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THE TOPOLOGICAL CLASSIFICATION OF CUBIC CURVES

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ABSTRACT. The real plane cubic curves given by $Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0$ are classified up to equivalence under the homeomorphism group of the plane.

Introduction. In a previous paper [**18**] the author briefly sketched the history of classifications of cubic curves and described the two major approaches to the problem: group theoretic and non-group theoretic. The author then presented a group theoretic classification based on the affine group of the plane. In this paper a group theoretic classification based on the homeomorphism group of the plane is presented. The non-group theoretic approach was dominated by Newton. The fact that Newton's early classifications were based on criteria far from topological can easily be seen by inspecting the graphs on pages 72-84 of Vol. II of Newton's works [**9**].

An important and fundamental problem associated with the Kleinian (group theoretic) approach to classification is the computation of invariants. In the present context the relevant question is: How can the equivalence class of the curve be determined from the coefficients? This question is not answered here and is left to a future paper. The classification is presented by exhibiting a complete set of equivalence class representatives.

In every analytic geometry course, it is taught that, for quadratic curves $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, there are precisely three (nondegenerate) affine classes: parabola, ellipse, and hyperbola. Even though the homeomorphisms of the plane form a much larger class of transformations, it is nevertheless the case that the (nondegenerate) homeomorphism classes of quadratic curves are the same as the affine classes. For cubic curves these classes are much different. For example,

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there is an infinite number of nondegenerate affine classes, but only a finite number of homeomorphism classes.

Let us now clarify another distinction between the affine and homeomorphism classifications of cubic curves. The affine group of the plane actually acts on the set of cubic curves in the well-known sense of group action. The homeomorphism group of the plane does not act on the set of cubic curves in this sense, but it still determines an equivalence relation in the following way: Two cubic curves Γ_1 and Γ_2 are equivalent if there exists a homeomorphism of the plane which takes Γ_1 onto Γ_2.

Let us give an example which illustrates an important distinction. Consider the hyperbola $y = 1/x$ and the straight line $y = x$ with the origin removed. These two curves are homeomorphic, but they are not equivalent via a homeomorphism of the plane because, although each curve consists of two components, the complement of the hyperbola consists of three components, while the complement of the punctured line consists of only one component.

The homeomorphism and diffeomorphism groups of the plane both yield a finite number of equivalence classes for cubic curves, but there are more diffeomorphism classes because, for example, $y = x^3$ and $y^2 = x^3$ are equivalent via a homeomorphism of the plane but not via a diffeomorphism of the plane. It turns out that the classification determined by the homeomorphism group of the plane yields 21 equivalence classes, where the genuine (irreducible) cubics fall into 15 of the classes, the remaining 6 coming from degenerate cases consisting of certain combinations of lines and conics.

The purpose of this paper is to present the

THEOREM. *The following list gives a complete set of equivalence class representatives for the classification of real plane cubic curves under the homeomorphism group of the plane:*

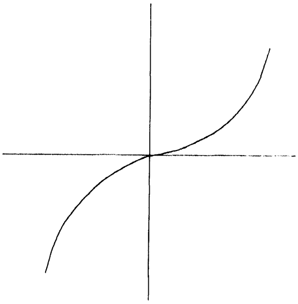

FIGURE 1. $y = x^3$

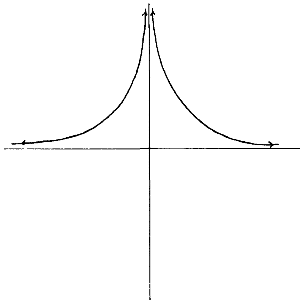

FIGURE 2. $x^2 y = 1$

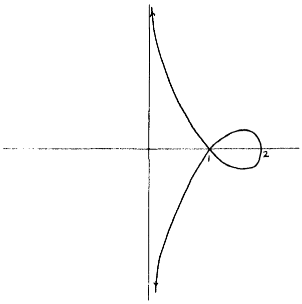

FIGURE 3. $y^2 = \frac{(x-1)^2(x-2)}{-x}$

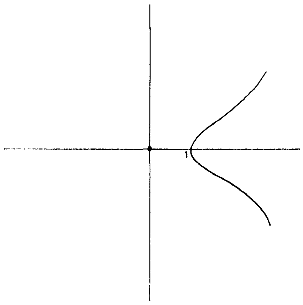

FIGURE 4. $y^2 = x^2(x-1)$

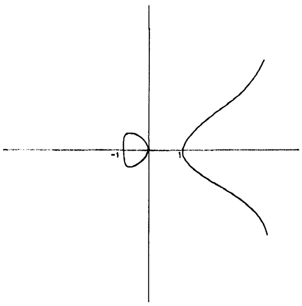

FIGURE 5. $y^2 = x(x+1)(x-1)$

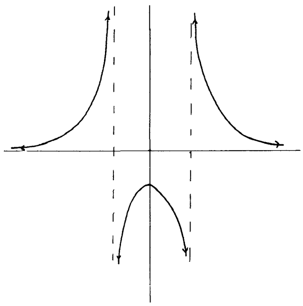

FIGURE 6. $y = \frac{1}{x^2-1}$

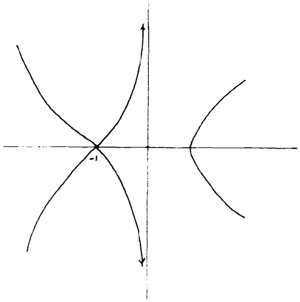

FIGURE 7. $y^2 = \frac{(x+1)^2(x-2)}{x}$

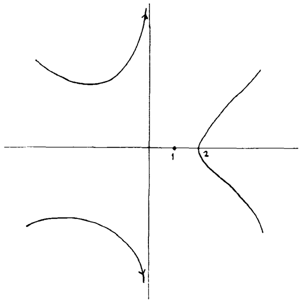

FIGURE 8. $y^2 = \frac{(x-1)^2(x-2)}{x}$

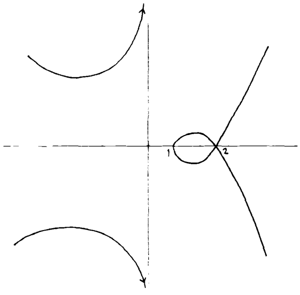

FIGURE 9. $y^2 = \frac{(x-2)^2(x-1)}{x}$

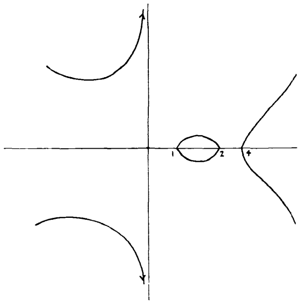

FIGURE 10. $y^2 = \frac{(x-1)(x-2)(x-4)}{x}$

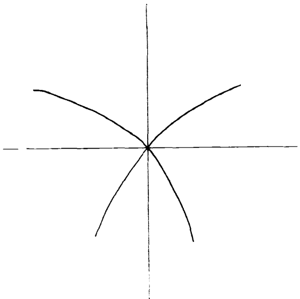

FIGURE 11. $x^2y - \frac{1}{4}x^2 + y^2 = 0$

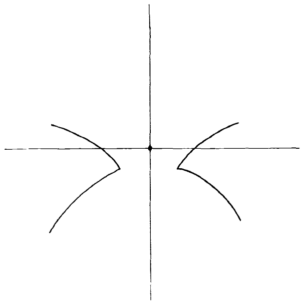

FIGURE 12. $x^2y + y^2 + \frac{1}{4}x^2 = 0$

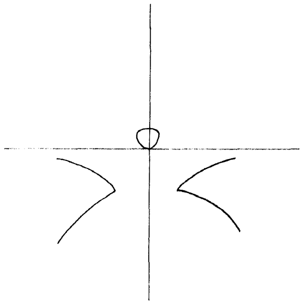

FIGURE 13. $x^2y + \frac{1}{2}x^2 + \frac{1}{2}y^2 - 2y = 0$
(i.e. $y = -(x^2 - 2) \pm \sqrt{(x^2 - 1)(x^2 - 4)}$)

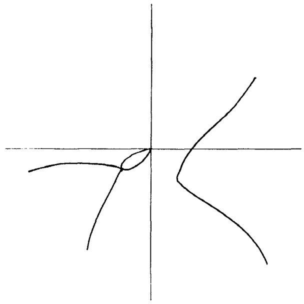

FIGURE 14. $y = \frac{-x^2 \pm \sqrt{x(x-2)(x+1)^2}}{2}$

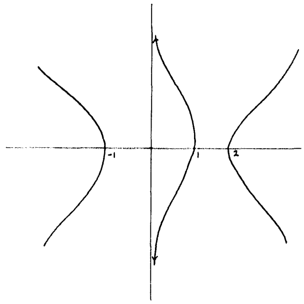

FIGURE 15. $y^2 = \frac{(x+1)(x-1)(x-2)}{x}$

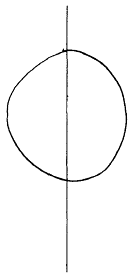

FIGURE 16. $x(x^2 + y^2 - 1) = 0$

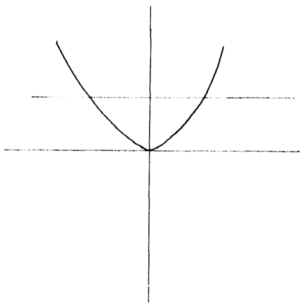

FIGURE 17. $(y - 1)(x^2 - y) = 0$

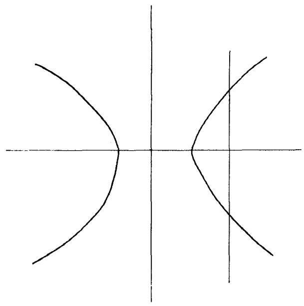

FIGURE 18. $(y - 2)(x^2 - y^2 - 1) = 0$

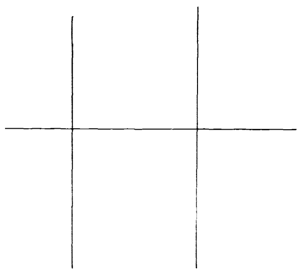

FIGURE 19. $xy(y-1)=0$

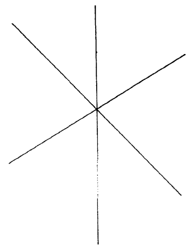

FIGURE 20. $(x-y)(x+y)x=0$

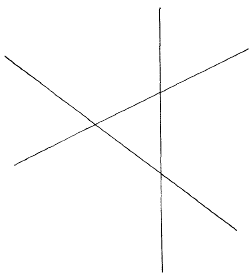

FIGURE 21. $(x-y)(x+y)(x-1)=0$

The theorem immediately implies the

COROLLARY. *The following is a complete set of topological invariants for real plane cubic curves: the numbers of bounded and unbounded components of the curve, the numbers of bounded and unbounded components of the complement of the curve, the existence of a* 6*-point, and the nesting invariant.*

A point on a curve is called an n-point if, upon removal from the curve, the component to which it belongs splits into n components. The nesting invariant gives, for each component C of the curve Γ and each component D of $\mathbf{R}^2/C$, the number of components of Γ/C which lie in D. Class 20 is distinguished as the only class possessing a 6-point (the point of intersection of the three lines). The nesting invariant is required only for the purpose of distinguishing between classes 6 and

15; the nesting invariant for class 6 is $((2,0),(2,0),(2,0))$, while that for class 15 is $((2,0),(1,1),(2,0))$. It is evident from the figures in the list of equivalence classes what the values of all the invariants are in each case.

Calculations. In the general cubic equation

$$(1)\quad Ax^3+Bx^2y+Cxy^2+Dy^3+Ex^2+Fxy+Gy^2+Hx+Iy+J=0$$

the y^3 term can be eliminated by applying an affine transformation of the plane, which is, of course, a homeomorphism. In fact, by the observation made in the previous paper [**18**], we may assume that the terms of degree three are in one of the following four forms: $x^3+xy^2, x^3-xy^2, x^3, x^2y$. Each case is examined separately.

Having eliminated y^3, the quadratic formula can now be applied to (1) to solve for y in terms of x and A,B,C,E,F,G,H,I,J. The result is:

$$(2)\quad \begin{aligned} y = &-(Bx^2+Fx+I) \pm \sqrt{x^4(B^2-4AC)} \\ &\overline{+x^3(2BF-4AG-4EC)+x^2(2BI+F^2} \\ &\overline{-4CH-4EG)+x(2FI-4CJ-4GH)} \\ &\overline{+(I^2-4GJ)}2(Cx+G) \end{aligned}$$

Evidently the topological type of the curve can now be determined by carefully examining the possibilities for the roots of the radicand and of the denominator.

The calculations are elementary, but very lengthy: The four major cases each have many sub-cases. Since each of the four major cases is similar in spirit, it is sufficient to present one of these cases. It is expedient to examine the irreducible cubics first and save the reducible (degenerate) cubics for the end. The pictures provided in this paper may not be perfectly accurate, but are certainly correct up to a homeomorphism of the plane.

Concerning the proof of equivalence within the various classes, although in some cases the homeomorphisms are trivial to construct, all cases follow from repeated application of the Schoenflies Theorem which says that any Jordan curve can be mapped onto a circle by a homeomorphism of the whole of $\mathbf{R}^2$.

Sample calculation-the case where the terms of degree three are $x^3 - xy^2$. By an affine transformation, we may assume that $G = 0$ and $F = 0$ in the general cubic equation (1). By referring to (2), the radicand is

$$x^4(4) + x^3(4E) + x^2(4H) + x(4J) + I^2 \tag{3}$$

while the denominator is $-2x$. The possibilities for the roots of (3) are now carefully examined:

(i) Quadruple root - the radicand has the form $4(x-a)^4$. The graph is degenerate.

(ii) One triple root, 1 single root - the radicand has the form $4(x-a)^3(x-b)$. Since $I^2 = 4a^3b, ab \geq 0$. The radicand is nonnegative for $x \leq b$ or $x \geq a$, where we assume $b < a$. The graph of the curve consists of three unbounded components, while the graph of the complement of the curve consists of four unbounded components.

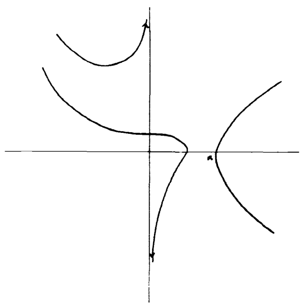

FIGURE 22.

(iii) Two double roots - the radicand has the form $4(x-a)^2(x-b)^2$. The graph is degenerate.

(iv) One double root, 2 single roots - the radicand has the form $4(x-a)^2(x-b)(x-c)$. Since $I^2 = 4a^2bc$, $bc \geq 0$. We may assume $b < c$.

If $0 < a < b < c$, then the graph is

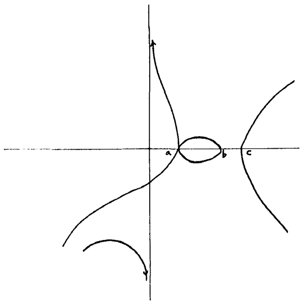

FIGURE 23.

If $a < 0 < b < c$, then the graph is

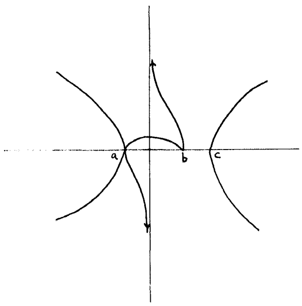

FIGURE 24.

If $0 < b < a < c$, then the graph is

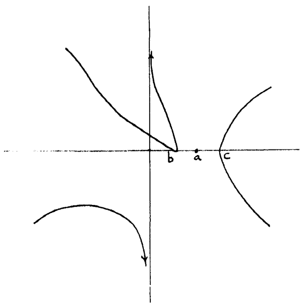

FIGURE 25.

If $b = 0$, then these types are repeated, depending on a and c. If $a = 0$, the graph is degenerate.

(v) One double root - the radicand has the form $4(x-a)^2(x^2+1)$. The graph is

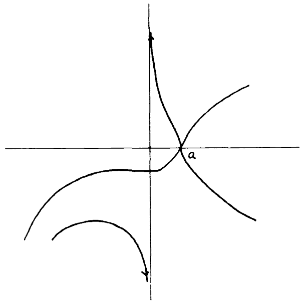

FIGURE 26.

If $a = 0$, the graph is degenerate.

(vi) Four single roots-the radicand has the form $4(x-a)(x-b)(x-c)(x-d)$. We may assume $a < b < c < d$. Since $I^2 = 4abcd$, $abcd \geq 0$. If $0 < a$, then the graph is

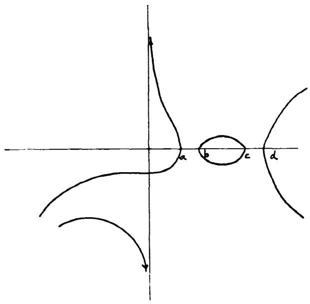

FIGURE 27.

If $a < b < 0 < c < d$, then the graph is

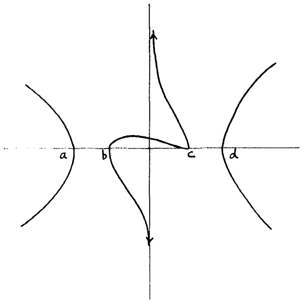

FIGURE 28.

These types are repeated if $abcd = 0$.

(vii) Two single roots - the radicand cannot have the form $4(x - a)(x - b)(x^2 + 1)(-1)$ since the coefficient of x^4 must be 4. Thus, the radicand must have the form $4(x - a)(x - b)(x^2 + 1)$. We may assume $a < b$. Since $I^2 = 4ab$, $ab \geq 0$. We may assume $0 < a$. The graph is

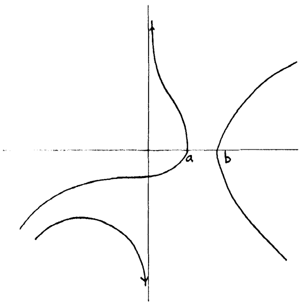

FIGURE 29.

The same type occurs if $a = 0$.

(viii) No real roots - the radicand has the form $4(x^2 + 1)(x^2 + 4)$. The graph is

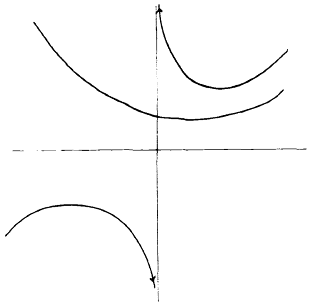

FIGURE 30.

By reviewing (i) - (viii), it follows that when the terms of degree three are $x^3 - xy^2$, the irreducible cubics fall into six equivalence classes under the homeomorphism group of the plane.

By examining the other three cases ($x^3 + xy^2, x^2y, x^3$), it follows that, up to a homeomorphism of the plane, the irreducible real plane cubic curves fall into fifteen equivalence classes.

In the case of a reducible plane cubic curve, the polynomial of degree three factors into either a linear factor and an irreducible quadratic factor or three linear factors. Now the graph of a linear factor is a straight line and the graph of an irreducible quadratic factor is either a parabola, ellipse, hyperbola, point, or nothing. The following illustrates all of the possibilities (up to a homeomorphism of the plane) for the graphs of reducible plane cubic curves. It is easy to recognize which of the possibilities were not encountered during the consideration of irreducible cubic curves.

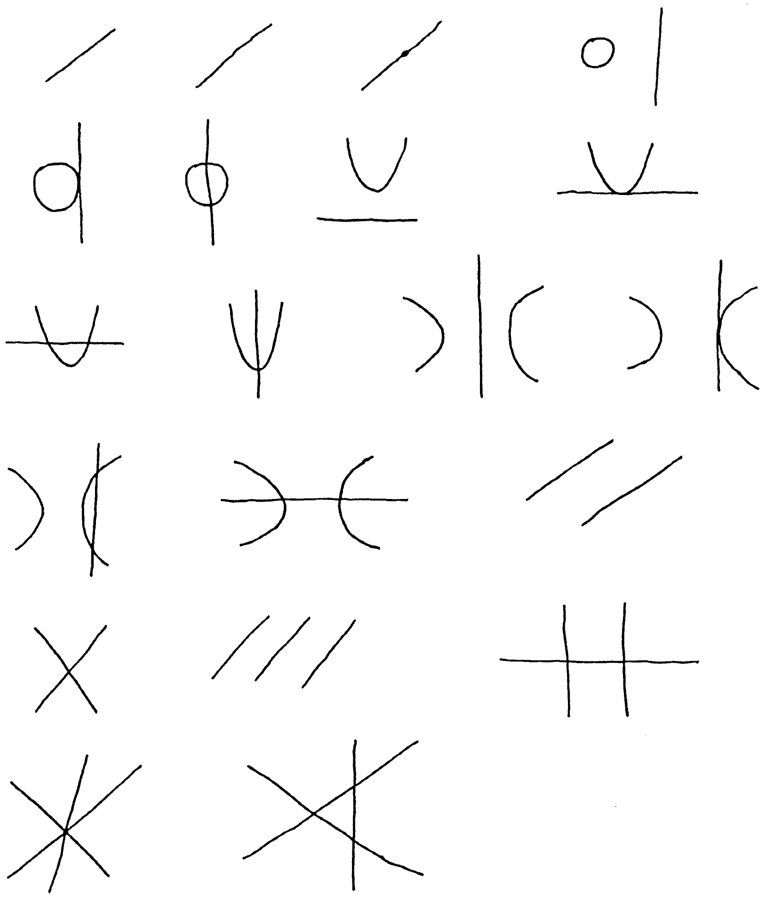

FIGURE 31.

The following are topological types not encountered during the consideration of irreducible cubic curves.

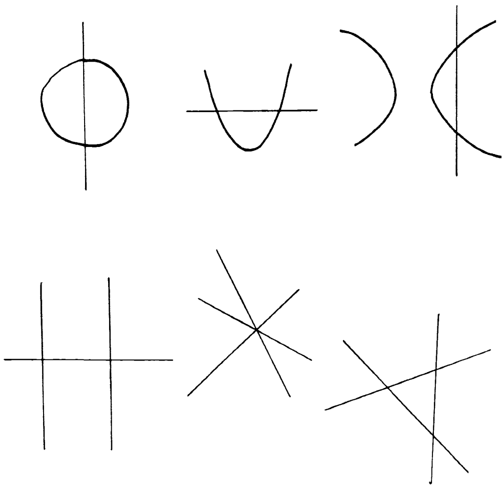

FIGURE 32.

Thus, there are a total of twenty-one equivalence classes for real plane cubic curves up to a homeomorphism of the plane. As a final remark, note that of the fifteen equivalence classes into which the irreducible cubics fall, precisely nine of them contain reducible representatives.

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