every degree in $\operatorname{cd}(G) - \{1\}$ and so $\Gamma(G)$ is a complete graph. In [**46**], we constructed for every pair of odd primes p and q where p is congruent to 1 modulo 3 and q is a prime divisor of $p+1$ a solvable group G with $\operatorname{cd}(G) = \{1, 3q, p^2q, 3p^3\}$. Observe that $\Gamma(G)$ is a complete graph and no single prime divides all the nontrivial degrees. It would be interesting to find other groups where $\Gamma(G)$ is complete, but no prime divides all the nontrivial degrees. Given that $\Gamma(G)$ can be a complete graph for solvable groups, we found the following new result to be surprising.

Theorem 7.3 (Bianchi, Chillag, Lewis and Pacifici [**13**]). *If $\Gamma(G)$ is a complete graph, then G is a solvable group.*

Recently, there have been other efforts at further understanding the structure of $\Gamma(G)$. Say that a graph $\mathcal{G}$ has a *connective* subset $\mathcal{C}$ if $\mathcal{C}$ induces a subgraph of $\mathcal{G}$ that is a complete graph and every vertex in $\mathcal{G}$ is adjacent to some vertex in $\mathcal{C}$. In working on [**59**], McVey noticed that such subsets were common in $\Gamma(G)$. In the papers [**60, 61**], he studied the groups G where $\Gamma(G)$ has such a subset. In particular, he proved that if G is a nonabelian group so that $\Gamma(G)$ is connected of diameter at most 2, then $\Gamma(G)$ has a connective subset. Obviously, if $\Gamma(G)$ is disconnected, then it will not have a connective subset. In [**59**], McVey constructs both solvable and nonsolvable groups G where $\Gamma(G)$

is connected but does not have a connective subset. On the other hand, it seems that such examples are rare, so McVey has asked whether one can classify the groups G where $\Gamma(G)$ does not contain a connective subset.

Given a vertex v of graph $\mathcal{G}$, we define $\mathcal{N}_*(v)$ to be the subgraph of $\mathcal{G}$ induced by the vertices that are adjacent to r. In his Ph.D. dissertation [**50**], Lo has proved when G is solvable, $r \in \mathrm{cd}\,(G)$, and $\mathcal{G} = \Gamma(G)$ that $\mathcal{N}_*(r)$ has at most two connected components. Notice that Pálfy's condition will imply a similar conclusion for $\mathcal{N}_*(p)$ when $p \in \rho(G)$ and $\mathcal{G} = \Delta(G)$. Most of the results for character degree graphs can be thought of as "global" results. This is the first result that is focused on the situation at a given vertex. That is, it is the first "local" result.

8. The graphs $\Gamma'(G)$ and $\Delta'(G)$. It is well known that there is a strong connection between the irreducible characters of a group and the conjugacy classes. For example, the number of irreducible characters of a group equals the number of conjugacy classes. Even though for a particular group there may not be a close connection between $\mathrm{cd}\,(G)$ and $\mathrm{cs}\,(G)$, there is a similarity to the theorems that can be proved for each set.

We will demonstrate this similarity by looking at the associated graphs. We let $\Delta'(G) = \Delta(\mathrm{cs}\,(G))$ and $\Gamma'(G) = \Gamma(\mathrm{cs}\,(G))$. The common divisor graph for conjugacy class sizes has been studied more closely than the prime vertex graph for conjugacy class sizes. This is the opposite of the situation with character degrees where the prime vertex graph has received more attention.

Technically, the graph studied in most of the literature is the graph $\Gamma'_*(G)$ whose vertex set is the set of noncentral conjugacy classes, and there is an edge between two classes if their sizes have a nontrivial common divisor. It is not difficult to see that $\Gamma'_*(G)$ can be transformed into $\Gamma'(G)$ by collapsing all the classes with the same size into a single vertex. Thus, these graphs have the same number of connected components. The connected components will have the same diameter except in the case where a connected component contains classes all of the same size. If there is more than one class in this component, then the component in $\Gamma'(G)$ has diameter 0, whereas in $\Gamma'_*(G)$, the component has diameter 1. This is similar to the original graph studied for character degrees in [**53**]. In that paper, they considered the graph $\Gamma_*(G)$ whose vertex set was all nonlinear irreducible characters and there was an edge between two characters if their degrees had a nontrivial common divisor. While $\Gamma_*(G)$ is similar to the common divisor graph for $\mathrm{cd}\,(G)$, in the proofs of [**53**] it is noted that the referee suggested studying the prime divisor graph for $\mathrm{cd}\,(G)$, which has been the practice for character degrees.

To make our presentation for $\mathrm{cs}\,(G)$ consistent with the presentation for $\mathrm{cd}\,(G)$, we have translated the results proved for $\Gamma'_*(G)$ into results for the common divisor graph for $\mathrm{cs}\,(G)$. Unfortunately, there are a few results that do not translate well from $\Gamma'_*(G)$. (See [**12**, Corollary 3] and the paper [**22**].)

The following are the main results for the common divisor graph of $\mathrm{cs}\,(G)$. We say G is a *quasi-Frobenius group* if $G/\mathbf{Z}(G)$ is a Frobenius group. We define the kernel and complements of G to be the preimages of the Frobenius kernel and Frobenius complements of $G/\mathbf{Z}(G)$. Apparently, the authors of [**12**] did not know of [**32**], and they reproved many of the results found by Kazarin in [**32**]. The main results of [**32**] are formulated without any mention of a graph. At the end of [**32**] a graph is mentioned, but it is the complement graph of $\Gamma'(G)$. (If $\mathcal{G}$ is a graph, then its complement graph $\overline{\mathcal{G}}$ is the graph with the same vertices, and there is an edge in $\overline{\mathcal{G}}$ between a and b if and only if there is not an edge between a and b in $\mathcal{G}$.)

Theorem 8.1. *Let G be a group.*

1. (Kazarin [**32**]; Bertram, Herzog, and Mann [**12**]). $n(\Gamma'(G)) \leq 2$.

2. (Kazarin [**32**]; Bertram, Herzog, and Mann [**12**]). $n(\Gamma'(G)) = 2$ *if and only if G is a quasi-Frobenius group with abelian kernel and complement. In this case, both connected components are isolated vertices.*

3. (Chillag, Herzog, and Mann [**17**]). *If $\Gamma'(G)$ is connected, then* $\mathrm{diam}\,(\Gamma'(G)) \leq 3$.

4. (Fisman and Arad [**23**]). *If G is a nonabelian simple group, then $\Gamma'(G)$ is a complete graph.*

5. (Chillag, Herzog, and Mann [**17**]). *Suppose G is a nontrivial perfect group. Then $\Gamma'(G)$ is connected and* $\mathrm{diam}\,(\Gamma'(G)) \leq 2$.

If you look at [**23**], it would appear that (4) is not related to the result proved in [**23**], but in [**12**, Proposition 5], they showed that (4) is an easy application of the result proved in [**23**].

Notice that they do not need to assume G is solvable to show that the common divisor graph for $\mathrm{cs}(G)$ has at most two connected components. The class of groups where the common divisor graph for class sizes is disconnected is much simpler to describe than the class of groups where common divisor graph for character degrees is disconnected. Related to this is the fact that, if the common divisor graph for $\mathrm{cs}(G)$ is disconnected, then it consists of two disconnected vertices. Whereas we saw that the common divisor graph for $\mathrm{cd}(G)$ can have either two or three connected components where the connected components may have more than one vertex, and in the two connected component case, one component can even have diameter 2.

We should note that in [**12, 17**], they show similar results for the common divisor graph for FC-groups (infinite groups where all conjugacy classes are finite). Not only can one characterize the groups G where $\Gamma'(G)$ is disconnected, it is possible to characterize groups G where $\Gamma'(G)$ is not a complete graph. In addition, one can study groups G where $\Gamma'(H)$ is a complete graph for every subgroup H of G.

Theorem 8.2. *Suppose G is a finite group.*

1. (Adami, Bianchi, Mauri, and Herzog [**1**]). *Then $\Gamma'(G)$ is not a complete graph if and only if there exist subgroups A and B and a set of primes π so that* (1) $G = AB$, (2) $\mathbf{C}_G(A) > \mathbf{Z}(G)$, (3) $\mathbf{C}_G(B) > \mathbf{Z}(B)$, (4) $|G|_\pi$ *divides* $|A|$ *and* (5) $|G|_{\pi'}$ *divides* $|B|$.
2. (Puglisi and Spiezia [**70**]). *Suppose $\Gamma'(H)$ is a complete graph for every subgroup H of G. Then G is solvable.*

Recall that if G is a nonabelian simple group, then the common divisor graph for $\mathrm{cs}(G)$ is a complete graph. This last result says that such a group G must have a subgroup H so that the common divisor graph for $\mathrm{cs}(H)$ is not a complete graph.

An interesting result related to the common divisor graph for $\mathrm{cs}(G)$ has been proved by Bianchi, Gillio and Casolo in [**15**]. They proved for a finite group G that $\Gamma'(G)$ is disconnected if and only if the two largest elements in $\mathrm{cs}(G) - \{1\}$ are relatively prime.

The prime vertex graph for $\mathrm{cs}\,(G)$ has been studied by Dolfi and Alfandary. Using Lemma 3.2, we see that $n(\Delta'(G)) \leq 2$ and $n(\Delta'(G)) = 2$ if and only if G is quasi-Frobenius with abelian complement and kernel. When $\Gamma'(G)$ is disconnected, each connected component is an isolated vertex, and so, if $\Delta'(G)$ is disconnected, then each connected component is a complete graph. When $\Delta'(G)$ is connected, $\Gamma'(G)$ can be used to get a bound on the diameter of $\Delta'(G)$, which can be improved as follows. In addition, Dolfi showed that $\Delta'(G)$ satisfies the same condition that Pálfy showed for $\Delta(G)$.

Theorem 8.3. *Let G be a group.*

1. (Alfandary [**2**]). *If $\Delta'(G)$ is connected, then* $\mathrm{diam}\,(\Delta'(G)) \leq 3$.
2. (Dolfi [**21**]). *If G is solvable, then $\Delta'(G)$ satisfies Pálfy's condition.*

We note that Dolfi proved (1) for solvable groups in [**21**].

It is an old result of Itô that, if p and q are primes that occur as vertices in $\Delta'(G)$ and are not adjacent in $\Delta'(G)$, then either G is p-nilpotent or q-nilpotent, see [**31**]. In [**21**] Dolfi proved that if G is solvable, then both the Sylow p-subgroups and the Sylow q-subgroups of G are abelian. In [**3**] Alfandary proved that if G is solvable and G does not contain a certain subgroup of an affine semi-linear group, then $\Delta'(G)$ is composed of two complete graphs. In [**16**] Casolo and Dolfi characterized the groups G where $\Delta'(G)$ has diameter 3. In particular, they showed that if G is not solvable, then $\mathrm{diam}\,(\Delta'(G)) \leq 2$. The solvable groups with $\mathrm{diam}\,(\Delta'(G)) = 3$ that they obtained have a much less complicated structure than the group found in [**40**] of a solvable group where $\mathrm{diam}\,(\Delta(G)) = 3$.

Perhaps the most amazing and surprising result regarding the prime vertex graph for $\mathrm{cs}\,(G)$ is the following direct connection to the prime vertex graph for $\mathrm{cd}\,(G)$ that was proved by Dolfi.

Theorem 8.4 (Dolfi [**20**]). *Let G be a solvable group. Then $\Delta(G)$ is a subgraph of $\Delta'(G)$.*

9. Other graphs of character degrees. In this section we discuss other graphs that have been studied in the context of character degrees.

Let G be a group, and let p be a prime. We set $\mathrm{cBr}_p(G) = \{\phi(1) \mid \phi \in \mathrm{IBr}_p(G)\}$ (the degrees of the irreducible p-Brauer characters of G). The prime vertex graph for $\mathrm{cBr}_p(G)$ is studied in [**54, 55**]. They proved that if G is a solvable group, then the prime vertex graph for $\mathrm{cBr}_p(G)$ has at most 2 connected components. In this case, if $\Delta(\mathrm{cBr}_p(G))$ is disconnected, then one component has diameter at most 3 and the other component has diameter at most 2, and if it is connected, then its diameter is at most 5. For general groups, the situation is much different. If $p > 2$ is a prime, it is known that the irreducible p-Brauer characters of $\mathrm{SL}(2, p)$ have degrees $1, 2, 3, \ldots, p-1, p$, see [**26**, Example VII.3.10]. For any integer m, we can find a prime p so that the number of connected components of the prime vertex graph for $\mathrm{cBr}_p(\mathrm{SL}(2, p))$ is at least m. (This is the only graph associated with a group that we encounter where the number of connected components cannot be bounded, or in fact can be greater than 3.)

Let A act coprimely via automorphisms on a group G. We define $\mathrm{cd}_A(G)$ to be the set of degrees of the irreducible A-invariant characters of G. We define $\Delta_A(G) = \Delta(\mathrm{cd}_A(G))$. Beltrán has proved that this graph shares some properties with $\Delta(G)$ in [**6**].

Theorem 9.1 (Beltrán [**6**]). *Suppose a group A acts coprimely via automorphisms on a solvable group G. Then $\Delta_A(G)$ has at most two connected components.*

Let π be a set of primes. We set $\mathrm{cd}_\pi(G)$ to be the subset of $\mathrm{cd}(G)$ consisting of those degrees that are π-numbers. We define $\Delta_\pi(G) = \Delta(\mathrm{cd}_\pi(G))$. Observe that $\rho(\mathrm{cd}_\pi(G)) \subseteq \rho(G) \cap \pi$, and we do not necessarily have equality. In [**45**], we studied this graph, and we proved the following.

Theorem 9.2 (Lewis, McVey, Moretó, and Sanus [**45**]). *Let π be a set of primes, and suppose that G is a π-solvable group. Then $\Delta_\pi(G)$ has at most two connected components.*

Since $\mathrm{cd}_\pi(G)$ appears to be an arbitrary subset of $\mathrm{cd}(G)$, it is somewhat surprising that $\Delta_\pi(G)$ shares any property with $\Delta(G)$. In [**45**], we also bound the diameter of $\Delta_\pi(G)$.

Theorem 9.3 (Lewis, McVey, Moretó, and Sanus [**45**]). *Let π be a set of primes, and suppose that G is a π-solvable group.*

1. *If $\Delta_\pi(G)$ is disconnected, then each connected component has diameter at most* 3.
2. *If $\Delta_\pi(G)$ is connected, then* $\operatorname{diam}(\Delta_\pi(G)) \leq 6$.

We do not know of any examples where the diameter of $\Delta_\pi(G)$ actually meets the bounds found in this theorem. In fact, we do not know of any examples where $\Delta_\pi(G)$ has diameter exceeding 3 (or 2 if $\Delta_\pi(G)$ is disconnected). One question that arises is: does there exist an invariant of G which is bounded when $\Delta_\pi(G)$ is disconnected?

We should mention that Theorem 9.2 generalizes Theorem 9.1. Let A act coprimely via automorphisms on a group G. Let $\pi = \rho(G)$. Looking at the semi-direct product GA, one can show that any character in $\operatorname{Irr}(GA)$ having π-degree is an extension of an A-invariant irreducible character of G and that any A-invariant irreducible character of G extends to an irreducible character of GA having π-degree. It follows that $\operatorname{cd}_A(G) = \operatorname{cd}_\pi(GA)$, and so $\Delta_A(G) = \Delta_\pi(GA)$.

Suppose N is a normal subgroup of a group G and $\theta \in \operatorname{Irr}(N)$. The set of irreducible constituents of θ^G is $\operatorname{Irr}(G \mid \theta)$ and $\operatorname{cd}(G \mid \theta) = \{\chi(1) \mid \chi \in \operatorname{Irr}(G \mid \theta)\}$. Now, define the graph $\Delta(G \mid \theta) = \Delta(\operatorname{cd}(G \mid \theta))$. Moretó and Sanus studied this graph in [**65**]. They obtained the following results.

Theorem 9.4 (Moretó and Sanus [**65**]). *Let N be a normal subgroup of a group G, and let $\theta \in \operatorname{Irr}(N)$.*

1. *Then $n(\Delta(G \mid \theta)) \leq 3$.*
2. *If G/N is solvable, then $n(\Delta(G \mid \theta)) \leq 2$.*
3. *If G/N is solvable, then* $\operatorname{diam}(\Delta(G \mid \theta)) \leq 4$.

Notice that, if $N = 1$, then $\theta = 1$ and $\operatorname{cd}(G \mid \theta) = \operatorname{cd}(G)$, so $\Delta(G \mid \theta)$ can be viewed as a generalization of $\Delta(G)$. In particular, it is not possible to improve (1) or (2) of Theorem 9.4. Moretó and Sanus use this theorem to study the following situation. Let G be a group, let p be a prime and let B be a Brauer p-block of G. We define $\operatorname{Irr}(B)$ to be the set of irreducible characters of G that lie in B and

$\mathrm{cd}(B) = \{\chi(1) \mid \chi \in \mathrm{Irr}(B)\}$. We define $\Delta(B) = \Delta(\mathrm{cd}(B))$. They have the following result.

Theorem 9.5 (Moretó and Sanus [**65**]). *Let p be a prime, and let B be a Brauer p-block of the p-solvable group G.*

1. *Then $n(\Delta(B)) \leq 3$.*
2. *If G is solvable, then $n(\Delta(B)) \leq 2$.*
3. *If G is solvable, then* $\mathrm{diam}(\Delta(B)) \leq 4$.

Let N be a normal subgroup of G. We define $\mathrm{cd}(G \mid N)$ to be the union of $\mathrm{cd}(G \mid \theta)$ as θ runs over the nonprincipal characters in $\mathrm{Irr}(N)$. Hence, $\mathrm{cd}(G) = \mathrm{cd}(G/N) \cup \mathrm{cd}(G \mid N)$. If $\theta \in \mathrm{Irr}(N)$ is not the principal character, then $\mathrm{cd}(G \mid \theta) \subseteq \mathrm{cd}(G \mid N)$. We define the graphs $\Gamma(G \mid N) = \Gamma(\mathrm{cd}(G \mid N))$ and $\Delta(G \mid N) = \Delta(\mathrm{cd}(G \mid N))$. Note that $\mathrm{cd}(G \mid G') = \mathrm{cd}(G) \setminus \{1\}$, and so $\Gamma(G \mid G') = \Gamma(G)$ and $\Delta(G \mid G') = \Delta(G)$. We can view these graphs as generalizations of the graphs for G. The first paper on this graph is [**29**], where Isaacs proved the following results, among others.

Theorem 9.6 (Isaacs [**29**]). *Let N be a solvable normal subgroup of a group G such that $N \subseteq G'$.*

1. *Then $n(\Gamma(G \mid N)) \leq 2$.*
2. *If $\Gamma(G \mid N)$ is disconnected, then some prime divides all the degrees in one connected component and the other connected component has diameter at most* 2.
3. *If $\Gamma(G \mid N)$ is connected, then* $\mathrm{diam}(\Gamma(G \mid N)) \leq 4$.
4. *If $\Gamma(G \mid N)$ is connected and N is nilpotent, then* $\mathrm{diam}(\Gamma(G \mid N)) \leq 3$.

In fact, Isaacs was able to prove the first conclusion and a weaker version of the second conclusion when he replaced N solvable by the weaker hypothesis that $N' < N$. We do not know any examples where $\Gamma(G \mid N)$ has diameter 4, and we would not be surprised if the correct bound is 3.

In [**29**], Isaacs was mainly concerned with the connection between $\mathrm{cd}\,(G \mid N)$ and the structure of N. However, Isaacs did study one connection between $\Gamma(G \mid N)$ and the structure of G. In [**29**], he included the conjecture that if $N \subseteq G'$, N is normal and solvable, and $\Gamma(G \mid N)$ is disconnected, then G is solvable. This was a conjecture that we had communicated to him when [**29**] was being prepared, and that paper includes strong evidence that the conjecture is true. Recently, Moretó has proved this conjecture, and in fact, he was able to prove a stronger result. In particular, he did not need to assume that N was solvable.

Theorem 9.7 (Moretó [**62**]). *Let N be a normal subgroup of a group G with $N \subseteq G'$. If $\Delta(G \mid N)$ is disconnected, then G/N is solvable.*

Motivated by Moretó's result, we further studied $\Delta(G \mid N)$. Among other things, we are able to remove the hypothesis that $N \subseteq G'$. We also have the following result:

Theorem 9.8 [**43**]. *Let N be a normal subgroup of a group G.*

1. *If $\Delta(G \mid N)$ is disconnected, $N \subseteq G'$ and N is not solvable, then $N = G'$.*
2. *If $\Delta(G \mid N)$ is disconnected and G is solvable, then G has Fitting height at most* 4*, and there is a normal subgroup B in G so that $|G : B|$ divides* 2 *and $\Delta(G \mid N) = \Delta(B)$.*
3. *If $G'' < G'$, then $\Delta(G \mid N)$ has diameter at most* 4.
4. *If G is solvable, then $\Delta(G \mid N)$ has diameter at most* 3.

At this time, most of the results regarding $\Gamma(G \mid N)$ and $\Delta(G \mid N)$ have shown the similarities between these graphs and the graphs $\Gamma(G)$ and $\Delta(G)$. In his Ph.D. dissertation [**56**], McQuistan has shown the first difference between $\Delta(G)$ and $\Delta(G \mid N)$. He has that, for every positive integer k and prime p, there exist distinct primes $p_1, p_2, \ldots, p_k$ and a solvable group G with normal subgroup N so that

$$\mathrm{cd}\,(G \mid N) = \{p^{nk}, p^{n(k-1)}, p_1 \cdot p^{n(k-1)}, p_2 \cdot p^{n(k-1)}, \ldots, p_k \cdot p^{n(k-1)}\}$$

for some integer n. In particular, the graph $\Delta(G \mid N)$ will not satisfy

Pálfy's condition, and in fact, this graph will not satisfy even a weaker version of Pálfy's condition.

Lo also studied $\mathcal{N}_*(r)$ when $r \in \mathrm{cd}\,(G \mid N)$ and $\mathcal{G} = \Gamma(G \mid N)$. He showed that if G is solvable, then $\mathcal{N}_*(r)$ has at most 2 connected components in this case. When G is any group with $N \subseteq G'$ and N is nilpotent, he is able to show that $\mathcal{N}_*(r)$ will have at most 4 isolated vertices.

As we were completing the research for this paper, we came across one other graph related to the character degrees of a group. Given a character $\chi \in \mathrm{Irr}\,(G)$, the co-degree of χ is the quotient $|G|/\chi(1)$. We define $\Sigma(G) = \{|G|/\chi(1) \mid \chi \in \mathrm{Irr}\,(G), \chi(1) \neq 1\}$. In [**18**], Chillag, Mann, and Manz study the graphs associated with $\Sigma(G)$. In particular, they proved that $\Gamma(\Sigma(G))$ is always a complete graph, see [**18**, Proposition 1.1]. It follows that $\Delta(\Sigma(G))$ has diameter at most 2. They show more in this case. In particular, they show that $\Delta(\Sigma(G))$ has a complete subgraph so that every other vertex is adjacent to some vertex in the subgraph. (In the terminology mentioned earlier, $\Delta(\Sigma(G))$ has a connective subset.)

10. Other graphs of conjugacy class sizes. We have just seen that several subgraphs and generalizations of the character degree graphs have been studied. In the same manner, several subgraphs and generalizations of the graphs associated with conjugacy class sizes have been considered. Let p be a prime, and let G be a group. The class analog of p-Brauer characters are the p-regular classes of G which we denote by $\mathrm{class}_p(G)$. We define $\mathrm{cs}_p(G) = \{|C| \mid C \in \mathrm{class}_p(G)\}$. We set $\Delta'_p(G) = \Delta(\mathrm{cs}_p(G))$ and $\Gamma'_p(G) = \Gamma(\mathrm{cs}_p(G))$. The prime vertex graph for $\mathrm{cs}_p(G)$ has been studied in [**51**]. They proved the following.

Theorem 10.1 (Lu and Zhang [**51**]). *Let p be a prime, and let G be a finite p-solvable group.*

1. *Then $n(\Delta'_p(G)) \leq 2$.*
2. *If $\Delta'_p(G)$ is disconnected, then each connected component is a complete graph.*

Lu and Zhang also obtained some additional information regarding the structure of G when $\Delta'_p(G)$ is disconnected. The common divisor graph for $\mathrm{cs}_p(G)$ has been studied by Beltrán and Felipe. In [**51**], Lu and Zhang had proved that if $\Delta'_p(G)$ is connected, then it has diameter at most 6. Beltrán and Felipe were able to improve this result.

Theorem 10.2 (Beltrán and Felipe). *Let p be a prime, and let G be a p-solvable group.*

1. [**8**]. *If $\Gamma'_p(G)$ is connected, then* $\operatorname{diam}(\Gamma'_p(G)) \leq 3$.
2. [**8**]. *If $\Gamma'_p(G)$ is disconnected, then each connected component is a complete graph.*
3. [**9**]. *If $\Delta'_p(G)$ is connected, then* $\operatorname{diam}(\Delta'_p(G)) \leq 3$.

In the paper [**10**], Beltrán and Felipe begin to characterize the p-solvable groups G where $\Gamma'_p(G)$ is disconnected, but their characterization is not complete.

We can also look at conjugacy classes when the group A acts coprimely on G. We define $\mathrm{cs}_A(G)$ to be the sizes of the noncentral A-invariant conjugacy classes of G. We set $\Gamma'_A(G) = \Gamma(\mathrm{cs}_A(G))$. In [**9**], Beltrán studied this graph. He proves the following.

Theorem 10.3 (Beltrán [**7**]). *Let A act coprimely via automorphisms on a group G.*

1. *Then $n(\Gamma'_A(G)) \leq 2$.*
2. *If $\Gamma'_A(G)$ is connected, then* $\operatorname{diam}(\Gamma'_A(G)) \leq 4$.
3. *If $\Gamma'_A(G)$ is disconnected, then G is solvable and each connected component has diameter at most* 2.

An interesting generalization of $\Gamma'(G)$ is the following. Let G act transitively on a set Ω. The subdegrees $D(G, \Omega)$ are the cardinalities of the orbits of a point stabilizer G_α on Ω. Since the action of G is transitive, the values in $D(G, \Omega)$ are independent of α. We set $\Gamma(G, \Omega) = \Gamma(D(G, \Omega))$. In [**30**], Isaacs and Praeger studied this graph and obtained the following results.

Theorem 10.4 (Isaacs and Praeger [**30**]). *Let G act transitively on Ω, and assume that all the elements of $D(G, \Omega)$ are finite.*

1. *Then* $n(\Gamma(G,\Omega)) \leq 2$.

2. *If* $\Gamma(G,\Omega)$ *is connected, then* $\mathrm{diam}\,(\Gamma(G,\Omega)) \leq 4$.

3. *If* $\Gamma(G,\Omega)$ *is disconnected, then one connected component is a complete graph and the other one has diameter at most* 2.

To see that $\Gamma(G,\Omega)$ is a generalization of $\Gamma'(G)$, we consider the following situation. Let H act via automorphisms on a group G, and let K be the resulting semi-direct product. We define Ω to be the set of conjugacy classes of H in K, and note that G acts transitively on Ω. It is not difficult to show that $D(G,\Omega) = \mathrm{cs}\,(G)$ when $H = G$.

11. Applications. We feel obligated to mention the applications of these graphs. We initially encountered these graphs in the first paper we wrote [**33**]. In that paper, we used the fact that $\Gamma(G)$ has at most two connected components to obtain a bound on $|\mathrm{cd}\,(G)|$ when G is a solvable group satisfying what we termed the one-prime hypothesis. We apply the classification in [**36**] of disconnected graphs of solvable groups to other character degree problems in [**39**] and [**46**]. The first of these two papers improves a bound on the derived length of a solvable group satisfying the one-prime hypothesis. The second paper is concerned with solvable groups where there is no divisibility among distinct nontrivial character degrees. McVey applies the facts about $\Gamma(G)$ when it is disconnected to obtain his results in [**57**].

Two open conjectures regarding character degrees have been proved when $\Gamma(G)$ is disconnected. In [**68**], Pálfy showed that the $\rho - \sigma$ conjecture is true when $\Gamma(G)$ is disconnected. We have shown that the Taketa problem has a positive answer when $\Gamma(G)$ is disconnected in [**34**]. We made use of this fact in [**38**] to show that $\mathrm{dl}\,(G) \leq |\mathrm{cd}\,(G)|$ when $|\mathrm{cd}\,(G)| = 5$. One example where $\Gamma'(G)$ has been applied to other problems regarding conjugacy classes is [**14**].

12. Further structure. The following has been proposed by Isaacs. As in Section 3, we let X be a set of positive integers. We associate two simplicial complexes with X. The first one we call the *prime complex*, and it has as its simplexes all sets of primes that divide some integer in X. The second one is the *common-divisor complex*, and its simplexes will be all subsets of X^* that have a nontrivial common divisor. For a nonnegative integer n, the n-simplexes are all simplexes of size $n+1$, and the n-skeleton is the simplicial complex obtained by taking all k-simplexes with $k \leq n$. The 0-simplexes form the vertices of the simplicial complex. The 1-simplexes of the prime complex for X will be the edges of $\Delta(X)$, and so the 1-skeleton of the prime complex is $\Delta(X)$. Similarly, the 1-skeleton of the common divisor complex for X will be $\Gamma(X)$.

As far as we know, no results have been proven regarding these complexes, but we would like to close this paper with some conjectures regarding these complexes and the sets considered in this paper. Recall that the dimension of a simplicial complex is one less than the size of the largest simplex. For an integer a, we define $\sigma(a)$ to be the number of distinct primes dividing a. For a set of positive integers X, the simplexes of the prime complex correspond to the set of distinct primes dividing elements of X^*, so the dimension of the prime complex for X will be one less than the maximum of $\sigma(a)$ for $a \in X^*$. The simplexes in the common divisor complex of X are sets divisible by a common prime, so the dimension of the common divisor complex is one less than the maximal number of elements of X^* divisible by a single prime.

When G is a group, we define $\sigma(G)$ to be the maximum of $\sigma(a)$ for $a \in \mathrm{cd}\,(G)$. Therefore, $\sigma(G) - 1$ will be the dimension of the prime complex for $\mathrm{cd}\,(G)$. The $\rho - \sigma$ conjecture states for any group G that $|\rho(G)| \leq 3\sigma(G)$ and when G is solvable that $|\rho(G)| \leq 2\sigma(G)$. If this conjecture is proven, then this says that the number of vertices in the prime complex for $\mathrm{cd}\,(G)$ is bounded in terms of the dimension of the complex. At this time, it is known that $|\rho(G)| \leq 3\sigma(G) + 2$ when G is solvable and $|\rho(G)| \leq 3\sigma(G)$ when G is simple, see [**4, 24, 55**]. Recently, Moretó has proved in [**64**] that $|\rho(G)| \leq 4\sigma(G)^2 + 6.5\sigma(G) + 13$ for a general group G.

Similarly, we define $\sigma'(G)$ to be the maximum of $\sigma'(a)$ for $a \in \mathrm{cs}\,(G)$. Notice that $\sigma'(G) - 1$ will be the dimension of the prime complex for $\mathrm{cs}\,(G)$. When G is solvable, Zhang has shown that $|\rho'(G)| \leq 4\sigma'(G)$ [**77**]. For nonsolvable groups G, Moretó showed in [**64**] that $|\rho'(G)| \leq 3\sigma'(G)^2 + 7.5\sigma'(G) + 11$. In this case, the number of vertices of the prime complex for $\mathrm{cs}\,(G)$ will be bounded in terms of the dimension of the complex.

In [**57**], McVey conjectured that some prime must divide at least one third of the elements in $\mathrm{cd}(G)$ when G is a solvable group. If we let $k(G)$ be the maximal number of degrees in $\mathrm{cd}(G) \setminus \{1\}$ divisible by any prime p, then the dimension of the common divisor complex for $\mathrm{cd}(G)$ is $k(G) - 1$. McVey's conjecture is that $|\mathrm{cd}(G)| \leq 3k(G)$ when G is solvable. This would say that the number of vertices in common divisor complex for $\mathrm{cd}(G)$ is bounded in terms of its dimension. In [**11**], Benjamin has shown that $|\mathrm{cd}(G)|$ is bounded by a quadratic function in $k(G)$ when G is solvable.

In closing, the questions regarding these simplicial complexes are the same as the questions regarding the graphs. Which simplicial complexes arise in these situations? If some group is known to have some simplicial complex that can be said about the group? The last several paragraphs seem to suggest that the number of vertices in the simplicial complexes is bounded in terms of the dimensions of the complexes. What other limits can be placed on the structure of these complexes?

Added in proof. Since this paper was accepted, I have been made aware of several results:

1. Dolfi has shown that the hypothesis that G is solvable is not necessary for Theorem 8.3.2. See S. Dolfi, "On independent sets in the class graph of a finite group," J. Algebra **303** (2006), 216–224.

2. Cassolo and Dolfi have shown that the hypothesis that G is solvable is not necessary for Theorem 8.4 in the preprint, "Products of primes in conjugacy class sizes and irreducible character degrees."

3. Cassolo and Dolfi have shown that the quadratic inequalities of Moretó in Section 12 can be improved to linear inequalities. In particular, if G is any group, they show that $|\rho(G)| \leq 7\sigma(G)$ and $|\rho'(G)| \leq 5\sigma'(G)$. See C. Cassolo and S. Dolfi, "Prime divisors of irreducible character degrees and of conjugacy class sizes in finite groups," J. Group Theory **10** (2007), 571–583.

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