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PRODUCT COMPLEX SUBMANIFOLD OF INDEFINITE COMPLEX SPACE FORMS

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ABSTRACT. We study product complex submanifolds of indefinite complex space forms and characterize a submanifold which corresponds to the Segre imbedding in a definite complex space form.

0. Introduction. It is well-known that for complex hyperbolic spaces an analogue to the Segre imbedding cannot be given. The key to this is that if there is a holomorphic isometric immersion of a product of two Kaehler manifolds into a complex space form, then the holomorphic sectional curvature of the ambient space has to be non-negative (see [8]). A careful observation to the proof of this fact tells us that the positivity of the metric in each normal space to the submanifold also must be taken into account. Thus the following problems arise in a natural way:

1. Does a product of two complex hyperbolic spaces admit a holomorphic isometric imbedding in an "indefinite" complex hyperbolic space?

2. If the answer is yes, find the smallest possible dimension and index of such an indefinite complex hyperbolic space.

3. Characterize this holomorphic isometric imbedding when dimension and index are as small as possible.

In this paper these problems will be solved in a more general context. In §1, an indefinite analogue to the Segre imbedding for indefinite complex projective (and hyperbolic) spaces is given. Using the relationship between indefinite complex hyperbolic spaces and indefinite complex projective spaces, we obtain a holomorphic isometric imbedding of a product of definite complex hyperbolic spaces into an "indefinite" complex hyperbolic space. In §2, some basic formulas are recalled for later use. In §3, an answer to the second problem is given (Corollary 3.2)

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as a consequence of a more general result (Theorem 3.1), which is obtained under the assumption that manifolds are complete. As one can see from the proofs, the corresponding local versions are also true, provided that this hypothesis is dropped. In §4 parallelism of the second fundamental form provides us a characterization of the indefinite Segre imbedding. Finally, the Segre imbedding of a product of two definite complex hyperbolic spaces into an indefinite complex hyperbolic space is characterized in §5 by using the length of its second fundamental form.

1. The indefinite Segre imbedding. Let $CP_s^n(c)$ be the indefinite complex projective space of complex dimension n, index $2s$ and constant holomorphic sectional curvature $c > 0$. We recall that $CP_s^n(C) = S_{2s}^{2n+1}(c/4)S^1$ where $S_{2s}^{2n+1}(c/4)$ is the $(2n+1)$-dimensional indefinite sphere with index $2s$ and of sectional curvature $c/4$. Thus a point of $CP_s^n(c)$ can be represented by $[(z,w)]$ where $z = (z_1, \dots z_s) \in \mathbf{C}^s$, $w = (w_1, \dots, w_{n-s+1}) \in \mathbf{C}^{n-s+1}, (z,w) \in S_{2s}^{2n+1} \subset \mathbf{C}_s^{n+1}$ and $[(z,w)]$ denotes the class $(z,w) \cdot S^1$.

We consider a mapping

$$\phi : CP_s^n(c) \times CP_t^m(c) \to CP_{R(n,m,s,t)}^{N(n,m)}(c)$$

with

$$N(n,m) = n + m + nm \text{ and } R(n,m,s,t) = s(m-t) + t(n-s) + s + t$$

given by

$$\phi([(z,w)],[(x,y)]) = [(z_i y_\alpha, w_k x_a, z_j x_b, w_l y_\beta)],$$

where

$$i,j = 1,2,\dots,s; \qquad k,\ell = 1,2,\dots,n-s+1$$

and

$$a,b = 1,2,\dots,t; \qquad \alpha,\beta = 1,2,\dots,n-t+1.$$

Then ϕ is a well-defined holomorphic mapping and, from the results in [1], it is easy to see that ϕ is also an isometric imbedding. It is called the "indefinite Segre imbedding" of $CP_s^n(c) \times CP_t^m(c)$ into $CP_{R(n,m,s,t)}^{N(n,m)}(c)$.

Note that if $s = t = 0$, then $R(n, m, s, t) = 0$ for all n, m and ϕ becomes the classical Segre imbedding (see [3] or [4]).

In the definite case $CP^1(c) \times CP^1(c)$ is the complex quadric CQ^2 in $CP^3(c)$. However $CP_1^1(c) \times CP_1^1(c)$ and $CP_1^1(c) \times CP^1(c)$ are mutually different complex quadric in $CP_2^3(c)$; in fact, they are respectively denoted CQ_2^2 and CQ_1^* in [7].

By using the fact that the indefinite complex hyperbolic space $CH_s^n(-c)$ of complex dimension n, index $2s$ and holomorphic sectional curvature $-c, c > 0$ is obtained from $CP_{n-s}^n(c)$ by changing the Kaehler metric of $CP_{n-s}^n(c)$ to its negative. Another indefinite Segre imbedding

$$\phi : CH_s^n(-c) \times CH_t^m(-c) \to CH_{s(n,m,s,t)}^{N(n,m)}(-c)$$

is given, where

$$S(n, m, s, t) = (n - s)(m - t) + st + s + t.$$

In particular, for $s = t = 0$ we have a holomorphic isometric imbedding ϕ of a product of definite complex hyperbolic spaces $CH^n(-c) \times CH^m(-c)$ into an indefinite complex hyperbolic space $CH_{nm}^{N(n,m)}(-c)$.

2. Some basic formulas. In this section some basic results will be outlined for later use.

Let M be an indefinite Kaehler manifold isometrically and holomorphically immersed in an indefinite complex projective space $CP_R^N(c)$. Let g and J be the Kaehler metric and the complex structure of $CP_R^N(c)$, also its induced ones on M. Let ∇ and ∇ denote the metric connections of $CP_R^N(c)$ and M. The pull back of the tangent bundle of the ambient space is expressed as an orthogonal sum $TM \oplus NM$, where TM and NM denote the tangent bundle and the normal bundle of M, respectively. Let D be the normal connection on NM induced from ∇. Then the Gauss and Weingarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{2.1}$$

$$\overline{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for any X, Y, TM and $\xi \in NM$. The tensors B and A_ξ are called the second fundamental form and the Weingarten endomorphism associated with ξ, respectively. $B(X,Y)$ is symmetric with respect to X and Y and related to A by

$$g(A_\xi X, Y) = g(B(X,Y), \xi). \tag{2.3}$$

Moreover, for a Kaehler submanifold of an indefinite Kaehler manifold the following formulas are given:

$$B(JX,Y) = B(X,JY) = JB(X,Y), \tag{2.4}$$

$$A_\xi J = -JA_\xi = -A_{J\xi}, \tag{2.5}$$

$$D_X J\xi = JD_X\xi. \tag{2.6}$$

The usual connection $\tilde{\nabla}$ induced from ∇ and D is defined by

$$(\tilde{\nabla}_X B)(Y,Z) = D_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for vector fields X, Y and Z tangent to M.

By using the fact that $CP^N_R(c)$ has constant holomorphic sectional curvature c, the Gauss equation for the curvature tensor R of M and the Codazzi equation are given respectively by

(2.7)
$$\begin{aligned} g(R(X,Y)Z,W) = \frac{c}{4}&(g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) \\ &+ 2g(X,JY)g(JZ,W)) + g(B(Y,Z), g(X,W)) \\ &- g(B(X,Z), B(Y,W)), \end{aligned}$$

$$(\tilde{\nabla}_X B)(Y,Z) = (\tilde{\nabla}_Y B)(X,Z), \tag{2.8}$$

and we have

$$\begin{aligned} (\tilde{\nabla}_{JX} B)(Y,Z) &= (\tilde{\nabla}_X B)(JY,Z) = (\tilde{\nabla}_X B)(Y,JZ) \\ &= J(\tilde{\nabla}_X B)(Y,Z). \end{aligned} \tag{2.9}$$

The curvature tensor $R^\perp$ of the normal connection D, since $CP^N_R(c)$ has constant holomorphic sectional curvature c, is given by the Ricci equation

$$g(R^\perp(X,Y)\xi,\eta) = g(X,JY)g(J\xi,\eta) + g([A_\xi, A_\eta]X,Y) \tag{2.10}$$

for any vector fields X, Y tangent to M and any vector fields ξ, η normal to M in $CP^N_R(c)$.

Let M^n_s and M'^m_t be indefinite Kaehler manifolds of complex dimensions n, m and index $2s, 2t$, respectively. Assume that the Riemannian product $M^n_s \times M'^m_t$ admits a holomorphic isometric immersion into $CP^N(c)$. Tangent vector fields to M^n_s or to M'^m_t can be regarded as one to $M^n_s \times M'^m_t$ in a natural way. Then we have

$$g(R(X,Y)Z,W) = 0 \tag{2.11}$$

if X, W are tangent to M^n_s and Y, Z tangent to M'^m_t or if X, Y, Z are tangent to M^n_s (resp. M'^m_t) and W tangent to M'^m_t (resp. M^n_s).

3. Product submanifolds. This section will be concerned with the following theorem:

THEOREM 3.1. *Let M^n_s and M'^m_t be complete indefinite Kaehler manifolds with complex dimensions n, m and index $2s, 2t$, respectively. Assume that there exists a holomorphic isometric immersion φ of $M^n_s \times M'^m_t$ into $CP^N_R(c)$, $c > 0$. Then*

(1) *$N \geq N(n,m)$ and $R \geq R(n,m,s,t)$.*

(2) *If $N = N(n,m)$, then $R = R(n,m,s,t)$, M^n_s is holomorphically isometric to $CP^n_s(c)$, M'^m_t is holomorphically isometric to $CP^m_t(c)$ and, by identifying $M^n_s \times M'^m_t$ with $CP^n_s(c) \times CP^m_t(c)$, the immersion φ is an imbedding obtained by the composition of the indefinite Segre imbedding*

$$\phi : CP^n_s(c) \times CP_t(c) \to CP^{N(n,m)}_{R(n,m,s,t)}(c)$$

and a rigid motion of $CP^{N(n,m)}_{R(n,m,s,t)}(c)$.

PROOF. We choose a local orthonormal frame of vector fields $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$ and $\{X'_1, \ldots, X'_1, JX'_1, \ldots, JX'_1\}$ respectively in M_s^n and $M_t'^m$, that is, they are mutually orthogonal vector fields and satisfy

$$\begin{aligned} &g(X_i, X_i) = -1 \text{ or } 1 \text{ according to } 1 \leq i \leq s \text{ or } s+1 \leq i \leq n \\ &g(X'_a, X'_a) = -1 \text{ or } 1 \text{ according to } 1 \leq a \leq t \text{ or } t+1 \leq a \leq m. \end{aligned} \tag{3.1}$$

From (2.11) it follows that

$$g(R(X_i, X'_a)(X'_b, X_j)) = g(R(X_i, JX'_a)JX'_b, X_j) = 0. \tag{3.2}$$

In this section the following convention on the range of indices is used unless otherwise stated: $i, j = 1, \ldots, n$ and $a, b = 1, \ldots m$.

The Gauss equation (2.7) and (3.2) imply

$$\begin{aligned} &g(B(X_i, X'_b), B(X_j, X'_a)) - g(B(X_i, X_j), B(X'_a, X'_b)) \\ &= \frac{c}{4} g(X_i, X_j) g(X'_a, X'_b) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &g(B(X_i, JX'_b), B(X_j, JX'_a)) - g(B(X_i, X_j), B(JX'_a, JX'_b)) \\ &= \frac{c}{4} g(X_i, X_j) g(X'_a, X'_b). \end{aligned} \tag{3.4}$$

So from (2.4), both (3.3) and (3.4) give us

$$g(B(X_i, X'_b), B(X_j, X'_a)) = \frac{c}{4} g(X_i, X_j) g(X'_a, X'_b). \tag{3.5}$$

A similar reasoning from

$$g(R(X_i, X'_a)JX'_b, X_j) = g(R(X_i, JX'_a)X'_b, X_j) = 0 \tag{3.6}$$

provides us with

$$g(B(X_i, X'_a), JB(X_j, X'_b)) = 0. \tag{3.7}$$

Thus if a vector field ξ_{ia} is defined by $\frac{2}{\sqrt{c}} B(X_i, X'_a)$, then it follows that $\{\xi_{ia}, J\xi_{ia}\}$ are orthogonal vector fields normal to $M_s^n \times M_t'^m$ which satisfy

$$\begin{aligned} &g(\xi_{ia}, \xi_{ia}) = -1 \\ &\text{if } 1 \leq i \leq s \text{ and } t+1 \leq a \leq m \\ &\text{or } s+1 \leq i \leq n 1 \leq a \leq t, \text{ and} \end{aligned} \tag{3.8a}$$

$$
\begin{aligned}
&g(\xi_{ia}, \xi_{ia}) = 1 \\
&\text{if } 1 \leq i \leq s \text{ and } 1 \leq a \leq t \text{ or} \\
&s+1 \leq i \leq n \text{ and } t+1 \leq a \leq m.
\end{aligned} \tag{3.8b}
$$

Hence the normal space at each point has complex dimension $\geq nm$ and index $\geq 2s(m-t)+2t(n-s)$, and the dimension N and the index $2R$ satisfy $N \geq N(n,m)$ and $R \geq R(n,m,s,t)$.

Now suppose that $N = N(n,m)$. In this case $\{\xi_{ia}, J\xi_{ia}\}$ is a basis of the normal space at each point and hence it follows from (3.8) that the index is given by $2R = 2R(n,m,s,t)$. In order to prove that M_s^n and $M_t'^m$ are holomorphically isometric to complex projective spaces, it is first shown that M_s^n and $M_t'^m$ are totally geodesic in $CP^{N(n,m)}_{R(n,m,s,t)}(c)$. In fact, again, (2.11) implies

$$
g(R(X_i, X_j)X'_a, X_k) = g(R(JX_i, X_j)X'_a, JX_k) = 0 \tag{3.9}
$$

$$
g(R(X_i, X_j)JX'_a, X_k) = g(R(JX_i, X_j)X'_a, JX_k) = 0 \tag{3.10}
$$

But from the Gauss equation (2.7), both (3.9) and (3.10) give

$$
g(B(X_i, X_k), B(X_j, X'_a)) = g(B(X_i, X_k), JB(X_j, JX'_a)) = 0. \tag{3.11}
$$

This equation means that $B(X_i, X_k)$ is orthogonal to each vector field of the basis $\{\xi ia, J\xi ia\}$ and hence $B(X_i, X_k) = 0$ for all $i, k = 1, \ldots, n$, that is, M_s^n is totally geodesic. By means of a similar reasoning, $M_t'^m$ is totally geodesic. By means of a result in [1], both M_s^n and $M_t'^m$ have constant holomorphic sectional curvature c. Therefore M_s^n (respectively $M_t'^m$) is an open set of $CP_s^n(c)$ (respectively $CP_t^m(c)$), but the assumption that M_s^n is complete (respectively $M_t'^n$ is complete) implies $M_s^n = CP_s^n(c)$ (respectively $M_t'^m = CP_t^m(c)$).

Finally, for each holomorphic isometric imbedding φ of $CP_s^n(c) \times CP_t^m(c)$ into $CP^{N(n,m)}_{R(nm,m,s,t)}(c)$ it is verified that there exists a rigid motion of F of $CP^{N(n,m)}_{R(nm,m,s,t)}(c)$ such that $\varphi = F \circ \phi$. In fact we choose a local orthonormal frame of vector fields $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$ and $\{X'_1, \ldots, X'_m, JX'_1, \ldots, JX'_m\}$ in $CP_s^n(c)$ and $CP_t^m(c)$ respectively satisfying (3.1). We denote by B and $\tilde{B}$ the second fundamental

forms of ϕ and φ respectively and define $\xi ia = \frac{2}{\sqrt{c}}B(X_i, X'_a), \tilde{\xi} ia = \frac{2}{\sqrt{c}}\tilde{B}(X_i, X'_a)$. Then $\{\xi_{ia}, J\xi_{ia}\}$ and $\{\tilde{\xi}_{ia}, J\tilde{\xi}_{ia}\}$ are orthonormal bases of the normal spaces of ϕ and φ, because both ϕ and φ induce totally geodesic immersions of $CP_s^n(c)$ and $CP_t^m(c)$ into $CP^{N(n,m)}_{R(n,m,s,t)}(c)$. In this case it is claimed that the Weingarten endomorphisms $A_{\xi_{ia}}$ and $\tilde{A}_{\tilde{\xi}_{ia}}$ corresponding to ξ_{ia} and $\tilde{\xi} ia$ according to ϕ and φ coincide. In fact, from the Gauss equation (2.7), we have

$$A_{B(Y,Z)}X - A_{B(X,Z)}Y = \tilde{A}_{\tilde{B}(Y,Z)}X - \tilde{A}_{\tilde{B}(X,Z)}Y \tag{3.12}$$

for all X, Y, Z tangent to $CP_s^n(c) \times CP_t^m(c)$. Changing X and Z in (3.12) into JX and JZ and taking into account (2.4) and (2.5) we have

$$A_{B(Y,Z)}X + A_{B(X,Z)}Y = \tilde{A}_{\tilde{B}(Y,Z)}X + \tilde{A}_{\tilde{B}(X,Z)}Y \tag{3.13}$$

(3.12) and (3.13) imply

$$A_{B(Y,Z)}X = \tilde{A}_{\tilde{B}(Y,Z)}X$$

for all X tangent to $CP_s^n(c) \times CP_t^m(c)$, and so

$$A_{B(Y,Z)} = \tilde{A}_{\tilde{B}(Y,Z)}$$

which yields

$$A_{\xi_{ia}} = \tilde{A}_{\tilde{\xi}_{ia}}. \tag{3.14}$$

Now, for any point p of $CP_s^n(c) \times CP_t^m(c)$, let U be an open neighbourhood of p on which ξia and $\tilde{\xi} ia$ are defined as above. A holomorphic linear isometry

$$L : T_{\phi(p)}CP^{N(n,m)}_{R(n,m,s,t)}(c) \to T_{\varphi(p)}CP^{N(n,m)}_{R(n,m,s,t)}(c)$$

is given by

$$\begin{aligned} &L(\xi_{ia}(p)) = \tilde{\xi}_{ia}(p), \qquad L(J\xi_{ia}(p)) = J\tilde{\xi}_{ia}(p) \\ &L(d\phi_p(X)) = d\varphi_p(X) \quad \text{for all } X \text{ in } T_p(CP_s^n(c) \times CP_t^m(c)), \end{aligned} \tag{3.15}$$

where $TqCP_R^N(c)$ denotes the tangent space of $CP_R^N(c)$ at the point q. As in [6] there exists a unique holomorphic motion F of $CP^{N(n,m)}_{R(n,m,s,t)}(c)$ such that $F(\phi(p)) = \varphi(p)$ and that the differential of F at $\phi(p)$ coincides with L, which means

$$F(\phi(p)) = \varphi(p). \qquad dF_{\phi(p)}.d\phi_p = d\varphi_p$$

and

$$dF_{\phi(p)}(\xi_{ia}(p)) = \tilde{\xi}_{ia}(p).$$

Moreover, by (3.14), an analogous result to the local rigidity theorem of Calabi [2] shows that the imbedding is determined up to within the group of motions in the ambient space, which implies that $F \circ \phi = \varphi$ on U. From considerations of analyticity it follows that the extension theorem for local mappings in [6] guarantees that the relation $F \circ \phi = \varphi$ remains true on $CP_s^n(c) \times CP_t^m(c)$.□0

By using the fact that an indefinite complex hyperbolic space can be obtained from an indefinite complex projective space we have

COROLLARY 3.2. *Let M^n and M'^m be complex Kaehler manifolds with complex dimensions n and m, respectively. Assume that there exists a holomorphic isometric immersion φ of $M^n \times M'^m$ into $CH_S^N(-c), c > 0$. Then*

(1) $N \geq N(n,m)$ and $S \geq nm$,

(2) *If $N = N(n,m)$, then $S = nm$, M^n is holomorphically isometric to $CH^n(-c)$, M'^m is holomorphically isometric to $CH^m(-c)$ and φ is given by the Segre imbedding of $CH^n(-c) \times CH^m(-c)$ into $CH_{nm}^{N(n,m)}(-c)$ and a rigid motion of $CH_{nm}^{N(n,m)}(-c)$.*

REMARK. The corresponding local versions of Theorem 3.1 and Corollary 3.2 are also true.

4. Parallel second fundamental form. From the Codazzi equation (2.8) it is easy to see that the indefinite Segre imbedding

$$\phi : CP_s^n(c) \times CP_t^m(c) \to CP^{N(n,m)}_{R(n,m,s,t)}(c)$$

has parallel second fundamental form. In the definite case this property gives a well-known characterization of the Segre imbedding (see [4] and [8] for instance). Then it seems natural to study an analogous indefinite case. In this case we have

THEOREM 4.1. *Let M_s^n and $M_t'^m$ be complete indefnite Kaehler manifolds with complex dimensions n, m, and indices $2s, 2t$, respectively. Assume that there exists a holomorphic isometric immersion φ of $M_s^n \times M_t'^m$ into $CP^N_{R(n,m,s,t)}(c), c > 0$, with parallel second fundamental form. Then M_s^n (respectively $M_t'^n$) is holomorphically isometric to $CP_s^n(c)$ (respectively $CP_t^n(c)$) and φ is given by the indefinite Segre imbedding ϕ of $CP_s^n(c) \times CP_t^m(c)$ into $CP^{N(n,m)}_{R(n,m,s,t)}(c)$ and a rigid motion of $CP^{N(n,m)}_{R(n,m,s,t)}(c)$. The corresponding local version is also true.*

PROOF Let $\nu = \mathrm{Span}\{B(X_i, X_a'), JB(X_i, X_a')\}$. From (3.8) and the assumption on the index it follows that the orthogonal complement $\nu^\perp$ of ν is a positive definite subspace of the normal space at each point.

By using (3.11), a linear subspace μ defined by

$$\mu = \mathrm{Span}\{B(X_i, X_j), JB(X_i, X_j)\}$$

satisfies

$$\mu \subset \nu^\perp \tag{4.1}$$

and hence μ is a positive definite subspace.

On the other hand, (2.5) and the Ricci equation (2.10) imply

$$\begin{aligned} &g(R^\perp(JX_a', X_a')B(X_i, X_j), JB(X_i, X_j)) \\ &= \frac{c}{2}||X'||^2||B(X_i, X_j)||^2 + 2g(A^2_{B(X_i,X_j)}X_a', X_a'). \end{aligned} \tag{4.2}$$

Now the assumption that the second fundamental form is parallel yields the relation

$$D_{X_a'}B(X_i, X_j) = B(\nabla_{X_a'}X_i, X_j) + B(X_i, \nabla_{X_a'}X_j) = 0, \tag{4.3}$$

because $\nabla_{X'_a} X_i = 0$. Therefore (4.3) says that the left hand side of (4.2) vanishes identically, and so from (2.3) it reduces to

$$\begin{aligned} &\frac{c}{2}||X'||^2||B(X_i, X_j)||^2 \\ &= -2g(B(A_{B(X_i,X_j)}X'_A, X'_a), B(X'_i, X'_j)). \end{aligned} \tag{4.4}$$

But according to (3.3) and (3.4) we have

$$g(A_{B(X_i,X_j)}X'_b, X'_c) = g(B(X_i, X_j), B(X'_b, X'_c)) = 0 \tag{4.5}$$

and also

$$g(A_{B(X_i,X_j)}X'_b, JX'_c) = g(B(X_i, X_j), JB(X'_b, X'_c)) = 0. \tag{4.6}$$

Both (4.5) and (4.6) imply that $A_{B(X_i,X_j)}X'_b$ lies in $\mathrm{Span}\{X_i, JX_i\}$, i.e., in the tangent space of M^n_s at each point. Hence, from (3.11), the right hand side of (4.4) vanishes. So

$$||B(X_i, X_j)||^2 = 0. \tag{4.5}$$

From (4.1) and (4.5) we conclude that M^n_s is totally geodesic. By means of a similar discussion, M'^m_t is also totally geodesic.

Now the normal space at every point is equal to $\nu^\perp \oplus \nu$, where

$$\nu^\perp = \mathrm{Span}\{\xi | A_\xi = 0\}. \tag{4.6}$$

If $\xi \in \nu^\perp$, then it follows from (2.6) that

$$g(D_{X_i}B(X_j, X'_b), \xi) = g(D_{X'_b}B(X_i, X_j), \xi) = 0, \tag{4.7}$$

$$g(D_{X'_a}B(X_j, X'_b), \xi) = g(D_{X_j}B(X'_a, X'_b), \xi) = 0, \tag{4.8}$$

and, by using (2.9), the following equations are obtained:

$$g(D_{JX_i}B(X_j, X'_b), \xi) = -g(D_{X'_a}B(X_i, X_j), J\xi) = 0, \tag{4.9}$$

$$g(D_{JX'_a}B(X_j, X'_a), \xi) = -g(D_{X_j}B(X'_a, X'_b), J\xi) = 0 \tag{4.10}$$

(4.7) $\sim$ (4.10) imply that ν is parallel with respect to the normal connection. From [5], there exists a totally geodesic $CP^{N(n,m)}_{R(n,m,s,t)}(c)$ in $CP^N_R(c)$ such that $\varphi(CP^n_s(c)\times CP^m_t(c))$ lies into $CP^{N(n,m)}_{R(n,m,s,t)}(c)$. From Theorem 3.1, φ is given by the indefinite Segre imbedding and a rigid motion of $CP^{N(n,m)}_{R(n,m,s,t)}(c)$.□

COROLLARY 4.2. *Let M^n and M'^m be complete positive definite Kaehler manifolds with complex dimension n and m. Assume that there exists a holomorphic isometric immersion φ of $M^n \times M'^m$ into $CH^N_{nm}(-c), c > 0$. If the second fundamental form of φ is parallel, then M^n is holomorphically isometric to $CH^n(-c), M'^m$ is holomorphically isometric to $CH^m(-c)$ and, by identifying $M^n_s \times m'^n_t$ with $CP^n_s(c) \times CP^n_t(c)$, the immersion φ is obtained from the Segre imbedding ϕ of $CH^n(-c) \times CH^m(-c)$ into $CH^{n+m+nm}_{nm}(-c) \subset CH^N_{nm}(-c)$ and a rigid motion of $CH^{n+m+nm}_{nm}(-c)$. The corresponding local version is also true.*

5. Length of the second fundamental form. Finally, this section is devoted to giving a characterization of the Segre imbedding

$$\phi : CH^n(-c) \times CH^m(-c) \to CH^{n+m+nm}_{nm}(-c)$$

in terms of the square of the length of its second fundamental form. We know that $CH^n(-c)$ and $CH^m(-c)$ are totally geodesic, so

$$||B||^2 = 4\sum_{i=1}^{n}\sum_{a=1}^{m} ||B(X_i, X'_a||^2 = -cnm$$

according to (3.5). Conversely this equality can characterize the Segre imbedding ϕ. So we have

THEOREM 5.1. *Let M^n and M'^m be complete positive definite Kaehler manifolds with complex dimensions n and m, respectively. Assume that there exists a holomorphic isometric immersion φ of $M^n \times M'^m$ into $CH^N_{nm}(-c), c > 0$. Let $||B||^2$ denote the square of the length of the second fundamental form of φ. Then*

$$||B||^2 \leq -cnm \tag{5.1}$$

and the equality holds if and only if M^n is holomorphically isometric to $CH^n(-c)$, M'^m is holomorphically isometric to $CH^m(-c)$ and the immersion φ is an imbedding obtained by composition of the Segre imbedding

$$\phi : CH^n(-c) \times CH^m(-c) \to CH_{nm}^{n+m+nm}(-c) \subset CH_{nm}^{N}(-c)$$

and a rigid motion of $CH_{nm}^{n+m+nm}(-c)$.

PROOF. We choose a local orthonormal frame of vector fields $\{X_i, \ldots, X_n, JX_1, \ldots, JX_n\}$ and $\{X'_1, \ldots, X'_m, JX'_1, \ldots JX'_m\}$ in M^n and M'^m, respectively. Then

$$\begin{aligned} ||B||^2 = 2 \sum_{i,j=1}^{n} ||B(X_i, X_j)||^2 + 2 \sum_{a,b=1}^{m} ||B(X'_a, X'_b)||^2 \\ + 4 \sum_{\substack{1 \le i \le n \\ 1 \le a \le m}} ||B(X_i, X'_a||^2. \end{aligned} \tag{5.2}$$

But an analogue to (3.5) gives

$$||B(X_i, X'_a)||^2 = -\frac{c}{4}. \tag{5.3}$$

By substituting (5.3) into (5.2), the square of the length of B can be reduced to

$$\begin{aligned} ||B||^2 = -cnm + 2 \sum_{i,j=1}^{n} ||B(X_i, X_j)||^2 \\ + 2| \sum_{a,b=1}^{m} ||B(X'_a, X'_b)||^2. \end{aligned} \tag{5.4}$$

Now note that $\nu = \text{Span}\{B(X_i, X'_a), JB(X_i, X'_a)\}$ is a complex nm-dimensional subspace of the normal space at each point and that ν is negative definite according to (5.3). On the other hand, it follows from (3.11) that $B(X_i, X_j)$ and $B(X'_a, X'_b)$ are both orthogonal to ν, hence $||B(X_i, X_j)||^2 \ge 0$ and $||B(X'_a, X'_b)||^2 \ge 0$, from which, together with (5.4), (5.1) holds true.

If the equality of (5.1) holds, (5.4) means

$$||B(X_i, X_j)||^2 = ||B(X'_a, X'_b)||^2 = 0. \tag{5.5}$$

Since the orthogonal complment $\nu^\perp$ of ν is a positive definite subspace of the normal space at each point, (5.5) implies

$$B(X_i, X_j) = B(X'_a, X'_b) = 0, \tag{5.6}$$

which shows that M^n and M'^m are totally geodesic in $CH^N_{nm}(-c)$. Again, a property in [1] implies that M^n and M'^m are definite complete space forms with constant holomorphic sectional curvature $-c$. Hence $M^n = CH^n(-c)$ and $M'^m = CH^m(-c)$.

Now the normal space at each point can be decomposed as $\nu^\perp \oplus \nu$. From (5.6) we have that ν is the first normal space at each point, and then

$$\nu^\perp = \mathrm{Span}\{\xi | A_\xi = 0\}. \tag{5.7}$$

If $\xi \in \nu^\perp$, then, from (2.6), it follows that

$$\begin{aligned} g(D_{X_i} B(X_j, X'_b), \xi) &= g((\tilde{\nabla}_{X_i} B)(X_j, X'_b), \xi) \\ &= g((\tilde{\nabla}_{X'_b} B)(X_i, X_j), \xi) \\ &= g(D_{X_b} B(X_i, X_j), \xi) = 0. \end{aligned} \tag{5.8}$$

In a similar way

$$g(D_{X'_a} B(X_j, X'_b), \xi) = g(D_{X_j} B(X'_a, X'_b), \xi) = 0, \tag{5.9}$$

and by using (2.9) one obtains

$$g(D_{JX_i} B(X_j, X'_b), \xi) = -g(D_{X'_b} B(X_i, X_j), J\xi) = 0, \tag{5.10}$$

$$g(D_{JX'_a} B(X_j, X'_b), \xi) = -g(D_{X_j} B(X'_a, X'_b), J\xi) = 0. \tag{5.11}$$

(5.8) $\sim$ (5.11) imply that ν is parallel with respect to the normal connection D. By means of [5], there exists a totally geodesic $CH^{n+m+nm}_{nm}(-c)$ in $CH^N_{nm}(-c)$ such that $\varphi(CH^n(-c) \times CH^m(-c)) \subset CH^{n+m+nm}_{nm}(-c)$. From Theorem 3.1, φ is the composition of the Segre imbedding and a rigid motion of $CH^{n+m+nm}_{nm}(-c)$.□

REMARK. Since the scalar curvature ρ of $M^n \times M'^m$ is $\rho = (-c)(n+m)(n+m+1) - ||B||^2$, condition (5.1) can be replaced by $\rho \geq (-c)(n^2 + m^2 + mn + m + n)$ and one obtains the same results as in Theorem 5.1.

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