ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

PROXIMINALITY IN $L_p(S, Y)$

W.A. LIGHT

Introduction. Throughout this work (S, Σ, μ) will be a finite measure space and Y a Banach space. For $1 \leq p < \infty, L_p(S, Y)$ is the Banach space consisting of strongly measurable functions $f : S \to Y$ such that $\int_S ||f(s)||^p ds$ is finite. In this case

$$||f||_p = \left\{ \int_S ||f(s)||^p \, ds \right\}^{1/p}.$$

Occasionally we shall consider the space $L_\infty(S, Y)$ which consists of all strongly measurable functions $f : S \to Y$ such that ess sup $\{||f(s)|| : s \in S\}$ is finite. Then

$$||f||_\infty = \operatorname{ess\,sup}\{||f(s)|| : s \in S\}.$$

A typical example of the questions we investigate here is the following. Suppose H is a proximinal subspace of Y. Does it follow that $L_p(S, H)$ is a proximinal subspace in $L_p(S, Y)$? By way of introduction we indicate some results which are easy consequences of known theorems about the structure of $L_p(S, Y)$. The two key results are as follows:

THEOREM 1. *Let (S, Σ, μ) be a finite measure space, $p \in [1, \infty)$ and Y be a Banach space. Then $L_p(S, Y)^* = L_q(S, Y^*)$, where $p^{-1} + q^{-1} = 1$, if and only if Y^* has the Radon-Nikodym property with respect to μ.*

THEOREM 2. *Let (S, Σ, μ) be a finite measure space and Y a uniformly convex Banach space. Then, for $1 < p < \infty, L_p(S, Y)$ is uniformly convex.*

A proof of the first of these results may be found in [**2**, p. 98] while the second is proved in [**10**]. A useful consequence of the first theorem is

Received by the editors on September 10, 1986.

COROLLARY 1. *Let (S,Σ,μ) be a finite measure space and Y a Banach space. Then $L_p(S,Y)$ is reflexive for $1 < p < \infty$ if and only if Y is reflexive.*

There are two immediate approximation-theoretic consequences of these results. First, it is well known that if H is a reflexive subspace of a Banach space Y then H is proximinal in Y. (See, for example [**11**, p. 100].) The following result is a consequence of this observation and Corollary 1.

THEOREM 3. *Let (S,Σ,μ) be a finite measure space, Y a Banach space and H a reflexive subspace of Y. Then, for $1 < p < \infty, L_p(S,H)$ is proximinal in $L_p(S,Y)$.*

For $p = 1$, Theorem 3 has already been established by Khalil [**7**]. The case $p = \infty$ will be dealt with later in our discussion. Part of the structure of uniformly convex Banach spaces is that closed subspaces are Chebyshev subspaces, that is, each element in the space has a unique element of best approximation in the subspace. The second result is really an observation based on the preceding remarks and Theorem 2.

THEOREM 4. *Let (S,Σ,μ) be a finite measure space and Y a uniformly convex Banach space. Then, for $1 < p < \infty$ and any closed subspace H in Y, $L_p(S,H)$ is a Chebyshev subspace of $L_p(S,Y)$.*

This concludes the harvesting of results as simple consequences of existing theorems.

Distance formulae. Progress in the discussion of proximinality when Y does not possess pleasant properties is greatly facilitated by the fact that the distance from an element $f \in L_p(S,Y)$ to a subspace $L_p(S,H)$ is computed rather easily using the following theorem.

THEOREM 5. *Let (S,Σ,μ) be a finite measure space, Y a Banach space and H a subspace of Y. Suppose $1 \leq p \leq \infty$. For $f \in L_p(S,Y)$, define $\phi : S \to \mathbf{R}$ by $\phi(s) = \operatorname{dist}(f(s),H)$. Then $\phi \in L_p(S)$ and*

$$\operatorname{dist}_p(f, L_p(S,H)) = ||\phi||_p.$$

PROOF. If $f \in L_p(S,Y)$, then f is strongly measurable, and so it is the limit almost everywhere of a sequence of simple functions $\{f_n\}$ in $L_p(S,Y)$. Since the distance function $d(y,H)$ is a continuous function of $y \in Y$, $||f_n(s) - f(s)|| \to 0$ implies $|\text{dist}(f_n(s),H) - \text{dist}(f(s),H| \to 0$. Furthermore, each function $\phi_n : S \to \mathbf{R}$ defined by $\phi_n(s) = \text{dist}(f_n(s),H)$ is a simple function, and so we may assume ϕ is measurable. Now, for any $g \in L_p(S,H)$, we have, for $1 \le p < \infty$,

$$\begin{aligned} ||f-g||_p &= \left\{ \int_S ||f(s)-g(s)||^p \, ds \right\}^{1/p} \\ &\ge \left\{ \int_S [\text{dist}(f(s),H)]^p \right\}^{1/p} = ||\phi||_p, \end{aligned}$$

while for $p = \infty$,

$$\begin{aligned} ||f-g||_\infty &= \operatorname*{ess\,sup}_{s\in S} ||f(s)-g(s)|| \\ &\ge \operatorname*{ess\,sup}_{s\in S} \{\text{dist}(f(s),H)\} = ||\phi||_\infty. \end{aligned}$$

These inequalities establish that $\phi \in L_p(S)$ and by taking an infimum on $g \in L_p(S,H)$ we obtain

$$\text{dist}_p(f, L_p(S,H)) \ge ||\phi||_p.$$

For the reverse inequality suppose that $\varepsilon > 0$ and let f' be a simple function in $L_p(S,Y)$ such that $||f - f'||_p < \varepsilon/3$. Write $f'(s) = \sum_{i=1}^n x_i(s)y_i$ where the x_i are characteristic functions of measurable sets A_i and $y_i \in Y$. We can assume $\sum_{i=1}^n x_i = 1$, $\mu(S) = 1$ and $\mu(A_i) > 0$. Select $h_i \in H$ so that

$$||y_i - h_i|| < \text{dist}(y_i,H) + \varepsilon/3.$$

Set $g \in L_p(S,H)$ as $g(s) = \sum_{i=1}^n x_i(s)h_i$. Now, for $1 \le p < \infty$,

$$\begin{aligned} \text{dist}(f, L_p(S,H)) &\le \varepsilon/3 + \text{dist}(f', L_p(S,H)) \le \varepsilon/3 + ||f'-g||_p \\ &= \varepsilon/3 + \left\{ \int_S ||f'(s)-g(s)||^p \, ds \right\}^{1/p} \\ &= \varepsilon/3 + \left\{ \sum_{i=1}^n \int_{A_i} ||y_i - h_i||^p \, ds \right\}^{1/p} \\ &= \varepsilon/3 + \left\{ \sum_{i=1}^n ||y_i - h_i||^p \mu(A_i) \right\}^{1/p} \\ &< \varepsilon/3 + \left\{ \sum_{i=1}^n \mu(A_i)[\text{dist}(y_i,H) + \varepsilon/3]^p \right\}^{1/p}. \end{aligned}$$

Now by using the triangle inequality for L_p-norms we obtain, writing $\mu(A_i)=\mu_i$,

$$
\begin{aligned}
&\operatorname{dist}(f, L_p(S,H))\\
&< \varepsilon/3 + \Big\{\sum_{i=1}^{n}[\mu_i^{1/p}\operatorname{dist}(y_i,H)]^p\Big\}^{1/p} + \Big\{\sum_{i=1}^{n}\Big(\mu_i^{1/p}\varepsilon/3\Big)^p\Big\}^{1/p}\\
&\le \frac{\varepsilon}{3} + \Big\{\sum_{i=1}^{n}\mu_i[\operatorname{dist}(y_i,H)]^p\Big\}^{1/p} + \varepsilon\Big(\sum_{i=1}^{n}\mu_i\Big)^{1/p}/3\\
&\le \frac{2\varepsilon}{3} + \Big\{\sum_{i=1}^{n}\int_{A_i}[\operatorname{dist}(y_i,H)]^p ds,\Big\}^{1/p}\\
&= \frac{2\varepsilon}{3} + \Big\{\int_S[\operatorname{dist}(f'(s),H)]^p\,ds\Big\}^{1/p}\\
&\le \frac{2\varepsilon}{3} + \Big\{\int_S[\operatorname{dist}(f(s),H]^p\,ds\Big\}^{1/p} + \Big\{\int_S||f(s)-f'(s)||^p\,ds\Big\}^{1/p}\\
&= \frac{2\varepsilon}{3} + ||\phi||_p + ||f-f'||_p\\
&\le ||\phi||_p + \varepsilon.
\end{aligned}
$$

For $p=\infty$ a similar analysis yields $\operatorname{dist}(f, L_\infty(S,H)) \le ||\phi||_\infty + \varepsilon$ and so the required formula is established. □

COROLLARY 2. *Let H be a closed subspace of a Banach space Y, (S,Σ,μ) a finite measure space and $1\le p<\infty$. In order that an element g of $L_p(S,H)$ be a best approximation to an element f in $L_p(S,Y)$ it is necessary and sufficient that $g(s)$ be a best approximation in H to $f(s)$ for almost all $s\in S$.*

Theorem 5 generalises 2.10 in [**8**, p. 39]. A much wider generalisation is given in [**3**]. There one considers the space $C(S,X)$ of continuous functions from a compact Hausdorff space S to a Banach space X. Then one imposes a monotone norm α on $C(S)$ and "lifts" it to $C(S,X)$ in the obvious way, so that if $f\in C(S,X)$ then $||f|| = ||Jf||_\alpha$ where $(Jf)(s) = ||f(s)||$. Then Theorem 5 holds for this lifted α-norm. However, Theorem 5 will only be a corollary of that work if S is a compact Hausdorff space and μ is a Borel measure on S. Corollary 2 shows that Theorem 4 is far more restrictive than it need be, since it is an immediate consequence of the following result.

COROLLARY 3. *Let H be a Chebyshev subspace of a Banach space Y.*

Let (S, Σ, μ) be a finite measure space and $1 < p < \infty$. If $L_p(S, H)$ is proximinal in $L_p(S, Y)$ then it is a Chebyshev subspace of $L_p(S, Y)$.

Corollary 3 and Theorem 3 combine to give the final result in this section.

COROLLARY 4. *Let H be a reflexive Chebyshev subspace of a Banach space Y. Let (S, Σ, μ) be a finite measure space and let $1 < p < \infty$. Then $L_p(S, H)$ is a Chebyshev subspace of $L_p(S, Y)$.*

Proximinality. Corollary 2 indicates that the choice of candidates for a best approximation to $f \in L_p(S, Y)$ from $L_p(S, H)$ is severely limited for $1 \leq p < \infty$. The next lemma shows that the real point at issue is always the strong measurability of the candidate.

LEMMA 1. *Let (S, Σ, μ) be a finite measure space, H a proximinal subspace of a Banach space Y and $1 \leq p \leq \infty$. Let $f \in L_p(S, Y)$ and suppose g is a strongly measurable function such that $g(s)$ is a best approximation to $f(s)$ from H for almost all s in S. Then g is a best approximation to f from $L_p(S, H)$.*

PROOF. Since $g(s)$ is a best approximation to $f(s)$ for almost all $s \in S$, we have $||g(s)|| \leq 2||f(s)||$. Thus $||g||_p \leq 2||f||_p$ and so $g \in L_p(S, H)$. Also, for almost all $s \in S$, $||f(s) - g(s)|| \leq ||f(s) - h||$ for all h in H, and so $||f - g||_p \leq ||f - g'||_p$ for all $g' \in L_p(S, H)$. □

We now make use of a measurable selection theorem due to Himmelberg and van Vleck [**4**]. For our purposes it is convenient to describe their result under slightly more restrictive hypotheses. Let (S, Σ) be a σ-algebra. Let X be a Hausdorff topological space which is the union of countably many compact metrizable subspaces. Let ϕ be a set-valued mapping from S onto the closed, non-empty subsets of X.

THEOREM 6. (HIMMELBERG AND VAN VLECK). *If the set-valued mapping ϕ has the property that*

$$\phi^{-1}(K) = \{s \in S : \phi(s) \cap K \text{ is non-empty }\}$$

is in Σ whenever K is compact in X, then ϕ has a measurable selector. That is, there is a single-valued mapping $f : S \to X$ such that

(i) $f(s) \in \phi(s)$ *for each* s *in* S

(ii) $f^{-1}(K)$ *is in* Σ *whenever* K *is compact in* X.

For our purposes we shall need to have the hypotheses on X in Theorem 6 holding on our subspace H.

DEFINITION. *Let* H *be a subspace of a Banach space* Y *and* τ *a Hausdorff topology on* Y. *We shall say* (H, Y) *has property* (HV) *with respect to the linear topology* τ *if*

(i) H *is* τ*-closed in* Y.

(ii) *The unit ball in* H *is* τ*-compact and metrizable.*

(iii) *Every* τ*-compact set in* H *is proximinal in* Y.

An immediate consequence of H having property (HV) is that H is the union of countable many metrizable subsets - for example $H = \cup_{n=1}^{\infty}\{h : ||h|| \leq n\}$. We need one technical result before passing on to our main theorem.

LEMMA 2. *Let* A *be an equivalence class of functions in* $L_p(S,Y)$. *Then there exists a strongly measurable function* $f \in A$ *such that, for every open or closed set* $B \subset Y$, $f^{-1}(B)$ *is measurable.*

PROOF. Suppose g is a strongly measurable function in A. Let $\{g_n\}$ be a sequence of simple functions such that $g_n(s) \to g(s)$ on the set $S\backslash N$ where N is a suitable null set. Now define

$$f(s) = \begin{cases} g(s) & s \in S\backslash N \\ 0 & s \in N \end{cases} \quad \text{and} \quad f_n(s) = \begin{cases} g_n(s) & s \in S\backslash N \\ 0 & s \in N \end{cases}.$$

Then $f \in A$ and f is strongly measurable since each f_n is a simple function and $f_n(s) \to f(s)$ for all $s \in S$. Now let B be closed in Y, and define

$$E_{nk} = \left\{s \in S : \operatorname{dist}(f_n(s), B) \leq \frac{1}{k}\right\}.$$

Each E_{nk} is measurable and

$$f^{-1}(B) = \cap_{k=1}^{\infty} \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} E_{nk}.$$

Hence $f^{-1}(B)$ is measurable. If B is an open set, then $Y\backslash B$ is closed and

$$f^{-1}(B) = S\backslash f^{-1}(Y\backslash B),$$

so $f^{-1}(B)$ is again measurable.

THEOREM 7. *Let (S,Σ,μ) be a finite measure space and $1 \le p \le \infty$. Let H be a separable, proximinal subspace of a Banach space Y. Suppose τ is a topology on H such that (H,Y) has property* (HV). *Then $L_p(S,H)$ is proximinal in $L_p(S,Y)$.*

PROOF. Using Lemma 2, we take an equivalence class A of functions in $L_p(S,Y)$ and extract from it a representative f which is of necessity strongly measurable and so that $f^{-1}(\mathcal{O}) \in \Sigma$ whenever $\mathcal{O}$ is open in Y. For each $s \in S$, define

$$\Phi(s) = \{h \in H : ||f(s) - h|| = \text{dist}(f(s), H)\}.$$

Then, for each s in $S, \Phi(s)$ is a τ-closed, non-empty subset of H. Take K as a τ-compact set in H. Then

$$\Phi^{-1}(K) = \{s \in S : \phi(s) \cap K \text{ is non-empty}\}.$$

Since K is proximinal in Y,

$$\Phi^{-1}(K) = \{s \in S : \inf_{h \in K} ||f(s) - h|| = \inf_{h \in H} ||f(s) - h||\}.$$

Now consider the mapping $s \to ||f(s) - h||$. If $\mathcal{O}$ is open in $\mathbf{R}$, then by the continuity of the norm and the fact that $f^{-1}(\mathcal{O}')$ is measurable for each $\mathcal{O}'$ open in Y, this mapping is measurable. Hence the mapping $s \to \inf_{h \in A} ||f(s) - h||$ is measurable whenever A lies in H. Thus $\Phi^{-1}(K)$ is the set of s in S at which two measurable mappings agree and so is itself measurable. By Theorem 6, there is a selection $\phi : S \to H$ such that $\phi(s) \in \Phi(s)$ for each s in S and $\phi^{-1}(K) \in \Sigma$ whenever K is τ-compact in H.

Since H is separable, take $h_1, h_2, \ldots,$ a countable dense set in H. Each open set $\mathcal{O} \subset H$ can be written as

$$\mathcal{O} = \cup_{n,m=1}^{\infty}\{C_{nm} : C_{nm} \subset \mathcal{O}\}$$

where

$$C_{nm} = \{h \in H : ||h - h_n|| \le 1/m\}.$$

Each C_{nm} is τ-compact in H and so $\phi^{-1}(C_{nm})$ is measurable. Hence $\phi^{-1}(\mathcal{O})$ is measurable for each $\mathcal{O}$ open in H. Since ϕ also has separable range, ϕ is strongly measurable [**8**, p. 114]. By Lemma 1, ϕ is a best approximation to f from $L_p(S,H)$. □

We now make some observations on property (HV), and on the basis of these deduce several new and old results about proximinality. First, if H is a separable subspace of a dual space, then the weak-star topology is Hausdorff on H and the unit ball in H is weak-star compact and metrizable [**6**, p. 286]. Also every weak-star compact subset of H is proximinal [**5**, p. 123] and so if H is weak-star closed in Y then (H,Y) has property (HV). This preamble establishes our first corollary.

COROLLARY 5. *Let (S,Σ,μ) be a finite measure space and $1 \leq p \leq \infty$. Let H be a separable, weak-star closed subspace of a dual space Y. Then $L_p(S,H)$ is proximal in $L_p(S,Y)$.*

This result is new for all the values of p indicated. The fact that H or Y is not presumed to be reflexive means that the duality theory as expressed in Theorem 3 is not applicable. In the following two results that theory is applicable and so they only achieve independent interest at the extremes of the range, that is $p = 1$ and $p = \infty$. The first result depends on the fact that a finite-dimensional space H always has property (HV) in any containing space Y with respect to the norm topology.

COROLLARY 6. *Let (S,Σ,μ) be a finite measure space and $1 \leq p \leq \infty$. Let H be a finite-dimensional subspace of a Banach space Y. Then $L_p(S,H)$ is proximinal in $L_p(S,Y)$.*

For $p = 1$ this result was first proved in [**7**]. For $p = \infty$, almost the same result appears in [**8**], although the completeness of the measure space was assumed there.

If H is a reflexive, separable Banach space, then H is Hausdorff with respect to its weak topology and the unit ball in H is weakly compact and metrizable. Also since the norm is a weakly lower semicontinuous functional, every weakly compact set in H is proximinal.

COROLLARY 7. *Let (S,Σ,μ) be a finite measure space and $1 \leq p \leq \infty$. Let H be a reflexive, separable subspace of a Banach space Y. Then $L_p(S,H)$ is proximinal in $L_p(S,Y)$.*

For $1 < p < \infty$, this corollary fails to capture the generality of Theorem 3 by virtue of the separability hypothesis. For $p = 1$, the same remark applies to the work of Khalil [**7**]. For $p = \infty$, the result is new and is a significant improvement on the existing best available theorem which is from [**8**] and was given in Corollary 6.

REFERENCES

1. J. Diestel, *Geometry of Banach Spaces*, Lecture Notes in Mathematics Vol. 485, Springer-Verlag, New York 1975.

2. ——— and J.J. Uhl, *Vector Measures*, The American Mathematical Society, Providence, Rhode Island, 1977.

3. M. von Golitschek, W.A. Light and E.W. Cheney, *Approximation with monotone norms*, Center for Numerical Analysis Preprint, University of Texas, Austin, Texas 1986.

4. C.J. Himmelberg and F.S. Van Vleck, *Some selection theorems for measurable functions*, Can J. Math. **21** (1969), 394-399.

5. R.B. Holmes, *A course on Optimisation and Best Approximation*, Lecture Notes in Mathematics, Vol. 257, Springer-Verlag, Berlin 1972.

6. G.J.O. Jameson, *Topology and Normed Spaces*, Chapman and Hall, London, 1974.

7. R. Khalil, *Best approximation in $L^p(I, X)$*, Math. Proc. Camb. Phil. Soc. **94** (1983), 277-279.

8. W.A. Light and E.W. Cheney, *Approximation Theory in Tensor Product Spaces*, Lecture Notes in Mathematics, Vol. 1169, Springer-Verlag, Heidelberg, 1985.

9. ——— and ——— , *Some best approximation theorems in tensor product spaces*, Math. Proc. Camb. Phil. Soc. **89** (1981), 385-390.

10. E.J. McShane, *Linear functionals on certain Banach spaces*, Proc. Amer. Math. Soc. **1** (1950), 402-408.

11. I. Singer, *Best approximation in normed linear spaces by elements of linear subspace*, Springer-Verlag, New York, 1970.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LANCASTER, LANCASTER LA1 4YL, U.K. d