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# DUAL MODULES AND GROUP ACTIONS ON EXTRA-SPECIAL GROUPS

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**1. Introduction.** When constructing examples or counter-examples in the theory of solvable groups, it is often the case that what is needed is some group which acts in an interesting way on an extra-special $p$-group. Specifically, what we have in mind is the following.

Let $G$ be a finite group and let $V$ be an irreducible $FG$-module where $F = GF(p)$. It is easy to construct an extra-special $p$-group $E$ acted on by $G$ such that $E = AB$ where $A$ and $B$ are $G$-invariant elementary abelian normal subgroups with $A \cap B = Z = \mathbf{Z}(E)$. This can be done so that $A/Z$ is $FG$-isomorphic to $V$ and $B/Z$ is $FG$-isomorphic to the "dual" or contragredient $FG$-module $V^*$. Furthermore, $G$ acts trivially on $Z$.

Now comes the more subtle part. Suppose $G \triangleleft \Gamma$ where $|\Gamma : G| = 2$ and where the conjugation action of $\Gamma$ on the set of isomorphism classes of $FG$-modules interchanges the classes of $V$ and $V^*$. (We allow the possibility that $V \simeq V^*$ and this isomorphism class is $\Gamma$-invariant.) The question is whether or not the action of $G$ on $E$ can be extended to a $\Gamma$-action in which the elements of $\Gamma - G$ interchange $A$ and $B$.

The answer is "yes".

THEOREM A. *Let $G \triangleleft \Gamma$ with $|\Gamma : G| = 2$ and let $V$ be an irreducible $FG$-module where $F = GF(p)$. Assume that $V$ is conjugate to $V^*$ in $\Gamma$. Then $\Gamma$ acts on an extra-special $p$-group $E$ and the following hold.*

a) *$E = AB$ where $A, B \triangleleft E$ are elementary abelian and $A \cap B = \mathbf{Z}(E)$.*

b) *$G$ centralizes $Z = \mathbf{Z}(E)$ and acts on $A/Z$ and $B/Z$ as it does on $V$ and $V^*$ respectively.*

c) *The elements of $\Gamma - G$ interchange $A$ and $B$ and either all of them centralize or else all of them invert $Z$. Furthermore, the choice of the*

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*action of $\Gamma - G$ on $Z$ may be specified in advance except in the case where an absolutely irreducible constituent of $V$ is $\Gamma$-conjugate to its contragredient module.*

Some explanation of the last sentence is probably appropriate here. Let $\overline{F}$ be an algebraic closure of $F$. Then $V \otimes_F \overline{F}$ is a direct sum of pairwise nonisomorphic irreducible $\overline{F}G$-modules which constitute a Galois conjugacy class over $F$. Suppose $W$ is one of these. Now $V^* \otimes_F \overline{F}$ is the contragredient module for $V \otimes_F \overline{F}$ and therefore $W^*$ is isomorphic to a constituent of it. Since $V$ is $\Gamma$-conjugate to $V^*$, it follows that $W$ is $\Gamma$-conjugate to some irreducible constituent of $V^* \otimes_F \overline{F}$, but this constituent is not necessarily $W^*$. The theorem asserts that in the case where $W$ is not $\Gamma$-conjugate to $W^*$ (and $p \neq 2$), we can find two actions of $\Gamma$ on $E$. In one of these, $G - \Gamma$ (and hence all of $\Gamma$) centralizes $Z$, and in the other $\Gamma - G$ inverts $Z$. (Note that since $G$ centralizes $Z$, it is a triviality that all the elements of $\Gamma - G$ act in the same way on $Z$.)

As an example of the setting of the theorem, consider Dade's construction [**2**] of an $M$-group having a non-$M$ normal subgroup. What was needed there was an action of $\Gamma$, a dihedral group of order 14, on an extra special group of order $2^7$ such that the involutions of $\Gamma$ interchanged two elementary abelian subgroups of order $2^4$. Dade's situation was exactly as in our theorem. (There is an error in [**2**] which Dade corrected in [**3**]. The error occured at precisely the interesting part of Theorem A: the extension of the $G$-action to $\Gamma$.)

The author would like to thank the referee for the suggestion that readers of this paper might appreciate information concerning other papers which deal with actions of groups on extra-special groups and on symplectic modules. Specifically, the referee mentioned papers [**1**], [**6**] and [**7**] (and in this context, I cannot resist including [**5**]). It should be stressed, however, that none of these papers bears directly on the present work which is concerned with synthesis rather than analysis and which is almost entirely self-contained.

**2. The construction of $E(V)$.** Let $V$ be a finite dimensional vector space over an arbitrary field $K$. We construct a group $E(V)$ which, in the case where $K = GF(p)$, will turn out to be an extra-special $p$-group of order $p|V|^2$. This construction is certainly not new.

Let $A$ be the direct sum of the additive groups of $V$ and $k$ and write $A$ multiplicatively. Let $V^* = \operatorname{Hom}_K(V, K)$ be the dual space of $V$ and let $\Lambda$ be a copy of the additive group of $V^*$, written multiplicatively. Now for $(v, k) \in A$ and $\lambda \in \Lambda$, write

$$(*) \qquad (v, k)^\lambda = (v, v\lambda + k).$$

It is trivial to check that this defines an automorphism of $A$. Since $v(\lambda\mu) = v\lambda + v\mu$, we have an action of $\Lambda$ on $A$ and we define $E(V) = A \rtimes \Lambda$, the semi-direct product.

Let us write $Z = (0, K) \subseteq A$. It is routine to check that $Z = \mathbf{Z}(E(V))$ and also that $Z = (E(V))'$. If we write $B = Z\Lambda$, then $A \simeq V^+ \oplus K^+ \simeq B$. Also, $AB = E(V)$ and $A \cap B = Z$. In particular, in the situation of Theorem A, if we take $K = F$, then $E = E(V)$ is extra-special and part (a) of the theorem holds.

Now suppose $V$ is a $KG$-module for some group $G$ and make $V^*$ into a $KG$-module via the contragredient action so that $(vg)(\lambda g) = v\lambda$ for all $v \in V$ and $\lambda \in V^*$. We can now let $G$ act on $A$ and $\Lambda$ by defining

$$(v, k)^g = (vg, k) \text{ and } \lambda^g = \lambda g.$$

We need to check that $(*)$ is preserved by these actions. Specifically, we need

$$(vg, k)^{(\lambda g)} = (v, v\lambda + k)^g$$

and this is clear by direct computation.

It follows that $G$ acts on $E(V)$ and in this action, $G$ centralizes $Z$ and normalizes $A$ and $B$. Also, the induced actions of $G$ on $A/Z$ and $B/Z$ agree (via the natural isomorphisms) with the original actions of $G$ on $V$ and $V^*$. In particular, part (b) of Theorem A is proved.

**3. Conditions for $\Gamma$-action on $E(V)$.** Assume $G \triangleleft \Gamma$ with $|\Gamma : G| = 2$ as in Theorem A. Let $V$ be a $KG$-module and let $G$ act on $E(V)$ as in the previous section. Fix some element $c \in \Gamma - G$ and write $s = c^2 \in G$.

LEMMA 3.1. *Suppose we can find additive group isomorphisms*

$$\alpha : V \to V^* \quad \beta : V^* \to V \quad \gamma : K \to K$$

*such that*

i) $v\alpha\beta = vs$ *and* $\lambda\beta\alpha = \lambda s$ *for all* $v \in V$ *and* $\lambda \in V^*$.

ii) $vx\alpha = v\alpha x^c$ *and* $\lambda x\beta = \lambda\beta x^c$ *for all* $v \in V$, $\lambda \in V^*$ *and* $x \in G$.

iii) $k\gamma^2 = k$ *for all* $k \in K$.

iv) $(v\lambda)\gamma = -(\lambda\beta)(v\alpha)$ *for all* $v \in V$, $\lambda \in V^*$.

*Then the action of $G$ on $E(V)$ can be extended to an action of $\Gamma$ for which*

$$(**) \qquad \begin{aligned} (v,0)^c &= v\alpha \in \Lambda \\ \lambda^c &= (\lambda\beta, 0) \in A \\ (0,k)^c &= (0,k\gamma). \end{aligned}$$

*In particular, the elements of $\Gamma - G$ interchange $A$ and $B$.*

PROOF. We use equations $(**)$ to define an action of $c$ on $E(V)$. To see that this does define an automorphism, recall that multiplication in $E(V)$ satisfies

$$(v,k)\lambda = \lambda(v, k + v\lambda)$$

by $(*)$ and it suffices to show that

$$(v,0)^c(0,k)^c\lambda^c = \lambda^c(v,0)^c(0,k+v\lambda)^c.$$

Writing

$$v\alpha = \mu \text{ and } \lambda\beta = w,$$

what we need becomes

$$\mu(0,k\gamma)(w,0) = (w,0)\mu(0,k\gamma + (v\lambda)\gamma).$$

By $(*)$,

$$(w,0)\mu = \mu(w, w\mu)$$

and the desired equation follows by (iv).

To show that we really have an action of $\Gamma$ on $E(V)$ we need to establish that $c^2$ acts like $s$ and that $xc$ acts like $cx^c$ for all $x \in G$. These follow by routine computations using (i), (ii) and (iii).

We shall impose the additional condition that the maps $\alpha$ and $\beta$ of Lemma 3.1 be $K$-linear. It follows by 3.1 (iv) that $\gamma$ will also be $K$-linear and so must be multiplication by some element $\varepsilon \in K$. By 3.1 (iii) we have $\varepsilon^2 = 1$ and so $\varepsilon = \pm 1$.

In view of Lemma 3.1, we see that in order to complete the proof of Theorem A, it suffices to show the following.

THEOREM 3.2. *Let* $|G \triangleleft \Gamma$ *with* $|\Gamma : G| = 2$. *Fix* $c \in \Gamma - G$ *and let* $s = c^2 \in G$. *Let* $K$ *be a finite field and* $V$ *an irreducible* $KG$*-module. Assume that* $V$ *and* $V^*$ *are conjugate in* $\Gamma$. *Then there exists* $\varepsilon = \pm 1$ *and vector space isomorphisms*

$$\alpha : V \to V^* \text{ and } \beta : V^* \to V$$

*such that*

a) $v\alpha\beta = vs$ *and* $\lambda\beta\alpha = \lambda s$ *for all* $v \in V$ *and* $\lambda \in V^*$.

b) $vx\alpha = v\alpha x^c$ *and* $\lambda x\beta = \lambda\beta x^c$ *for all* $v \in V, \lambda \in V^*$ *and* $x \in G$.

c) $(\lambda\beta)(v\alpha) = (v\lambda)\varepsilon$ *for all* $v \in V$ *and* $\lambda \in V^*$.

*Furthermore,* $\varepsilon = \pm 1$ *can be prespecified except in the case where an absolutely irreducible constituent of* $V$ *is* $\Gamma$*-conjugate to its dual. In that case,* $\varepsilon$ *is uniquely determined.*

**4. Preliminaries.** We begin work toward Theorem 3.2 with some elementary linear algebra. Fix a field $K$ and let $V$ be a finite dimensional vector space over $K$ with dual space $V^* = \mathrm{Hom}_K(V, K)$.

LEMMA 4.1. *Let* $\Theta : V \to V^*$ *be an arbitrary* $K$*-isomorphism. Then*

a) *There exists a unique* $K$*-isomorphism* $\varphi : V^* \to V$ *such that*

$$(A) \qquad (\lambda\varphi)(v\Theta) = v\lambda$$

*for all* $\lambda \in V^*$ *and* $v \in V$.

b) *If* $\alpha : V \to V$ *is any linear transformation, there exists a unique transformation* $\alpha^\tau : V \to V$ *such that*

$$(B) \qquad (w)(v\alpha\Theta) = (w\alpha^\tau)(v\Theta)$$

*for all* $v, w \in V$.

c) *The map* $\tau$ *is a* $K$*-linear antiautomorphism of the ring* $R = Hom_K(V, V)$.

PROOF. Fix a basis for $V$ and its corresponding dual basis for $V^*$. We may now identify $V$ with the space of row vectors over $K$ and $V^*$ with the column vectors. With this identification, the computation of $v\lambda$ for $v \in V$ and $\lambda \in V^*$ is simply matrix mulitplication. Also, if $[\Theta]$ denotes the matrix of $\Theta$, then $v\Theta = (V[\Theta])^t$. If $\varphi : V^* \to V$ is any linear transformation, and its matrix is $[\varphi]$, then $\lambda\varphi = \lambda^t[\varphi]$.

Equation (A) now reads

$$(\lambda^t[\varphi])(v[\Theta])^t = v\lambda$$

or equivalently

$$\lambda^t[\varphi][\Theta]^t v^t = v\lambda = \lambda^t v^t.$$

We see that the unique $\varphi$ which works is determined by the matrix $[\varphi] = ([\Theta]^t)^{-1}$ and part (a) is proved.

If $\alpha$ and $\alpha^\tau$ are any two elements of $R = \mathrm{Hom}_k(V, V)$, equation (B) translates into matrix language as

$$w(v[\alpha][\Theta])^t = w[\alpha^\tau](v[\Theta])^t$$

and this is equivalent to

$$w[\Theta]^t[\alpha]^t v^t = w[\alpha^\tau][\Theta]^t v^t.$$

We see then that $\alpha^\tau$ is uniquely determined by the matrix equation

$$[\alpha^\tau] = [\Theta]^t[\alpha]^t([\Theta]^t)^{-1}.$$

Part (b) is now proved and (c) follows since the map $\tau$, when viewed on the matrix level, is the composition of the transpose map with conjugation by $([\Theta]^t)^{-1}$ and so is a $K$-linear antiautomorphism of $R$ as desired.

We shall also need the following easy result on finite fields.

LEMMA 4.2. *Let $\Delta$ be a finite field and let $\tau \in \mathrm{Aut}(\Delta)$ have order 2. Suppose $\gamma \in \Delta^\times$ with $\gamma^\tau = \gamma^{-1}$. Then there exists $\delta \in \Delta^\times$ such that*

$$\delta^{-1}\delta^\tau = \gamma.$$

PROOF. Let $|\mathrm{Fix}(\tau)| = q$ so that $|\Delta| = q^2$ and $\delta^\tau = \delta^q$ for all $\delta \in \Delta$. We have

$$\gamma^{-1} = \gamma^\tau = \gamma^q$$

and so $\gamma^{q+1} = 1$. However, $\Delta^\times$ is a cyclic group of order $(q+1)(q-1)$ and it follows that $\gamma = \delta^{q-1}$ for some $\delta \in \Delta^\times$. Now

$$\delta^{-1}\delta^\tau = \delta^{-1}\delta^q = \gamma$$

as required.

We need one more preliminary result.

LEMMA 4.3. *Let $V$ be an irreducible $KG$-module where $G$ is a finite group and $K$ is a finite field. Let $\Delta = \mathrm{Hom}_{KG}(V, V)$ (and note that $\Delta$ is a finite field).*

a) *Viewing $V$ as a $\Delta G$-module, it is isomorphic to an absolutely irreducible constituent of $V \otimes_K \Delta$.*

b) *The dual $KG$-module $V^*$ can be made into a $\Delta G$ module by defining $\lambda\delta \in V^*$ according to the formula*

$$(v)\lambda\delta = (v\delta)\lambda$$

*for $v \in V$, where $\lambda \in V^*$ and $\delta \in \Delta$*

c) *The $\Delta G$-module $V^*$ is $\Delta G$-isomorphic to the $\Delta$-dual of the $\Delta G$-module $V$.*

PROOF. We have $\mathrm{Hom}_{\Delta G}(V, V) = \Delta$ and this implies that $V$ is an absolutely irreducible $\Delta G$-module by Theorem 9.2 of [**4**]. As such, it is a constituent of $V \otimes_K \Delta$ by Lemma 9.18 of [**4**]. This completes the proof of (a).

It is clear (since $\Delta$ is commutative) that the action of $\Delta$ on $V^*$ defined in (b) makes $V^*$ into a $\Delta$-space and we need only check that the $\Delta$-action commutes with the $G$-action. For $x \in G$, $\lambda \in V^*$, $\delta \in \Delta$ and $v \in V$ we have

$$(v)(\lambda x\delta) = (v\delta)(\lambda x) = (v\delta x^{-1})\lambda = (vx^{-1}\delta)\lambda = (vx^{-1})(\lambda\delta) = (v)(\lambda\delta x)$$

as desired.

To prove (c), let $\tilde{V}$ be the $\Delta$-dual of $V$, viewed as a $\Delta G$-module and let $T : \Delta \to K$ be any nonzero $K$-linear map. For each $\alpha \in \tilde{V}$, the composition $\alpha T : V \to K$ is $K$-linear and thus $\alpha \mapsto \alpha T$ defines a map $\tilde{V} \to V^*$. We claim that this is a $\Delta G$-module isomorphism.

This map is clearly additive. To see that it is $\Delta$-linear, let $v \in V$ and $\delta \in \Delta$ and compute

$$(v)(\alpha\delta)T = (v\delta)(\alpha T) = v((\alpha T)\delta).$$

Also, if $x \in G$, then

$$(v)(\alpha x)T = (vx^{-1})(\alpha T) = (v)(\alpha T)x$$

and so our map $\alpha \mapsto \alpha T$ is a $\Delta G$-module homomorphism. Since any nonzero $\alpha \in \tilde{V}$ maps onto $\Delta$, we have $\alpha T \neq 0$ and the map is one-tō-one. We see that it maps onto $V^*$ by a dimension argument.

## 5. Proving the theorem.

PROOF OF THEOREM 3.2. We are assuming that the $KG$-modules $V$ and $V^*$ are conjugate in $\Gamma$. This means that there exists a $K$-isomorphism $\Theta : V \to V^*$ such that

(1) $$(vx)\Theta = v\Theta x^c \text{ for } x \in G.$$

(Recall that $c$ is some fixed element of $\Gamma - G$). Fix $\Theta$ and let $\varphi : V^* \to V$ be as in Lemma 4.1 (a). Our object is to produce certain maps $\alpha : V \to V^*$ and $\beta : V^* \to V$ and we shall do this with suitable modifications of $\Theta$ and $\varphi$.

Our first goal is to prove the analog of (1) for the map $\varphi$. We claim

(2) $$(\lambda x)\varphi = \lambda\varphi x^c \text{ for } x \in G$$

To see this, let $v \in V$ and compute

$$((\lambda x)\varphi)(v\Theta) = v(\lambda x) = (vx^{-1})\lambda = (\lambda\varphi)(vx^{-1}\Theta)$$

using (A) of Lemma 4.1. By (1), this yields

$$((\lambda x)\varphi)(v\Theta) = (\lambda\varphi)(v\Theta(x^{-1})^c) = (\lambda\varphi x^c)(v\Theta)$$

and since $v\Theta$ runs over all of $V^*$, (2) follows.

Now, as in Lemma 4.1, write $R = \mathrm{Hom}_K(V, V)$ and let $\tau$ be the antiautomorphism of $R$ given by 4.1 (b,c). Let $\Delta = \mathrm{Hom}_{KG}(V, V) \subseteq R$ so that $\Delta$ is a finite field.

Suppose we fix $\varepsilon = \pm 1$ and $\delta \in \Delta^\times$. Let

(3)
$$\alpha = \delta\Theta : V \to V^*$$
$$\beta = \varphi(\delta^\tau)^{-1}\varepsilon : V^* \to V.$$

We will show that for suitable choices of $\varepsilon$ and $\delta$, these maps satisfy the conclusion of the theorem.

To check condition (c), compute

$$(\lambda\beta)(v\alpha) = (\lambda\varphi(\delta^\tau)^{-1})(v\delta\Theta)\varepsilon = (\lambda\varphi(\delta^\tau)^{-1}\delta^\tau)(v\Theta)\varepsilon = (\lambda\varphi)(v\Theta)\varepsilon = v\lambda\varepsilon$$

as required. (We have used equation (B) of 4.1.) Thus (c) holds with $\delta$ and $\varepsilon$ arbitrary.

Next, we check (b) with $\alpha$ and $\beta$ defined by (3). We have

$$vx\alpha = vx\delta\Theta = v\delta x\Theta = v\delta\Theta x^c$$

by (1) and thus $vx\alpha = v\alpha x^c$ as required. To prove the second part of (b), we will need to know.

(4) $\tau$ maps $\Delta$ to $\Delta$.

Assuming this for the moment, we compute

$$\lambda x\beta = \lambda x\varphi(\delta^\tau)^{-1}\varepsilon = \lambda\varphi(\delta^\tau)^{-1}\varepsilon x^c = \lambda\beta x^c$$

where we have used (2) and (4).

To establish (4), let us write $\overline{x} \in R$ to denote the linear transformation of $V$ induced by $x \in G$. Then $\Delta$ is the centralizer in $R$ of $\overline{G} = \{\overline{x} | x \in G\}$ and it will suffice to show that $\tau$ maps $\overline{G}$ to itself. In fact, we claim that

$$(5) \qquad (\overline{x})^{\tau} = \overline{(x^c)^{-1}}.$$

To see this, compute for $v, w \in V$ that

$$w(vx\Theta) = w(v\Theta x^c) = w((x^c)^{-1})(v\Theta).$$

Comparison of this with the defining property (B) of $\tau$ in 4.2 proves (5). We have now shown that (b) holds for arbitrary $\delta$ and $\varepsilon$ in (3).

Before we can prove (a), we need to obtain some information about the map $\Theta\varphi : V \to V$. For $x \in G$ and $v \in V$ we compute

$$vx\Theta\varphi s^{-1} = v\Theta\varphi s^{-1}x$$

for all $v \in V$ and $x \in G$. In other words, setting $\gamma = \Theta\varphi\overline{s}^{-1}$, we have

$$(6) \qquad \gamma = \Theta\varphi\overline{s}^{-1} \in \Delta.$$

Now let us check to see if we can make (a) hold. By (3) we have

$$v\alpha\beta = v\delta\Theta\varphi(\delta^{\tau})^{-1}\varepsilon = v\delta\gamma\overline{s}(\delta^{\tau})^{-1}\varepsilon = (vs)\delta\gamma(\delta^{\tau})^{-1}\varepsilon$$

and so we need

$$(7) \qquad \delta(\delta^{\tau})^{-1}\gamma\varepsilon = 1$$

for the first part of (a). For the second part of (a) we compute

$$\lambda\beta\alpha = \lambda\varphi(\delta^{\tau})^{-1}\varepsilon\delta\Theta = \lambda\varphi(\delta^{\tau})^{-1}\varepsilon\delta\gamma s\varphi^{-1}$$

and if (7) holds, this yields

$$\lambda\beta\alpha = \lambda\varphi s\varphi^{-1} = \lambda s^{c^{-1}}\varphi\varphi^{-1} = \lambda s$$

using (2) and the fact that $c$ centralizes $s$.

To complete the proof of the theorem, we need to show that $\delta \in \Delta^\times$ and $\varepsilon = \pm 1$ can be chosen so that (7) holds and that $\varepsilon = \pm 1$ is uniquely determined if and only if an absolutely irreducible constituent of $V$ is $\Gamma$-conjugate to its dual.

Since $\Delta$ is commutative, it follows by (4) that the antiautomorphism $\tau$ defines an automorphism of $\Delta$. We claim that

$$\delta^{\tau^2} = \delta \text{ for } \delta \in \Delta. \tag{8}$$

To see this let $v, w \in V$ and compute

$$(w)(v\delta\Theta) = (w\delta^\tau)(v\Theta) = (v\Theta\varphi)(w\delta^\tau\Theta) = (v\Theta\varphi\delta^{\tau^2})(w\Theta)$$

using (A) and (B) of 4.1. By (6) it follows that $\Theta\varphi$ centralizes $\Delta$ and so we have

$$(w)(v\delta\Theta) = (v\delta^{\tau^2}\Theta\varphi)(w\Theta) = (w)(v\delta^{\tau^2}\Theta)$$

and (8) follows.

We wish to use Lemma 4.2 to solve (7) and so we need to establish

$$\gamma^\tau = \gamma^{-1} \tag{9}$$

where $\gamma$, of course, is as in (6). Let $v, w \in V$ and compute

$$(w)(v\Theta) = (v\Theta\varphi)(w\Theta) = (w\Theta\varphi)(v\Theta\varphi\Theta) = (w\Theta\varphi(\Theta\varphi)^\tau)(v\Theta).$$

It follows that $\Theta\varphi(\Theta\varphi)^\tau = 1$ and $(\Theta\varphi)^\tau = (\Theta\varphi)^{-1}$. Therefore

$$\gamma^\tau = (\Theta\varphi\bar{s}^{-1})^\tau = (\bar{s}^{-1})^\tau(\Theta\varphi)^\tau = \bar{s}(\Theta\varphi)^{-1} = \gamma^{-1}$$

where we have used (5).

Now that (9) is established, it follows by Lemma 4.2 that if the automorphism induced on $\Delta$ by $\tau$ has order 2, then for either choice of $\varepsilon = \pm 1$, we can find $\delta \in \Delta^\times$ with

$$\delta^{-1}\delta^\tau = \varepsilon\gamma$$

and (7) is satisfied. The remaining possibility (by (8)) is that $\tau$ induces the trivial automorphism on $\Delta$. In that case, we have $\delta(\delta^\tau)^{-1} = 1$ and also $\gamma = \pm 1$ by (9). It follows that (7) will be satisfied for any choice of $\delta \in \Delta^\times$ provided $\varepsilon = \gamma$. If $\varepsilon \neq \gamma$,there is no solution.

Now (7) is necessary as well as sufficient for the existence of the maps $\alpha$ and $\beta$ of the theorem. This is because any pair of maps $\alpha$ and $\beta$ which satisfy 3.2 (b,c) are in fact given by (3) for some choice of $\delta \in \Delta^\times$. To see this, observe that $\alpha\Theta^{-1} \in \Delta$ by (b) and (1) and so $\alpha = \delta\Theta$ for some $\delta$. For $v \in V$ and $\lambda \in V^*$, condition (c) yields

$$(\lambda\varphi\varepsilon)(v\Theta) = v\lambda\varepsilon = (\lambda\beta)(v\delta\Theta) = (\lambda\beta\delta^\tau)(v\Theta)$$

and so $\varphi\varepsilon = \beta\delta^\tau$. Therefore, (3) is satisfied, as claimed.

What remains to be shown is that the case where $\tau$ is the identity on $\Delta$ happens if and only if an absolutely irreducible constituent of $V$ is $\Gamma$-conjugate to its dual. By Lemma 4.3, it suffices to show that the $\Delta G$-modules $V$ and $V^*$ are conjugate in $\Gamma$ if $\tau$ is trivial on $\Delta$. Note that the conjugacy of $V$ and $V^*$ is equivalent to the existence of a $\Delta$-space isomorphism $\psi : V \to V^*$ such that

$$(vx)\psi = (v\psi)x^c \tag{10}$$

for $v \in V$ and$x \in G$.

In view of (1), we see that (10) is equivalent to the assertion that $\psi\Theta^{-1} \in \Delta$ and so we need to show that $\tau$ is trivial on $\Delta$ if and only if some map of the form $\psi = \delta\Theta : V \to V^*$ is $\Delta$-linear for some choice of $\delta \in \Delta$. Since $\Delta$ is commutative, our condition reduces to the $\Delta$-linearity of $\Theta$. Now for $v, w \in V$ and $\delta \in \Delta$, we have

$$w(v\delta\Theta) = (w\delta^\tau)(v\Theta) = (w)(v\Theta\delta^\tau)$$

where the last equality is by the definition of the $\Delta$-action on $V^*$. We now have

$$v\delta\Theta = v\Theta\delta^\tau$$

and so $\Theta$ is $\Delta$-linear if and only if $\tau$ is trivial on $\Delta$. □

**6. Concluding remarks.** In the situation of Theorem A, let $\Delta = \operatorname{Hom}_{FG}(V, V)$. If $|\Delta : F|$ is odd, then necessarily the absolutely irreducible constituents of $V$ are $\Gamma$-conjugate to their duals and we cannot hope to specify whether $\Gamma - G$ is to centralize or invert $Z = \mathbf{Z}(E)$. (Except, of course, when $p = 2$ where it makes no difference.) In particular, this occurs if $V$ is absolutely irreducible or if $\dim_F(V)$ is odd.

For example, suppose $G$ is cyclic of order 4. Let $p \equiv 1 \bmod 4$ and let $V$ be a faithful $FG$-module of dimension 1 (where $F = GF(p)$). If we take $\Gamma = D_8$ or $Q_8$, then $V$ is $\Gamma$-conjugate to $V^*$ and so $\Gamma$ will act on $E$, extra-special of order $p^3$ and exponent $p$. In this situation, $\Gamma - G$ necessarily inverts $Z$ if $\Gamma = D_8$ and centralizes $Z$ if $\Gamma = Q_8$.

On the other hand, suppose $p \equiv 3 \bmod 4$. In this case there is a unique faithful $FG$-module $V$ and it has dimension 2. We have $V \simeq V^*$ and there are four possibilities for $\Gamma$. In addition to $D_8$ and $Q_8$, there are two abelian groups: $Z_8$ and $Z_4 \times Z_2$. In this case, $|E| = p^5$ and again $D_8 - G$ inverts and $Q_8 - G$ centralizes. Each of the abelian possibilities, however, can act in more than one way and $\Gamma - G$ can be made to invert or centralize, as desired.

Note that at first glance, it seems unlikely that if $\Gamma - G$ contains an element $c$ of order 2 that $c$ can centralize $Z = E'$ since if $e \in E$, then

$$[e, e^c]^c = [e^c, e] = [e, e^c]^{-1}$$

and this seems to imply an inverting action. Of course, what must happen in this case is that $[e, e^c] = 1$ for all $e \in E$.

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