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NASH FUNCTIONS AND THE STRUCTURE SHEAF

FELIPE CUCKER

Let $\mathbf{R}$ be a real closed field and $U \subset \mathbf{R}^n$ an open semialgebraic (s.a.) set. A Nash function over U is a function of class C^∞ and s.a. If $\mathbf{R} = \mathbb{R}$, this definition agrees with the usual one [**1**, Chapter 8].

If $A = \mathbf{R}[X_1, \ldots, X_n]$ and $\tilde{U}$ is the constructible set in $\operatorname{Spec}_r A$ associated to U, the ring of Nash functions over U is canonically isomorphic to the ring $\mathcal{N}_A(\tilde{U})$ of global sections over $\tilde{U}$ of the structure sheaf of $\operatorname{Spec}_r A$. This is a consequence of the Artin-Mazur description of Nash functions and the behavior of the operator $\sim$. For this result and other basic properties of the structure sheaf see M.-F. Roy's article [**8**].

Now, let $V \subset \mathbf{R}^n$ be an algebraic variety (not necessarily smooth) and let A be its coordinate ring. In the above quoted article, we observe that if N_V is the sheaf obtained by restriction and identification of elements of $\mathcal{N}_{R[X_1,\ldots,X_n]}$ over $\operatorname{Spec}_r A$, this sheaf does not necessarily coincide with $\mathcal{N}_A$. Moreover, an example of a variety for which these sheaves differ is given, the study of the relationship between them is proposed and it is conjectured (for $\mathbf{R} = \mathbb{R}$) that the set of points of V, for which the stalks of both sheaves are isomorphic, is the set of quasi-regular points of V in Tognoli's sense.

To answer these questions, our first results are the following theorems.

THEOREM 1. ([**2**, II. 1.5]. or [**3**, 1.7]) *For every $\alpha \in \operatorname{Spec}_r A$ the stalk $N_{V,\alpha}$ is naturally isomorphic to $\mathcal{N}_{A,\alpha}/\operatorname{rad}_r(0)$.*

THEOREM 2. ([**2**, II. 2.1.], or [**3**, 2.1]) *Let $x \in V$. The following statements are equivalent:*

(i) *x is quasi-regular (i.e., the complexification of the Nash germ V_x coincides with the complex Nash germ at x of the algebraic complexification V_c of V);*

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(ii) *The henselization* ${}^h(A_x)$ *of the local ring* A_x *is real.*

Since the stalk at x of the structure sheaf $\mathcal{N}_{A,x}$ is just the ring ${}^h(A_x)$, the answer to Roy's question is now evident from the theorems above.

With respect to the relationship between both sheaves - which is related to isoalgebraic functions (see [**6**]) - we can summarize our results in

THEOREM 3. ([**2**, II. §1] or [**3**])

(i) *There is a surjective sheaf morphism* $\varphi : \mathcal{N}_A \to N_V$ *which induces in the stalks the natural projections given in Theorem* 1.

(ii) *Let* $U \subset V$ *be a s.a. open subset and* $\varphi_U : \mathcal{N}_A(\tilde{U}) \to N_V(\tilde{U})$ *the ring morphism on the global sections rings. Then* φ_U *is surjective if and only if* $\forall f \in N_V(\tilde{U})$, $\exists\ U' \subset V_c$ *an open s.a. subset containing* U, *and* $\exists\ g$ *an isoalgebraic function on* U' *such that* $\forall x \in U f(x) = g(x)$. *This is, for example, the case if* V *is a curve.*

Our last result concerns algebraic properties of the ring of global sections of the structure sheaf.

THEOREM 4. ([**2**, III.1.7., III 2.3. and III 3.4] or [**4**]) *Let* $U \subset V$ *be an open s.a. subset. Then*

(i) $\max \mathcal{N}_A(\tilde{U}) \approx U$;

(ii) $\forall x \in U\ {}^h(\mathcal{N}_A(\tilde{U})_m) \approx \mathcal{N}_{A,x}$;

(iii) *If* $\dim U$ *is* $\max\{\dim A_x / x \in U\}$, *the ring* $\mathcal{N}_A(\tilde{U})$ *is an excellent ring of dimension equal to* $\dim U$.

REMARKS. (i) Part (iii) of this last result is not known for $N_V(\tilde{U})$ for an arbitrary U, since even the dimension of this ring is unknown. For some positive results see [**7**].

Our results give part (iii) of theorem 4 for $N_V(\tilde{U})$ in case V is a curve or U is contained in the set of quasi-regular points of V.

(ii) Several nullstellensätze and the positive answer for Hilbert's 17$^{\text{th}}$

problem can be shown for $\mathcal{N}_A(\tilde{U})$. See [**5**].

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DEPT. DE LLENGUATGES I SISTEMES INFORMÀTICS, FACULTAT D'INFORMÀTICA DE BARCELONA, PAU GARGALLO 5, BARCELONA 08028, SPAIN