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WEIGHTED INEQUALITIES FOR A VECTOR-VALUED STRONG MAXIMAL FUNCTION

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ABSTRACT. We show weighted weak type and strong type norm inequalities for a vector analogue of the strong maximal function.

1. Let f be a locally integrable function on $\mathbf{R}^n$, the strong maximal function $M_s f$ is defined by

$$M_s f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)| dy,$$

where the supremum is taken over all rectangles R in $\mathbf{R}^n$, with edges parallel to the coordinate axes. We shall denote this class of rectangles by $\mathcal{R}$.

If $1 < q < \infty$ and $f = (f_1, \ldots, f_k, \ldots)$ is a sequence of functions defined on $\mathbf{R}^n$, we say that f is ℓ^q-valued if $f(x) \in \ell^q$, that is

$$|f(x)|_q = \left\{ \sum_{k=1}^{\infty} |f_k(x)|^q \right\}^{1/q} < \infty.$$

For such f we define $M_s f = (M_s f_1, \ldots, M_s f_k, \ldots)$.

A weight function w will be a non-negative, locally integrable function on $\mathbf{R}^n$ and for measurable $E \subset \mathbf{R}^n$ we write $w(E) = \int_E w(x) dx$. We say $w \in A_p(\mathcal{R})$, $1 \leq p < \infty$, if there is a constant C such that

$$\left(\frac{1}{|R|} \int_R w(x) dx\right) \left(\frac{1}{|R|} \int_R w(x)^{-1/(p-1)} dx\right)^{p-1} \leq C$$

for all $R \in \mathcal{R}$. For $p = 1$ the second factor on the left is understood to be ess $\sup_{x \in R} w(x)^{-1}$.

In this note we shall prove the following:

THEOREM. *Let* $1 < q < \infty$.

i) *If* $w \in A_1(\mathcal{R})$ *then there is a constant* C_1 *such that*

$$w\Big\{x \in \mathbf{R}^n : |M_s f(x)|_q > \lambda\Big\} \\ \leq C_1 \int_{\mathbf{R}^n} \frac{|f(x)|_q}{\lambda}\Big(1 + \log^+ \frac{|f(x)|_q}{\lambda}\Big)^{n-1} w(x)dx,$$

for all $\lambda > 0$.

ii) *If* $1 < p < \infty$, *there is a constant* C_2 *such that*

$$\int_{\mathbf{R}^n} |M_s f(x)|_q^p w(x)dx \leq C_2 \int_{\mathbf{R}^n} |f(x)|_q^p w(x)dx$$

if and only if $w \in A_p(\mathcal{R})$.

For real valued functions this theorem has been proved in [**2**]. Also a particular case of the inequality in ii) has been considered in [**3**]. If instead of M_s we have the Hardy-Littlewood maximal operator then similar results have been proved by K.F. Andersen and R.T. John in [**1**].

The proof is based on the fact that M_s can be dominated by the composition of n one-dimensional operators and then it will follow by applying lemma 2.1 below.

As usual the letter C will denote a constant, not necessarily the same at each occurrence.

2. Let $1 \leq p \leq \infty$, $1 < q < \infty$ and let w be a weight function. By $L_w^p(\ell^q)$ we denote the class of w-measurable functions f defined on R^n, ℓ^q-valued, such that

$$||f||_{L_w^p(\ell^q)} = \Big(\int_{\mathbf{R}^n} |f(x)|_q^p w(x)dx\Big)^{1/p} < \infty.$$

When $p = \infty$, we make the obvious modifications.

LEMMA 2.1. *Let $1 < p \leq \infty$, $1 < q < \infty$ and let k be a positive integer. Suppose that T_1, T_2 are sublinear operators and there are constants C_1, C_2, M_1 and M_2 such that*

$(a) ||T_i f||_{L^p_w(\ell^q)} \leq ||f||_{L^p_w(\ell^q)}$, $i = 1, 2$.
$(b)\, w\{x : |T_1 f(x)|_q > t\}$

$$\leq M_1 \int_{\mathbf{R}^n} \frac{|f(x)|_q}{t}\Big(1 + \log + \frac{|f(x)|_q}{t}\Big)^{k-1} w(x)dx,$$

for all $t > 0$.

$(c)\, \{x : |T_2 f(x)|_q > t\} \leq \frac{M_2}{t} ||f||_{L^1_w(\ell^q)}$, for all $t > 0$.

Then

$Tf = T_2 T_1 f$ satisfies

$$w\{x : |Tf(x)|_q > t\} \leq C \int_{\mathbf{R}^n} \frac{|f(x)|_q}{t}\Big(1 + \log^+ \frac{|f(x)|_q}{t}\Big)^k w(x)dx,$$

for all $t > 0$, where C is a constant independent of f.

PROOF. For each $t > 0$ write $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f(x)$ whenever $|f(x)|_q > t/2$, and $f_1(x) = 0$ otherwise. We assume $p < \infty$, for $p = \infty$ the proof is similar. To simplify the notation we put $|\cdot|_q = |\cdot|$. Then (a) and (b) imply that

$$\begin{aligned} &w\{x : |T_1 f(x)| > t\} \\ &\leq Ct^{-1} \int_{|f(x)|>t/2} |f(x)|\Big(1 + \log^+ \frac{2|f(x)|}{t}\Big)^{k-1} w(x)dx \\ &\quad + Ct^{-p} \int_{|f(x)|\leq t/2} |f(x)|^p w(x)dx. \end{aligned} \tag{2.2}$$

We shall show that

$$\begin{aligned} &\int_{|T_1 f(x)|>t/2} |T_1 f(x)| w(x)dx \\ &\quad \leq C \int_{\mathbf{R}^n} |f(x)|\Big(1 + \log^+ \frac{|f(x)|}{t}\Big)^k w(x)dx, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}(2.4)\qquad &\int_{|T_1 f(x)|<t/2} |T_1 f(x)|^p w(x)dx \\ &< Ct^{p-1} \int_{\mathbf{R}^n} |f(x)| \Big(1+\log^+ \frac{|f(x)|}{t}\Big)^k w(x)dx,\end{aligned}$$

for each $t > 0$.

To prove (2.3) write

$$\begin{aligned}&\int_{|T_1 f(x)|>t/2} |T_1 f(x)| w(x)dx \\ &= \frac{t}{2} w\{x : |T_1 f(x)| > t/2\} + \int_{t/2}^{\infty} w\{x : |T_1 f(x)| > \lambda\} d\lambda \\ &= I_1 + I_2.\end{aligned}$$

An application of (2.2) shows that

$$\begin{aligned}I_2 \leq & C \int_{|f(x)|>t/4} |f(x)| \int_{t/2}^{2|f(x)|} \lambda^{-1} \Big(1+\log \frac{2|f(x)|}{\lambda}\Big)^{k-1} d\lambda w\ (x)dx \\ & + C \int_{\mathbf{R}^n} |f(x)|^p \int_{2|f(x)|}^{\infty} \lambda^{-p} d\lambda\ w(x)dx.\end{aligned}$$

Now (2.3) follows by performing the integration

Since

$$\int_{|T_1 f(x)|\leq t/2} |T_1 f(x)|^p w(x)dx = p \int_0^{t/2} \lambda^{p-1} w\{x : |T_1 f(x)| > \lambda\} d\lambda$$

and $\big(1+\log^+ \frac{|f(x)|}{\lambda}\big) < \big(1+\log^+ \frac{|f(x)|}{t}\big)\big(1+\log^+ \frac{t}{\lambda}\big)$, from (b) we obtain

$$\begin{aligned}&\int_{|T_1 f(x)|\leq/t/2} |T_1 f(x)|^p w(x)dx \\ &\leq C \Big[\int_0^{t/2} \lambda^{p-2} \big(1+\log^+ \frac{t}{\lambda}\big)^{k-1} d\lambda\Big] \\ &\times \int_{\mathbf{R}^n} |f(x)| \big(1+\log^+ \frac{|f(x)|}{t}\big)^{k-1} w(x)dx.\end{aligned}$$

Hence (2.4) follows.

Now (a) and (c) imply that

$$w\{x : |T_2T_1f(x)| > t\}$$
$$\leq Ct^{-1}\int_{|T_1f(x)|>t/2}|T_1f(x)|w(x)dx + Ct^{-p}\int_{|T_1f(x)|\leq t/2}|T_1f(x)|^p w(x)dx.$$

Then, from (2.3) and (2.4) the lemma follows.

3. PROOF OF THE THEOREM. We recall the following result from [**2**].

LEMMA 3.1. *Let $1 \leq p < \infty$. Then $w \in A_p(\mathcal{R})$ if and only if there is a constant C, such that for almost every fixed $(n-1)$−tuple $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), w(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots\dots, x_n) \in A_p$ on $\mathbf{R}$ with constant bounded by C.*

For $1 \leq j \leq n$ and g a real-valued function defined on $\mathbf{R}^n$, the partial maximal operator M_j is defined by

$$M_jg(x) = \sup_{h,k>0}\frac{1}{h+k}\int_{x_j-h}^{x_j+k}|g(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)|dt.$$

For $f = (f_1, \dots, f_k, \dots)$ we write $M_jf = (M_jf_1, \dots, M_jf_k, \dots)$.

Since the inequality in ii) holds for the Hardy-Littlewood maximal operator (see Theorem 3.1 of [**1**]) then from lemma 3.1 we get

$$\int_{-\infty}^{\infty}|M_jf(x)|_q^p w(x)dx_j \leq C\int_{-\infty}^{\infty}|f(x)|_q^p w(x)dx_j,$$

for almost every $(n-1)$-tuple $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, where $1 < p, q < \infty, w \in A_p(\mathcal{R})$ and C is a constant depending only on p, q and the constant that appears in the definition of A_p. Analogously, when $w \in A_1(\mathcal{R})$ we have

$$\int_{\{x_j:|M_jf(x)|_q>\lambda\}} w(x)dx_j \leq \frac{C}{\lambda}\int_{-\infty}^{\infty}|f(x)|_q w(x)dx_j.$$

Integrating both sides with respect to $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$, we get

$$||M_j f||_{L^p_w(\ell^q)} \leq C||f||_{L^p_w(\ell^q)} \tag{3.2}$$

and

$$w\{x : |M_j f(x)|_q > \lambda\} \leq \frac{C}{\lambda}||f||_{L^1_w(\ell^q)}. \tag{3.3}$$

On the other hand, it is well known that

$$M_s f_k(x) \leq M_n M_{n-1} \dots M_1 f_k(x), k = 1, 2, \dots. \tag{3.4}$$

Therefore sufficiency in part ii) follows from (3.2) and (3.4). The necessity is a consequence of the fact that if the inequality in ii) is valid for real valued functions then $w \in A_p(\mathcal{R})$ (see **[2]** for a proof).

It follows from (3.2) and (3.3) that $T_1 = M_{n-1} \dots M_1, T_2 = M_n$ and k_{n-1} satisfy the hypothesis of lemma 2.1. Hence, from (3.4), we get i).

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