ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 3, Summer 1988

TWO THEOREMS ON INVERSE INTERPOLATION

ALAN L. HORWITZ AND L.A. RUBEL

ABSTRACT. The usual task of interpolation theory is, given a function f, or some of its properties, to find out what properties the set $\mathcal{L}(f)$, of all Lagrange interpolants of f, must have. What we mean by *inverse interpolation* is to reverse this body of problems. Namely, given the set $\mathcal{L}(f)$ or some of its properties, to recover f or some of its properties. We stress that $\mathcal{L}(f)$ is considered as an unstructured set of polynomials.

Our first result asserts that if f is analytic on the unit interval, then f is completely determined by the set $\mathcal{L}(f)$. Our second result constructs a large class of infinitely differentiable functions f on the unit interval, such that $\mathcal{L}\mathcal{L}(f) = \mathcal{P}$, the set of all polynomials. In other words, every polynomial in the world is a Lagrange interpolant of a Lagrange interpolant of f. Thus, such an f is in no wise recoverable from $\mathcal{L}(\mathcal{L}(f))$. So on the one hand, $\mathcal{L}(f)$ determines f if f is analytic on $[0,1]$, while on the other hand, $\mathcal{L}(\mathcal{L}(f))$ does not determine f if f is only assumed C^∞ on $[0,1]$. There is clearly a gap in our knowledge here that should be closed-see the problems at the end of the paper. In several further papers we are now preparing, we pursue such related questions as, "if we assume a uniform bound on all the Lagrange interpolants of f, what does this tell us about f"?

If f is a real-valued function on a set S, we say that a polynomial p, say of degree n, is a Lagrange interpolant of f, if there exist $n+1$ distinct numbers $x_0, x_1, \ldots x_n$ in S such that $f(x_i) = p(x_i)$ for $i = 0, 1, \ldots, n$. Of course there may be other points x where $p(x) = f(x)$. Then p must be given by the usual Lagrange interpolation formula

$$p(x) = \sum_{k=0}^{n} f(x_k) l_k(x),$$

AMS 1980 *Subject Classification Number* : 41A05.
The research of the second author was paritally supported by a grant from the National Science Foundation.
Received by the editors on April 29, 1986 and in revised form on October 31, 1986.

where, for $k = 0, 1, \dots, n$,

$$l_k(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}.$$

We denote, by $\mathcal{L}(f)$, the set of all such p's.

THEOREM 1. *Suppose f and g are analytic on $[0,1]$ and that $\mathcal{L}(f) = \mathcal{L}(g)$. Then $f = g$. (We stress that the hypothesis $\mathcal{L}(f) = \mathcal{L}(g)$ is an assertion about equality as unordered sets.)*

PROOF. The theorem will follow immediately from the following lemma.

LEMMA. *Suppose that $f, g \in C^\infty[0,1]$ and that $\mathcal{L}(f) = \mathcal{L}(g)$. Then either f and g agree at countably many distinct points in $[0,1]$, or $f - g$ has a zero of infinite multiplicity.*

PROOF OF THE LEMMA. Let p_n be the best uniform approximant to f on $[0,1]$, taken from π_n, the set of all polynomials of degree $\leq n$. We can assume that f is not a polynomial, since otherwise the lemma is trivial. Hence, as $n \to \infty, \deg p_n \to \infty$. By the classical Chebychev Alternation Theorem (see [**1**]), $p_n \in \mathcal{L}(f)$. By a theorem of Roulier (see [**5**]), for each positive integer j, we have

$$p_n^{(j)} \to f^{(j)} \text{ uniformly on } [0,1]. \tag{1}$$

From the hypothesis that $\mathcal{L}(f) = \mathcal{L}(g), p_n$ must be a Lagrange interpolant of g¡ so that there exist $m+1$ distinct points $(x_{1,m}, \dots, x_{m+1,m})$, where $m = m_n = \deg p_n$, such that

$$p_n(x_{k.m}) = g(x_{k,m}) \quad \text{for } k = 1, 2, \dots, m+1. \tag{2}$$

For each positive integer k, choosing a subsequence if necessary, let $x_k = \lim_{m\to\infty} x_{k,m}$.

Case 1. *There are infinitely many distinct* x_k. In this case, since $p_n \to f$ uniformly, and hence continuously,

$$\lim_{n\to\infty} p_n(x_{k,m}) = f(x_k). \tag{3}$$

But we also have, from (2),

$$\lim_{n\to\infty} p_m(x-k,m) = g(x_k). \tag{4}$$

Hence, from (3) and (4), $f(x_k) = g(x_k)$, and the lemma is proved in this case.

Case 2. *There are only finitely many different* x_k. In this case, infinitely many of the sequences $(x_{k,m})$ converge to the same number. For convenience, assume that

$$\lim_{m\to\infty} x_{k,m} = c \quad \text{for } k = 1, 2, \ldots. \tag{5}$$

Then it is easy to show that

$$f^{(j)}(c) = g^{(j)}(c) \quad \text{for } j = 0, 1, 2, \ldots, \tag{6}$$

and the lemma is proved in this case. The details of the proof of (6) are as follows.

Let r be a fixed positive integer, and consider

$$a_n = p_n[x_{1,m}, x_{2,m}, \ldots, x_{r+1,m}],$$

the r-th order divided difference of p_n at $\{x_{1,m}, x_{2,m}, \ldots, x_{r+1,m}\}$, remembering that $m = m_n$ depends on n. By (2) and standard properties of divided differences [**1**; Corollary 3.4.3, p. 65],

$$a_n = g[x_{1,m}, x_{2,m}, \ldots, x_{r+1,m}] \to g^{(r)}(c)/r!.$$

Also, we have that [**1**, Corollary 3.4.2, P. 65]

$$a_n = p_n^{(r)}(\xi_n)/r!$$

for some ξ_n in the smallest closed interval $I_{n,r}$ that contains $\{x_{1,m}, x_{2,m}, \ldots, x_{r+1,m}\}$. But since p_n converges uniformly to $f^{(r)}$by (1), it follows that $\xi_n \to c$ as $n \to \infty$ and since $p_n^{(r)}(\xi_n) \to f^{(r)}(c)$. Since r is arbitrary, we see that $f - g$ has a zero of infinite order at c.

There is an attractive alternative approach to a proof of this lemma, namely through fixed-point theorems for multi-valued functions. Unfortunately, the mappings φ we construct below do not seem to fit the hypotheses of any of the fixed-point theorems known to us (see, e.g., [**7**] and [**10**]). The idea, though, is as follows. Fix n and let $I = [0,1]^n$ be the unit cube in $\mathbf{R}^n$. For each point $\tilde{x} = (x_1, \ldots, x_n) \in I$ look at the Lagrange interpolant p, interpolates to g at, say, $\tilde{y} = (y_1, \ldots, y_n) \in I$.

(Here, we come to a second difficulty with this line of proof, namely, that if $m = \deg p$, we might well have $m < n - 1$ so that p need only interpolate to g at $m + 1$ points instead of at n points. However, there are perhaps ways to circumvent this difficulty.)

Let φ be the (possibly multi-valued) map $\varphi(\tilde{x}) = \tilde{y}$. (If some of the x_i, or some of the y_i coincide, then the usual conventions about interpolation of derivatives are to be used.) The idea is to show that φ must have a fixed point in I. (This is what we do not know how to do.) Once this is done, the rest is proved as above.

REMARK. It is interesting to note that Theorem 1 holds if we only assume that one of the functions, say f, is analytic on $[0,1]$, while the other function g is merely defined on $[0,1]$. This follows directly from the following

LEMMA. *A function f is analytic on* $[0,1]$ $\Leftrightarrow$ *there exists a finite constant K such that, for all* $n = 0, 1, 2, \ldots$,

$$(\#) \qquad ||p||_{\infty,[0,1]} \leq K^n$$

for all $p \in \mathcal{L}(f)$ with $\deg p = n$.

PROOF. The $\Rightarrow$ implication follows directly from Corollary 3.6.2 on p. 68 of [**1**]. For the $\Leftarrow$ implication, suppose that (#) holds. By a very slight modification of the argument used to prove Theorem 2 of [**6**], we see that $f \in C^\infty[0,1]$. For $x_0 \in [0,1]$ let

$$s_n(x:x_0) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

be the n-th partial sum of the Taylor series expansion of $f(x)$ around x_0. We may think of this as an interpolating polynomial with $n+1$ nodes coaslescing at x_0. By a passage to the limit in (#), then, we have (unless f is a polynomial, which certainly implies what we want) $|s_n(x:x_0)| \le K^n$ and also $|s_{n-1}(x:x_0)| \le K^{n-1}$. Supposing, with no loss of generality, that $K \ge 1$, we have

$$\frac{|f^{(n)}(x_0)|}{n!}|x-x_0|^n = |s_n(x:x_0) - s_{n-1}(x:x_0)| \le 2K^n.$$

Now choosing either $x = 0$ or $x = 1$, depending on which is furthest from x_0, so that $|x - x_0| \ge 1/2$, we get

$$\frac{|f^{(n)}(x_0)}{n!} \le 2^{n+1}K^n,$$

and since x_0 is arbitrary, the analyticity of f on $[0,1]$ follows immediately.

Thus we see that if we know that the Lagrange interpolants to f of degree n grow no faster than exponentially in n, then we may in principle recover f from these interpolants.

REMARK. For a problem related to Theorem 1, see [**3**].

THEOREM 2. *There exists an $f \in C^\infty[0,1]$ such that, for any real polynomial p whatever, there exists a real polynomial q such that q is a Lagrange interpolant of f on $[0,1]$, and p is a Lagrange interpolant of q on $[0,1]$.*

REMARK 1. It will be evident from the construction of f that we can make f coincide on $[0/10, 1[$, say, with any preassigned C^∞-function h on that interval. It follows that the set of such functions f has the cardinal number of the continuum. For any two distinct such f, say f_α and f_β, we have

$$\mathcal{L}(\mathcal{L}(f_\alpha)) = \mathcal{L}(\mathcal{L}(f_\beta)).$$

REMARK 2. If we consider $[0, \infty)$ instead of $[0, 1]$, then, on choosing $f(x) = e^x \sin x$¡ say,we see that every real polynomial p is actually a Lagrange interpolant of f on $[0, \infty)$. Indeed $p(x) = f(x)$ for an infinite sequence of x is tending to $+\infty$. Using similar arguments to the one in Remark 1, we can easily construct distinct analytic functions f and g on $[0, \infty)$ such that $\mathcal{L}(f) = \mathcal{L}(g)$. We do not know whether this is possible for a continuous or C^∞-function on the interval $[0, 1]$. Clearly, if f is bounded on a set S, then there will be many polynomials p such that $p(x) = f(x)$ for no $x \in S$.

PROOF OF THEOREM 2. For each positive integer n, say $n > 5$ for safety, construct a real polynomial q_n so that both

(7) throughout the interval $\dfrac{1}{n+1} + \dfrac{1}{n^4} \le x \le \dfrac{1}{n} - \dfrac{1}{n^4}$,

every one of the first n derivatives of q_n has absolute value $\le 2^{-n}$, say, and

(8) on the interval $1 - \dfrac{1}{n} \le x \le 1 - \dfrac{1}{n+1}$,

q_n oscillates at least $2n$ times from below $-n$ to above $+n$.

We don't care what q_n does outside these two intervals, or how large its degree is.

One way to construct such a q_n is via Runge's theorem, to get a polynomial Q_n, possibly complex on $\mathbf{R}$, that approximates 0 on $I_n = [\frac{1}{n+1} + \frac{1}{n^4}, \frac{1}{n} - \frac{1}{n^4}]$, with its first n derivatives small on that interval, and that approximates $n^2 \sin e^n x$, say, on $J_n = [1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$. We then take

$$q_n(z) = \frac{Q_n 9z) + \overline{Q}_n(\overline{z})}{2}$$

so that q_n is real on $\mathbf{R}$.

Now let $h(x)$ be a C^∞-function on $\mathbf{R}$ such that

(9) $$h(x) = 1 \text{ for all } \frac{1}{3} \le x \le \frac{2}{3}$$

and

$$h(x) = 0 \text{ whenever } x \leq 0 \text{ or } x \geq 1. \tag{10}$$

Such an h can be easily constructed by piecing together integrals of nonnegative C^∞-functions of compact support. Writing $I_n = [a_n, b_n]$ for convenience, we shall let

$$f(x) = \sum h_n(x) q_n(x), \tag{11}$$

where

$$h_n(x) = h\Big(\frac{x - a_n}{b_n - a_n}\Big). \tag{12}$$

To prove that $f \in C^\infty[0,1]$, we need only prove that $f^{(r)}90) = 0$ for all non-negative integers r. It will suffice for this to prove that if $x_n \to 0$, then $f^{(r)}(x_n) \to 0$. We may suppose that $x_n \in I_n$. But, on I_n,

$$\begin{aligned} |f^{(r)}(x_n)| &= |\sum_{j=0}^{r} \binom{r}{j} h_n^{(j)}(x_n q_n^{(r-j)}(x_n)| \\ &= |\sum_{j=0}^{r} \binom{r}{j} (\frac{1}{b_n - a_n})^j h^{(j)} (\frac{x_n - a_n}{b_n - a_n}) q_n^{(r-j)}(x_n)| \\ &\leq \sum_{j=0}^{r} \binom{r}{j} (n+1)^{2j} ||h^{(j)}|| 2^{-n} \leq 2^{-n} \\ &\quad \sum_{j=0}^{r} \binom{r}{j} (n+1)^{2j} ||h^{(j)}||, \end{aligned} \tag{13}$$

(where $||\cdot||$ is the supreme norm) since $b_n - a_n \geq (n+1)^{-2}$. So it is clear that $f^{(r)}(x_n) \to 0$ as $n \to \infty$, for each fixed r. This completes the construction of f.

Now, given any real polynomial p, say of degree k, it is clear that if we choose n large enough (say $n > k$ and $n > \max\{|p(x)| : 0 \leq x \leq 1\}$), then $q_n(x) - p(x)$ will have at least $k + 1$ zeros on the interval J_n, because of the wild oscillations of q_n there. But q_n is abviously a Lagrange interpolant of f since it actually coincides with f throughout an open interval. Thus the theorem is proved.

REMARK 3. If f is any transcendental entire function, then every polynomial is a Lagrange interpolant (on **C**) of a Lagrange interpolant (on **C**) of f. This follows by THeorem 2.5 on p. 47 of [**2**], which implies that there are at most two polynomials, say p_1 and p_2, for which $f - p_1$ and $f - p_2$ haveonly finitelymany zeros. So every polynomial other than p_1 and p_2 is surely a Lagrange interpolant of f and, as a Lagrange interpolant of itself, becomes a Lagrange interpolant of a Lagrange interpolant of f. Now to handle $p - 1$ and p_2, let P be some Lagrange interpolant to f, with $\deg P > \max(\deg p_1, \deg p_2)$. By the Fundamental Theorem of Algebra, p_1 and p_2 are Lagrange interpolants of P.

PROBLEM 1. Is it possible to choose the universal function f of Theorem 2 to be real-analytic on $[0,1]$?

PROBLEM 2. If f and g are analytic on $[0,1]$ and $\mathcal{L}(\mathcal{L}(f)) = \mathcal{L}(\mathcal{L}(g))$, must $f = g$?

PROBLEM 3. If f and g are C^∞ (or perhaps merely continuous) on $[0,1]$ and $\mathcal{L}(f) = \mathcal{L}(g)$, must $f = g$?

PROBLEM 4. What are the conditions on a set P of polynomials that make $P = \mathcal{L}(f)$ for some f?

Added in Proof. Since this paper was written, V. Totik has answered Problem 1 in the affirmative and Problem 2 in the negative (see [**9**]).

REFERENCES

1. P.J. Davis, *Interpolation and Approximation,* Blaisdell Publishing Company, New York, Toronto, and London, 1963.

2. W.K. Hayman, *Meromorphic Functions,* Oxford University Press, Osford, 1964.

3. A.L. Horowitz, *Reconstructing a function from its set of tangent lines,* Am. Math., Mo, to appear.

4. ——— and L.A. Rubel, *Restrictions on the zeroes of Lagrange interpolants to analytic functions,* Complex Var. **6** (1986), 149-157.

5. ———, *The space of totally bounded analytic functions,* Proc. of the Edinburgh Math. Soc. **30** (1987), 229-246.

6. A.L. Horwitz and L.A. Rubel, *Totally positive and totally bounded functions on* $[-1,1]$, J. Approx. Theory **52** (1988), 204-216.

7. R.P. Jerrad, *Homology with multiple-calued functions applied to fixed points,* Trans. Amer. Math. Soc. **213** (1975), 407-427 (also erratum **218** (1976), 406).

8. J. Roulier, *Best approximation to functions with restricted derivatives,* J. Approximation Theory **17** (1976), 344-347.

9. V. Totik, *Solution to two problems in inverse interpolation,* to appear in Rocky Mtn. J. of Math.

10. R. Wegrzyk, *Fixed point theorems for multi-valued functions and their applications to functional equations,* Dissertationes Math. **201** (1982), 31pp.

PENNSYLVANIA STATE UNIVERSITY, DELAWARE COUNTY CAMPUS, 25 YEARSLEY MILL ROAD, MEDIA, PA 19063

DEPARTMENTOF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, IL 61801