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FUNDAMENTALS OF ANALYSIS OVER SURREAL NUMBERS FIELDS

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ABSTRACT. The Tarski Principle informs us that, concerning first order statements, all real-closed fields are created equal. Thus the field $\mathbf{R}$ of all real-numbers and the field $\mathbf{R}_0$ of all real-algebraic numbers have the same first order theory; however, their higher order theories are quite different. For example, $\mathbf{R}$ is Dedekind-complete and is a vast transcendental extension of its prime field, whereas $\mathbf{R}_0$ is not Dedekind-complete and is an algebraic extension of its prime field. The surreal number fields $\xi\mathbf{No}$ are all real-closed. They have extraordinary higher order properties, which allow one to do analysis over them, as we will see below.

0. Introduction. The construction of the class, On, of all von Neumann ordinals is, in many ways, quite similar to some of the most instructive constructions of the surreal numbers. Let us recall von Neumann's definition. (For convenience, let us work within Kelley-Morse set theory. See, e.g., [**11**, Chapter 2] for details.)

A class A will be called *ε-transitive* if, for all sets x, y, for which $x \in y$ and $y \in A$, then $x \in A$. A is called an *ordinal* if it and each element in it is ε-transitive. Let On be the class of all ordinals. It is easy to see that the empty set is an ordinal, which is defined to be 0. Given $\alpha \in On$, let $\alpha + 1$ be defined to be the union of α and $\{\alpha\}$; then $\alpha + 1$ is in On. Clearly $0, 1, \ldots, n$ are finite ordinals. Let ω be defined to be the union of the set of all finite ordinals. ω is the smallest infinite ordinal. For all α, $\beta \in On$, one and only one of the following hold: $\alpha \in \beta$, $\beta = \alpha$, or $\beta \in \alpha$. (See, e.g., [**11**, pp. 68-75] for proofs and details.) One defines $\alpha < \beta$ in On if $\alpha \in \beta$, and one finds that, under this ordering, On is a well-ordered class. $\beta \in On$ is called a *limit ordinal* if there is no $\alpha \in On$ such that $\beta = \alpha + 1$. β is called a *non-limit ordinal* if it is not a limit ordinal. Thus, for example, ω is a limit ordinal, whereas 2 is a non-limit ordinal.

One interesting property of the (von Neumann) ordinals is that they are canonical objects in set theory. This follows from the fact that there

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is only one empty set and that, using transfinite induction, the ordinals are built up from the empty set, in a canonical fashion, as suggested above.

Let us define a *cardinal number* to be an element $\beta \in On$ such that β is not equipotent with any $\alpha \in \beta$. Thus $0, 1, \ldots, n$ and ω are cardinal numbers. However, $\omega + 1$ is not a cardinal number. For any ordinal number α let α^+ denote the least cardinal that is greater than α; thus, e.g., $2^+ = 3$. Assume, for some $\gamma \in On$, that $(\omega_\alpha)_{\alpha<\gamma}$ has been defined. If $\gamma = 0$, let $\omega_0 = \omega$. If $\gamma = \beta + 1$, for some $\beta \in On$, let $\omega_\gamma = \omega_\beta^+$. If γ is a non-zero limit ordinal, let ω_γ be defined to be the union of $(\omega_\alpha)_{\alpha<\gamma}$. Thus $\alpha \in On \to \omega_\alpha$ is a strictly order-preserving mapping of On onto the class of all infinite cardinal numbers. For example, ω_1 is the least uncountable ordinal.

Given any set S, there is a unique cardinal number σ with which S is equipotent. Let us also denote it by $|S|$, and call it the *cardinal number* (or the *power*)of S. ω_α is defined to be *regular* if every cofinal subset of ω_α has power ω_α, and *singular* if it is not regular. (Regular infinite cardinals may be thought of as irreducible infinite cardinals.) Note that, for all $\alpha \in On, \omega_{\alpha+1}$ is regular [**11**, 21.14]. Thus, given any infinite cardinal number κ, there exists a least $\xi \in On$ such that $\kappa \leq \omega_\xi$, for which $\xi > 0$ and ω_ξ is regular.

Let L and R be subclasses of an ordered class T. One writes $L < R$, and says that L is *less than* R, if, for all $x^L \in L$ and $x^R \in R, x^L < x^R$. Note that $\emptyset < R, L < \emptyset$, and that $\emptyset < \emptyset$. Following Hausdorff [**10**, pp. 172-185], T is called an η_ξ*-class* if, given any subsets L and R of T, for which $L < R$ and $|L| + |R| < \omega_\xi$, there exists $x \in T$ such that $L < \{x\} < R$. Any ordered set of power $\leq \omega_\xi$ can be embedded in an η_ξ-set by means of a strictly order-preserving map [**10**, p. 181].

Let T be an ordered set. By a *Cuesta Dutari cut* in T is meant a pair, (L, R), of subsets of T such that $L < R$, for which T is the union of L and R [**7**]. Let CD (T) be the set of *Cuesta Dutari* cuts in T. Note that L or R may be empty thus $(\emptyset, T)$ and $(T, \emptyset)$ are *Cuesta Dutari* cuts in T. Hence CD (T) is never empty.

Let $\chi(T)$, the *Cuesta Dutari completion* of T, be the union of T and CD (T), be ordered as follows. Let $x, y \in \chi(T)$. If x and y are in T, let them be ordered in $\chi(T)$ as they were ordered in T. If $x \in T$ and $y = (L, R)$ is in CD (T), let $x < y$ if $x \in L$, and $y < x$ if $x \in R$. If

$x = (L, R)$ and $y = (L', R')$ are in $\mathrm{CD}\,(T)$, let $x < y$ if L is a proper subset of L' (or equivalently if R' is a proper subset of R). The following hold:

(0) *$\chi(T)$ is an ordered set. Given $t_0 < t_1$ in T, there exists $c \in \mathrm{CD}\,(T)$ for which $t_0 < c < t_1$. Given $c_0 < c_1$ in $\mathrm{CD}\,(T)$, there exists $t \in T$ for which $c_0 < t < c_1$. $(\emptyset, T)$ is the least and $(T, \emptyset)$ is the greatest element of $\chi(T)$.* (See, e.g., [**5**, 4.02] for a proof.)

Assume, for some $\beta \in On$, that a family $(T_\alpha)_{\alpha<\beta}$ has been defined. If $\beta = 0$, let T_0 be the empty set. If there is an $\alpha \in On$ such that $\beta = \alpha + 1$, then let T_β be $\chi(T_\alpha)$; and if β is a non-zero limit ordinal, let T_β be the union of $(T_\alpha)_{\alpha<\beta}$. Then $(T_\alpha)_{\alpha \in On}$ is defined. Note that T_{ω_0} is a countable η_0-set, a set that is order-isomorphic to (for example) the set of all dyadic numbers.

(1) *If ω_ξ is regular, then T_{ω_ξ} is an η_ξ-set* [**9**, Satz 1].

Let **No**, the class of all *surreal numbers*, be defined to be the union of $(T_\alpha)_{\alpha \in On}$ (cf. [**6**, pp. 4, 15], [**3**], and [**5**, 4.02]). First note that **No** is a proper class. If $\alpha < \beta$, then T_α is a proper subset of T_β. Given $x \in \mathbf{No}$, there exists a least $\beta \in On$, such that $x \in T_{\beta+1}$, called the *birthday* of x, and denoted by $b(x)$. Note $b(x)$ is the unique element in On such that $x \in T_{b(x)+1} - T_{b(x)}$. The function b is called the *birth-order function* on **No** (cf. [**2**, Section 8], [**3**, p. 243], [**4**, p. 304], and [**5**, p. 360]).

CONWAY'S SIMPLICITY THEOREM. *Let L and R be subsets of* **No** *for which $L < R$, and let $I = \{y \in \mathbf{No} : L < \{y\} < R\}$. Then* (i) *$I$ is non-empty, and* (ii) *there exists a unique $x \in I$ such that $b(x) \leq b(y)$, for all $y \in I$* (cf. [**6**, p. 23], [**3**, p. 243], [**4**, p. 304], and [**5**, p. 124]).

Note that, in Conway's Simplicity Theorem, a unique element x is chosen having an interesting property. Following Conway's very useful compact notation [**6**, p. 4], let the element x, chosen in Conway's Simplicity Theorem, be denoted by $\{L|R\}$. Given such subsets L and R of **No**, let x^L denote a typical element of L, and let x^R denote a typical element of R. Thus, given $x \in \mathbf{No}, x$ may be written as $\{x^L|x^R\}$. Further, L and R may always be chosen so that $b(L), b(R) < \{b(x)\}$ [**5**, p. 125].

Conway then made the following inductive definitions: (i) $x + y = \{x^L + y, x + y^L | x^R + y, x + y^R\}$, (ii) $-x = \{-x^R | -x^L\}$, and (iii) $xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R | x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}$; the induction being with respect to birth-order [**6**, p. 5].

Conway gives a brilliant series of brief proofs that show that **No** is a real-closed field, having many additional interesting properties [**6**, pp. 4-44]. (See [**5**, Chapters 4-6] for a more detailed analysis.) Note, e.g., that $0 = \{\emptyset|\emptyset\}$, $1 = \{\{0\}|\emptyset\}$, and thus $-1 = \{\emptyset|\{0\}\}$ [**6**, p. 7].

Let ξ be a fixed ordinal number, with $\xi > 0$ and ω_ξ regular. Let ξ**No** (first defined in [**2**, p. 381]) be defined to be T_{ω_ξ} [**5**, p. 191]. Among its properties are the following:

(2) (i) ξ**No** *is a real-closed field that is an* η_ξ*-set* [**5**, pp. 246, 191].

(ii) *Any ordered field* K *of power* $\leq \omega_\xi$ *can be embedded, by means of a strictly order-preserving homomorphism, in any real-closed field that is an* η_ξ*-set* [**8**, p. 193]; *thus in* ξ**No**.

(3) ξ**No** *has a canonical valuation* V, *the* ω*-valuation, whose value group is the additive group* (ξ**No**$, +)$ *of* ξ**No**. *Finally,* ξ**No** *contains a canonical copy of* R *that maps onto its residue class field* ([**2**, p. 382] *and* [**5**, pp. 191-196]).

(4) ξ**No** *has a canonical formal power series structure, under which it is* R*-isomorphic to the field* $\xi R(((\xi\mathbf{No}, +)))$ *of all formal power series with coefficients in* R, *and exponents in* $(\xi\mathbf{No}, +)$, *of length less than* ω_ξ. *Further, the isomorphism and the valuations commute* [**5**, p. 246]. *Finally, every pseudo-convergent sequence in* ξ**No** *of length less than* ω_ξ *has a pseudo-limit in* ξ**No** [**5**, 6.41].

(5) ξ**No** *is order-isomorphic to Hausdorff's normal* η_ξ*-type* [**9**,Satz 9].

REMARK. It is well to note that the fact that the various structures cited in (3) (= displayed expression (3) above) and (4) are canonical follows from the Conway Simplicity Theorem, which may be derived from the birth-order structure on ξ**No**. Further, the canonical presence of these structures is very unusual and reflects the birth-order structure on **No**. See also [**4**] and [**5**, 4.03 and 4.60], for other abstract definitions of classes of surreal numbers.

1. The ξ-topology on $\xi\mathbf{No}$. Let $\xi\mathbf{No}!$ be defined to be the union of $\xi\mathbf{No}$ and $\{\pm\infty\}$, ordered so that $-\infty < x < +\infty$, for all $x \in \xi\mathbf{No}$. For a, $b \in \xi\mathbf{No}!$, let $(a,b) = \{x \in \xi\mathbf{No} : a < x < b\}$, and let (a,b) be called a *principal open interval* in $\xi\mathbf{No}$. Let B be the set of all principal open intervals in $\xi\mathbf{No}$. Clearly B is a base for the interval topology on $\xi\mathbf{No}$.

The phrase "not too many" here will mean that the cardinal numbers of the sets in question are less than ω_ξ. A subset U of $\xi\mathbf{No}$ will be called a *ξ-open* subset of $\xi\mathbf{No}$ if it can be written as the union of not too many principal open intervals of $\xi\mathbf{No}$. Although the set ξB, of all ξ-open subsets of $\xi\mathbf{No}$, is not a topology on $\xi\mathbf{No}$, it has many properties which resemble those of a topology. Let ξB be defined to be the *ξ-topology* on $\xi\mathbf{No}$ *generated* by B. Many definitions and theorems from topology go over virtually verbatim to the theory of ξ-topologies. Among these are the definitions of the relative ξ-topology, ξ-connected spaces, and ξ-continuous maps, as well as the first few theorems about these ideas. (See [**1**] and [**5**, Chapters 2, 3] for details.) For example,

(6) *A subspace of $\xi\mathbf{No}$ is ξ-connected if and only if it is an interval* ([**1**]*and* [**5**, 2.20]).

A subclass$X$ of $\xi\mathbf{No}$ is called *ξ-compact* if any cover of it by not too many ξ-open subsets of X has a finite subcover ([**1**], [**5**, p. 101]).

(7) *Let X be an interval in $\xi\mathbf{No}$ that either has a greatest* (*respectively least*) *element or has no cofinal* (*respectively coinitial*) *subset of power less than ω_ξ; then X is ξ-compact* ([**1**] *and* [**5**, 2.30]). *Hence all the intervals $[a,b], [a,b), (a,b]$, and (a,b) in $\xi\mathbf{No}$ are ξ-compact.*

2. Neumann's theorem, and extensions. Let K be a field and let G be an ordered Abelian group, both being sets. Let $F = K((G))$ (respectively $\xi K((G))$ (4)); then F is a field with a valuation, having value group G, valuation ring A, maximal ideal M, and residue class field K. B.H. Neumann proved a much more general version of

NEUMANN'S THEOREM. *For all $(a_n)_{n<\omega}$ in K, and all $x \in M$, $\sum_{n=0}^{\infty} a_n \cdot x^n$ is a well-defined element in F,* [**12**, 4.7]. (See also [**5**, 7.22].)

For all $x \in M$, we will say that the formal power series $\sum_{n=0}^{\infty} a_n \cdot X^n$, defined above, is *hyper-convergent.* Neumann's Theorem admits further generalization. First of all, it may be generalized to formal power series in several variables [**5**, 7.41]. This method of evaluation of formal power series has many properties, e.g., [**5**, 7.41].

Since $\xi\mathbf{No}$ has a canonical power series structure (4), we may proceed as follows. Let $C(X) = \sum_{k=0}^{\infty}(\sum_{\mathrm{sum}(v)=k} C(v)X^v)$ be in $\xi\mathbf{No}\,[[X_1, \ldots, X_n]]$, the ring of formal power series in n determinates over $\xi\mathbf{No}$. Since $\xi\mathbf{No}$ is R-isomorphic to $\xi R(((\xi\mathbf{No}, +)))$ (4), we may define the smallest convex subgroup B of $(\xi\mathbf{No}, +)$ (3), that contains all the exponents of the power series expansions of all of the $C(v)$'s. Since each $|\mathrm{supp}\,(C(v))| < \omega_\xi$ and $(\xi\mathbf{No}, +)$ is an η_ξ-set, and since $\xi > 0$, B is a proper subgroup of $(\xi\mathbf{No}, +)$. Since $(\xi\mathbf{No}, +)$ has a canonical Hahn group structure (3), we can define a canonical direct summand A of B in $(\xi\mathbf{No}, +)$. Further, $(\xi\mathbf{No}, +)$ is canonically R-isomorphic to the lexicographically ordered direct sum $A + B$ [**5**, 7.80]. It is not difficult to prove that

(8) $\xi\mathbf{No}$ *is R-isomorphic to* $\xi(\xi R((B))((A)))$, [**5**, 7.80].

Let $P = \{x \in \xi\mathbf{No} : \{V(x)\} > B\}$ (3); then P is a non-zero (convex) prime ideal of A. On applying Neumann's Theorem to $\xi(\xi R((B))((A)))$, we arrive at the

MAIN THEOREM. *For all $x = (x_1, \ldots, x_n) \in P^n$, the following is a well-defined element of* $\xi\mathbf{No}$: $\sum_{k=0}^{\infty}(\sum_{\mathrm{sum}(v)=k} C(v)x^v)$ [**5**, 7.82].

NOTE. This theorem is much more sweeping then the generalization of Neumann's Theorem to several variables mentioned above, in that the $C(v)$ may be in $\xi\mathbf{No}$, and are not restricted to lie in R.

3. Applications to analysis. We know that, over the field of complex numbers, the following classical theorems hold.

(A) *Locally, the simple roots of a polynomial $\phi(X)$ with coefficients in C, are analytic functions of these coefficients.*

(B) *Let* $\phi^{\wedge}(X,Y) \in C[X,Y]$ *be of degree* $m > 0$ *in* Y*, and let* x_0 *and* y_0 *be in* C *such that* $(d\phi\hat{}/dY)(x_0,y_0) \neq 0$*, and* $\hat{\phi}(x_0,y_0) = 0$*. Then, locally about* (x_0,y_0)*, the function* y *for which* $\hat{\phi}(x,y) = 0$*, is an analytic function of* x*.*

(C) *Let* $\phi_1\hat{}, \ldots, \phi_n\hat{} \in C[X_1,\ldots,X_n]$ *define a map* $\hat{\phi}$ *from* C^n *to* C^n*, taking* 0 *to* 0*, that is non-singular at the origin. Then* $\hat{\phi}$ *has an analytic inverse, defined on some neighborhood of the origin in the range space.*

Let $\xi C\mathbf{x} = \xi\mathbf{No}(\mathrm{i})$, and let it be called a *surcomplex number field.* $\xi C\mathbf{x}$ is an algebraically closed field, has a canonical copy of C in it which maps onto residue class field, and has $(\xi\mathbf{No}, +)$ as its value group under the extension of the ω-valuation (3) to $\xi C\mathbf{x}$. Further, every pseudo convergent sequence in $\xi C\mathbf{x}$ of length less than ω_ξ has a pseudo-limit in $\xi C\mathbf{x}$ [**5**, pp. 255-260].

Theorems A, B, and C all admit generalizations over the surcomplex number fields and–appropriately restricted–over the surreal number fields.

The hyper-local versions of (A), (B), and (C) hold for a substantial class of formal power series fields, in particular for all surreal and surcomplex number fields. Extensions of these hyper-local results to local results (as measured in the residue class field) employ classical analytic function theory, and thus require that the complex field or the real field be the field of coefficients; which is the case for $\xi C\mathbf{x}$ of $\xi\mathbf{No}$.

For the sake of simplicity, let $F = \xi C\mathbf{x}$ (although some of this can be done under very much weaker assumptions, as we plan to show in a subsequent publication). Let A be the valuation ring of F, M its maximal ideal, and let ρ be the place associated with A. For $a \in A$, let $\hat{a} = \rho(a)$. Clearly ρ extends to a C-linear homomorphism, also denoted by ρ, of $A[X_1,\ldots,X_n]$ onto $C[X_1,\ldots,X_n]$, having kernel $M[X_1,\ldots,X_n]$. Let $\phi(X_1,\ldots,X_n)$ (or simply $\phi(X)$) be in $A[X_1,\ldots,X_n]$, of degree $m > 0$. Let $\hat{\phi}(X)$ be defined to be $\rho(\phi(X))$. Note that $\hat{\phi}(X)$ is in $C[X_1,\ldots,X_m]$. $\theta(X)$ and $\phi(X)$ in $A[X_1,\ldots,X_m]$ will be called *infinitesimal perturbations* of one another if $\hat{\theta}(X) = \hat{\phi}(X)$. Clearly the set of all such $\theta(X)$ is $\hat{\phi}(X)+M[X_1,\ldots,X_n]$, a coset, in $A[X_1,\ldots,X_m]$. Further, $\hat{\phi}(X)$ is a canonical representative of that coset.

Let us now give a proof of an extension of Theorem B.

(9) *Let* $\phi(X,Y) = Y^m + \sum_{j=0}^{m-1}(\sum_{i=0}^{n(j)} a_{ij}X^i) \cdot Y^j \in A[X,Y]$, *such that* a_{01} *is a unit in* A, *and* a_{00} *is in* M.

Let $\Delta = \{(i,j) : j \in \{0,\ldots,m-1\}$ and $i \in \{0,\ldots,n(j)\}\}$, and let d be defined to be $\sum_{j=0}^{m-1} n(j)$. For $(i,j) \in \Delta$, let $a_{ij}\hat{} = c_{ij} \in C$ and $\mu_{ij} = a_{ij} - c_{ij}$; then $\mu_{ij} \in M$ and $a_{ij} = c_{ij} + \mu_{ij}$. Note also that, since a_{01} was assumed to be a unit in A and that a_{00} was assumed to be in M (9), then $c_{01} \neq 0$ and $c_{00} = 0$.

(10) $\Phi((X_{i,j})_{(i,j)\in\Delta}, X, Y) = Y^m + \sum_{j=0}^{m-1}(\sum_{i=0}^{n(j)}(c_{ij} + X_{ij}) \cdot X^i) \cdot Y^j$, *is in the polynomial ring,* $C[(X_{ij})_{(i,j)\in\Delta}, X, Y]$.

(11) *Clearly,* $\Phi((0)_{(i,j)\in\Delta}, X, Y) = \hat{\phi}(X,Y)$ *and* $\Phi((\mu_{ij})_{(i,j)\in\Delta}, X, Y) = \phi(X,Y)$.

Hence $\Phi((0)_{(i,j)\in\Delta}, X, Y)$ and $\Phi((\mu_{ij})_{(i,j)\in\Delta}, X, Y)$ are infinitesimal perturbations of one another. Note also that $(\partial\Phi/\partial Y)((0)_{(i,j)\in\Delta}, 0, 0) = c_{01} \neq 0$, and that $\Phi((0)_{(i,j)\in\Delta}, 0, 0) = c_{00} = 0$. By the classical Implicit Mapping Theorem, there exists a (classical) polydisc U in C^{d+1}, about the origin, and there exists a unique complex-valued function $G((X_{ij})_{(i,j)\in\Delta}, X)$, analytic over U, with $G((0_{ij})_{(i,j)\in\Delta}, 0) = 0$, such that the following identity holds:

(12) $\Phi((X_{ij})_{(i,j)\in\Delta}, X, G((X_{ij})_{(i,j)\in\Delta}, X)) = 0$, *over* U.

Let U° be defined to be $U + M^{d+1}$ [**5**, 7.65]. Using Neumann's Theorem, we may extend G to form a mapping G^0, which maps U^0 into $\xi C\mathbf{x}$ [**5**, 7.65]. Further, using Theorem 7.41 [**5**], we know that Φ and G^0 satisfy (12) over U^0. We will denote this extended version of (12) by (12^0). We then have

THEOREM. $\phi(X, G^0((\mu_{ij})_{(i,j)\in\Delta}, X)) = 0$, *for all* $X \in U^0$.

PROOF. By (12^0) and (11), $0 = \Phi((\mu_{ij})_{(i,j)\in\Delta}X, G^0((\mu_{ij})_{(i,j)\in\Delta}, X)) = \phi(X, G^0((\mu_{ij})_{(i,j)\in\Delta}, X))$, over U^0. □

Let W^0 be the image of the projection of U^0 onto the last coordinate in $\xi C\mathbf{x}^{d+1}$. Thus, the map $X \in W^0 \to G^0((\mu_{ij})_{(i,j)\in\Delta}, X)) = Y$ is a map from W^0 to $\xi C\mathbf{x}$ for which $\phi(X,Y) = 0$.

This completes the proof of the extension of Theorem B to $\xi C\mathbf{x}$. □

Bibliographic note. Part of the argument that Conway used on page 41 of [**6**] was very useful to the author in initiating the line of reasoning that led to this proof.

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