ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 4, Fall 1989

A LOCALIZATION OPERATOR FOR RATIONAL MODULES

JAMES LI-MING WANG

Let X be a compact subset of the complex plane $\mathbf{C}$ and let g be a continuous function on X. We denote by $\mathcal{R}(X, g)$ the rational module

$$\{r_0(z) + r_1(z)g(z)\},$$

where each r_i denotes a rational function with poles off X.

In the case that $g(z) = \bar{z}$, the closure of $\mathcal{R}(X, \bar{z})$ in various topologies was first considered by O'Farrell [**4**] and was applied to rational approximation problems in Lipschitz norm. Later, several authors (e.g., Carmona, Trent, Verdera and Wang) have gone into the subject. A question which arose from these investigations concerned the characterization of $R(X, g)$, the uniform closure of $\mathcal{R}(X, g)$ in $C(X)$ when X has empty interior $\dot{X}$. This was settled in [**5**] (also see [**1**]) by showing that $R(X, g) = C(X)$ if and only if $R(Z) = C(Z)$ where Z is the subset of X on which $\bar{\partial} g$ vanishes. Here $\bar{\partial}$ is the usual Cauchy-Riemann operator in the complex plane.

The existence of interior points, however, makes the problem more difficult. It is natural to ask the following question: Is

$$R(X, g) = \{f \in C(X) : \bar{\partial}(\bar{\partial} f / \bar{\partial} g) = 0 \text{ in } \dot{X}\}$$

whenever $\bar{\partial} g \neq 0$ on an arbitrary compact set X? In particular, when $g(z) = \bar{z}$, this should be viewed as the Mergelyan approximation problem for the operator $\bar{\partial}^2 = \bar{\partial} \circ \bar{\partial}$:

$$(*) \qquad \text{Is } R(X, \bar{z}) = \{f \in C(X) : \bar{\partial}^2 f = 0 \text{ in } \dot{X}\}$$

for an arbitrary compact set X?

For the case when X is a compact set whose complement is connected, the approximation problem is not too difficult. In [**1**], a standard

Received by the editors on November 10, 1986 and in revised form on February 12, 1987.

argument by Mergelyan [3] is extended to obtain a positive result for question (*). Because the module $R(X, \bar{z})$ is local (see [4], also [5] for other local modules), (*) is also true for any compact set X whose complement has a finite number of components, or for those compact sets X such that the diameters of the components of the complement are bounded away from zero. However, the general case remains unknown.

In this note, we examine the localization operator for $\bar{\partial}^2$. We can improve the localization theorem to handle isolated bad points (cf. [2, p. 52]).

We denote by m the Lebesgue measure on the complex plane $\mathbf{C}$. Let μ be a compactly supported Borel measure on $\mathbf{C}$. We write $\hat{\mu}(z) = \int \frac{d\mu(\xi)}{\xi - z}$ for the Cauchy transform and $\tilde{\mu}(z) = \int \frac{\bar{\xi}-\bar{z}}{\xi - z} d\mu(\xi)$. If $\phi \in L^1_{\ell oc}(m)$ has compact support, then we write $\hat{\phi} = \hat{\phi m}$ and $\tilde{\phi} = \tilde{\phi m}$.

Let f be a continuous function on S^2, the Riemann sphere, and ϕ be a twice-continuously differentiable function on $\mathbf{C}$ with compact support. We define the localization operator V_ϕ by

$$V_\phi(f) = f\phi + \frac{2}{\pi}(f \cdot \bar{\partial}\phi)\hat{} + \frac{1}{\pi}(f \cdot \bar{\partial}^2\phi)^{\sim}.$$

We then have $\bar{\partial}^2 V_\phi(f) = \phi \cdot \bar{\partial}^2 f$ in the sense of distribution and $V_\phi(f)$ is again continuous on $\mathbf{C}$.

LEMMA. *Let $f \in C(S^2)$ and ϕ be a twice-continuously differentiable function supported on the disk $\Delta(z_0; \delta)$ with center z_0 and radius δ. Then*

$$||V_\phi(f)||_\infty \leq Cw(f; \delta)(||\phi||_\infty + ||\bar{\partial}\phi||_\infty \cdot \delta + ||\bar{\partial}^2\phi||_\infty \cdot \delta^2),$$

where $w(f; \delta)$ is the modulus of continuity of f and $|| \ ||_\infty$ is the usual sup *norm.*

PROOF. Note that

$$|(f \cdot \phi)(z)| \leq \sup_{|\xi - z_0| \leq \delta} |f(\xi)| \cdot ||\phi||_\infty,$$

$$\begin{aligned} |(f \cdot \bar{\partial}\phi)\hat{}(z)| &= \left| \int \frac{(f \cdot \bar{\partial}\phi)(\xi)}{\xi - z} dm(\xi) \right| \\ &\leq C \sup_{|\xi - z_0| \leq \delta} |f(\xi)| \cdot ||\bar{\partial}\phi||_\infty \delta \end{aligned}$$

and

$$\begin{aligned} |(f \cdot \bar{\partial}^2\phi)^\sim (z)| &= \left| \int \frac{\bar{\xi} - \bar{z}}{\xi - z} (f \cdot \bar{\partial}^2\phi)(\xi) dm(\xi) \right| \\ &\leq C \sup_{|\xi - z_0| \leq \delta} |f(\xi)| \cdot ||\partial^2\phi||_\infty \cdot \delta^2. \end{aligned}$$

The lemma follows because $V_\phi(f - \alpha) = V_\phi(f)$ for any constant α. □

THEOREM. *Let $f \in C(S^2)$ such that $\bar{\partial}^2 f = 0$ on the open subset U of $\mathbf{C}$. Let $z_0 \in \mathbf{C}$. Then there is a sequence $\{f_n\}$ of continuous functions such that $\bar{\partial}^2 f_n = 0$ on U and a neighborhood of z_0, and $f_n \to f$ uniformly on $\mathbf{C}$.*

PROOF. We can assume that $z_0 = 0$. Let $\{\phi_n\}$ be a sequence of twice-continuously differentiable functions such that $g_n(z) = 0$ when $|z| \geq 2/n$, $g_n(z) = 1$ when $|z| \leq 1/n, |\bar{\partial} g_n| \leq 2n$ and $|\bar{\partial}^2 g_n| \leq 4n^2$. Then the lemma implies $V_{\phi_n}(f)$ tends uniformly to zero on $\mathbf{C}$ and the functions $f_n = f - V_{\phi_n}(f)$ do the trick. □

COROLLARY. *Let X be a compact set obtained from the closed unit disk by deleting a sequence of open disks where radii tend to zero, and whose centers accumulate on a set E which is at most countable. Then*

$$R(X, \bar{z}) = \{f \in C(X) : \bar{\partial}^2 f = 0 \textit{ in } \dot{X}\}.$$

PROOF. Let F be the set of points in X which have no neighborhood U satisfying $R(X \cap \overline{U}, \bar{z}) = \{f \in C(X \cap \overline{U}) : \bar{\partial}^2 f = 0 \text{ in } \dot{X} \cap U\}$.

Evidently F is closed, and $F \subseteq E$. In view of the Theorem and the fact that $R(X, \bar{z})$ is local, F has no isolated points. By the Baire category theorem, F must be empty and hence

$$R(X, \bar{z}) = \{f \in C(X) : \bar{\partial}^2 f = 0 \text{ in } \dot{X}\}.$$

REMARK. 1) The main lemma used by Mergelyan [**3**] can be extended so that $(*)$ is also true for any compact set satisfying the following capacity condition (see [**1**]):

$$\gamma(\Delta(z, r) - X) \geq Cr$$

for some positive constant C, for every point z on the boundary of X, and for all sufficiently small r, where γ is the analytic capacity [**2, 6**]. Following a similar scheme for approximation used by Vitushkin [**6**] one can prove that $(*)$ is true if the inner boundary of X is empty, where the inner boundary of X is the set of boundary points of X not belonging to the boundary of a component of the complement of X. Thus the localization argument shows that $(*)$ is true if the inner boundary of X is at most countable.

2) The argument used in this note can be generalized to other rational modules $R(X, g)$ whenever $R(X, g)$ is local.

REFERENCES

1. J.J. Carmona, *Mergelyan's approximation theorem for rational modules,* J. Approx. Theory **44** (1985), 113-126.

2. T.W. Gamelin, *Uniform Algebra,* Prentice-Hall, Englewood Cliffs, N.J., 1969.

3. S.N. Mergelyan, *Uniform approximations to functions of a complex variable,* Amer. Math. Soc. Transl. **101** (1954), 294-391.

4. A.G. O'Farrell, *Annihilators of rational modules,* J. Functional Analysis, **19** (1975), 373-389.

5. T.T. Trent and J. Wang, *The uniform closure of rational modules,* Bull. London Math. Soc. **13** (1981), 415-420.

6. A.G. Vitushkin, *Analytic capacity of sets and problems in approximation theory,* Russian Math. Surveys **22** (1967), 139-200.

MATHEMATICS DEPARTMENT, UNIVERSITY OF ALABAMA, TUSCALOOSA, AL 35487