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TRANSLATION BY TORSION POINTS AND RATIONAL EQUIVALENCE OF O-CYCLES ON ABELIAN VARIETIES

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ABSTRACT. Let A be an abelian variety defined over an algebraically closed field. We give an elementary proof of the following result(Theorem 1): If γ is a 0-cycle of degree 0 on A, and $c \in A$ is a point of finite order, then γ is rationally equivalent to γ_c, the translate of γ under c. From this follows Theorem 2: Given any effective 0-cycle $\eta = (a_1) + \cdots + (a_r)$ on A, and any points of finite order $c_1, \ldots c_r \in A$ satisfying $c_1 + \cdots + c_r = o =$ the identity of A, we have that η is rationally equivalent to the 0-cycle $(a_1 + c_1) + \cdots + (a_r + c_r)$. Consequently, for $r \geq 2$, the set $E[\eta]$ of effective 0-cycles rationally equivalent to η is always at least a countably infinite set (Corollary 1). Further corollaries of Theorem 2 are given, including a generalization of Theorem 1 to higher dimensional cycles (Corollary 4).

0. Introduction. The purpose of this paper is to prove some elementary results concerning rational equivalence on abelian varieties, with an eye toward the problem of describing explicitly the set of all effective 0-cycles rationally equivalent to a given effective 0-cycle (see [**4**, pp. 133-135] for an indication of why this problem is interesting). We begin by establishing notation and then provide a summary of the contents and organization of the paper. Note that all varieties ($\Rightarrow$ irreducible) considered are defined over algebraically closed fields, and points are always closed points.

Let X be a nonsingular projective variety. If γ is a (pure) s-dimensional cycle on X, we write $[\gamma]$ for the set of all s-dimensional cycle γ' which are rationally equivalent to γ, written $\gamma \sim \gamma'$ (discussions of rational equivalence and related matters may be found in, e.g., [**1,3,4**]). The (Chow) group of s-dimensional cycles on X modulo rational equivalence is denoted $\mathrm{CH}_s(X)$. We write $E[\gamma] \subseteq [\gamma]$ for the set of all effective cycles $\eta \sim \gamma$.

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Now let $X = A$ be an abelian variety. Given a point $a \in A$, we write (a) for the 0-cycle determined by a; for example, $(a+b)+(a+c)$ is an effective 0-cycle of degree 2 on A, where $a, b, c \in A$ and the $+$ signs within the parentheses denote addition in A. As the title of the paper indicates, we shall be concerned with translation of 0-cycles on A, an example of which we have just seen: $(a+b)+(a+c)$ is the translate of the 0-cycle $(b)+(c)$ by $a \in A$. The additive identity element of A is denoted by $o \in A$.

We now discuss the results and organization of the paper. Our main result, on which everything else hangs, is

THEOREM 1: *If γ is a 0-cycle of degree 0 on A, and $c \in A$ has finite order, then γ is rationally equivalent to its translate γ_c.*

This theorem is stated and proved in §2; the proof is based on results of Bloch's paper [**1**], which we recall in §1.

Theorem 1 is the key to the proof of the following result, given in §3.

THEOREM 2. *Let $\eta = (a_1)+\cdots+(a_r)$ be an effective 0-cycle on A, and let $c_1, \ldots, c_r \in A$ be points of finite order such that $c_1+\cdots+\cdots+c_r = o \in A$. Then the o-cycles $(a_1)+\cdots+(a_r)$ and $(a_1+c_1)+\cdots+(a_r+c_r)$ are rationally equivalent.*

As a simple corollary, we find that $E[\eta]$ is at least a countably infinite set, provided that η has degree ≥ 2 (Corollary 1).

In §4, we restrict consideration to abelian varieties defined over the complex numbers. By confronting Corollary 1 with a result of Roitman ((3)of §4), we find that the "typical" set $E[\eta]$ is (exactly) a countably infinite set (Corollary 2).

Three further corollaries of Theorem 2 are presented in §5, including a generalization of Theorem 1 to cycles of higher dimension on A (Corollary 4).

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1. Statement of needed results of Bloch. The proof of Theorem 1 in §2 is based on results of Bloch's paper [**1**], which "app[lies] the calculus of algebraic cycles on abelian varieties as developed by Weil [**12**] and Lang [**5**] in their study of divisors, to cycles of codimension greater than one." In particular, Bloch exploits the Pontryagin product of cycles on an abelian variety A, which is defined as follows [**1**, p. 216]: Let $\mu : A \times A \to A$ be the addition map. Given arbitrary (not necessarily 0-) cycles α and β on A, their Pontryagin product $\alpha * \beta$ is defined to be the cycle $\mu_*(\alpha \times \beta)$, which is well-defined on rational equivalence classes. Note that the Pontryagin product of 0-cycles is again a 0-cycle (for example, $((a) + (0)) * (b) = (a + b) + (b))$, so that $\mathrm{CH}_0(A)$ becomes a ring under $*$, with identity element $[(0)]$. The subgroup $I \subseteq \mathrm{CH}_0(A)$ of cycle classes of degree 0 is an ideal in this ring; the m^{th} Pontryagin power of I will be denoted byI^{*m}. Thus, for example, if $a, b \in A$, then$[(a) - (0)], [(b) - (0)] \in I$, and $[(a) - (0)] * [(b) - (0)] = [(a + b) - (a) - (b) + (0)] \in I^{*2}$.

We may now state the results that are needed for the proof of Theorem 1:

(1) *If A has dimension n, then $I^{*n+1} = 0$.* [**1**, Theorem 0.1, p. 216]

(2) *The groups I^{*m} are divisible for all $m \geq 1$.* [**1**, Lemma 1.3, p. 219].

2. Translation of 0-cycles of degree 0 by torsion points. As above, let A be an abelian variety (defined over an algebraically closed field). A acts by translation on its group of 0-cycle as follows: given a 0-cycle γ on A, the image (or translate) of γ under $a \in A$ is $T_a(\gamma) = \gamma_a = (a) * \gamma$. By passing to rational equivalence classes, translation by $a \in A$ defines a $\mathbf{Z}$-linear map $T_a : \mathrm{CH}_0(A) \to \mathrm{CH}_0(A)$ which preserves degree; in particular, T_a maps I to I. The following theorem implies that if $c \in A$ has finite order, then $T_c|I : I \to I$ is the identity map.

THEOREM 1. *If γ is a 0-cycle of degree 0 on A, and $c \in A$ has finite order, then γ is rationally equivalent to its translate γ_c.*

To facilitate the proof, we recast the theorem in a slightly different form. We are out to show that $\gamma_c = (c) * \gamma$ is rationally equivalent to $\gamma = (0) * \gamma$, or, equivalently, that $((c) - (0)) * \gamma$ is rationally equivalent to 0. For any $a \in A$, we let $U_a : \mathrm{CH}_0(A) \to \mathrm{CH}_0(A)$ denote the $\mathbf{Z}$-linear map $U_a([\gamma]) = [(a) - (0)] * [\gamma] = [\gamma_a - \gamma]$ (note that the image of U_a lies in $I \subseteq \mathrm{CH}_0(A)$). Then Theorem 1 is equivalent to the statement that $U_c|I : I \to I$ *is the zero map whenever* $c \in A$ *has finite order*; we shall prove the theorem in this form.

We need the following simple

LEMMA. *For any* $a, b \in A$, *we have that* $U_{a+b} = U_a \circ U_b + U_a + U_b$.

PROOF OF LEMMA. For any $[\gamma] \in \mathrm{CH}_0(A)$,

$$\begin{aligned} U_a \circ U_b([\gamma]) &= U_a([\gamma_b - \gamma]) \\ &= [\gamma_{a+b} - \gamma_a - \gamma_b + \gamma] \\ &= [\gamma_{a+b} - \gamma] + [\gamma - \gamma_a] + [\gamma - \gamma_b] \\ &= U_{a+b}([\gamma]) - U_a([\gamma]) - U_b([\gamma]); \end{aligned}$$

the lemma follows at once.

PROOF OF THEOREM 1. Let $c \in A$ be a point of finite order. Recall that we must prove that U_c restricts to the zero map on I. To do this, consider the chain of powers of I:

$$I \supseteq I^{*2} \supseteq \cdots \supseteq I^{*n} \supseteq I^{*\eta+1} = 0$$

(the last term is zero by Bloch's theorem (1)).

Trivially, U_c is the zero map on $I^{*\eta+1}$. To finish up, we proceed by descending induction: Assume that we have shown that $U_{c'}$ is the zero map on I^{*s}, $s \geq 2$, for all points $c' \in A$ of finite order. We must show that U_c is forced to be zero on I^{*s-1}. Let c have order m in A. By repeated application of the lemma, we obtain the following identities:

$$\begin{aligned} U_0 = U_{mc} &= U_{(m-1)c+c} \\ &= U_{(m-1)c} \circ U_c + U_{(m-1)c} + U_c \\ &= U_{(m-1)c} \circ U_c + U_{(m-2)c} \circ U_c + U_{(m-2)c} + 2U_c \\ &\;\vdots \\ &= U_{(m-1)c} \circ U_c + U_{(m-2)c} \circ U_c = \cdots + U_c \circ U_c + mU_c. \end{aligned}$$

We now observe that each term of the form $U_{(m-i)c} \circ U_c$ is zero on any $[\gamma] \in I^{*s-1}$, since $U_c([\gamma]) = [(c) - (0)] * [\gamma] \in I^{*s}$ and $U_{(m-i)c}$ kills I^{*s} by our induction hypothesis. Therefore, $U_0 = mU_c$ on I^{*s-1}. But I^{*s-1} is a divisible group, by (2), so the following computation shows that U_c must kill any $[\gamma] \in I^{*s-1}$:

$$U_c([\gamma]) = U_c(m[\gamma']) = mU_c([\gamma']) = U_0([\gamma']) = [\gamma' - \gamma'] = 0.$$

The induction is complete and the theorem is proved.

3. A lower bound for the cardinality of $E[\eta]$. Let $\eta = (a_1) + \cdots + (a_r)$ be an effective 0-cycle on the abelian variety A. Recall that we write $E[\eta] \subseteq [\eta]$ for the set of all effective 0-cycles $\eta' \sim \eta$. The next theorem shows how to generate 0-cycles in $E[\eta]$ by "altering" η by points of finite order.

THEOREM 2. *Let $\eta = (a_1) + \cdots + (a_r)$ be an effective 0-cycle of degree r on A, and let $c_1, \dots, c_r \in A$ be points of finite order such that $c_1 + \cdots + c_r = o \in A$. Then the o-cycles $(a_1) + \cdots + (a_r)$ and $(a_1 + c_1) + \cdots + (a_r + c_r)$ are rationally equivalent.*

PROOF. When $r = 1$, the theorem asserts that $(a_1) \sim (a_1)$, which is trivially true. When $r = 2$, the theorem asserts that $(a_1) + (a_2) \sim (a_1 + c) + (a_2 - c)$, for any $c \in A$ of finite order. To prove this case, we translate the degree-zero 0-cycle $(a_1) - (a_2 - c)$ by c, and apply Theorem 1: $(a_1) - (a_2 - c) \sim T_c((a_1) - (a_2 - c)) = (a_1 + c) - (a_2)$, whence $(a_1) + (a_2) \sim (a_1 + c) + (a_2 - c)$. When $r > 2$, the desired conclusion follows by repeated application of the previous case:

$$
\begin{aligned}
&(a_1) + (a_2) + \cdots + (a_{r-1}) + (a_r) \\
&\sim (a_1 + c_1) + (a_2) + \cdots + (a_{r-1}) + (a_r - c_1) \\
&\sim (a_1 + c_1) + (a_2 + c_2) + (a_3) + \cdots + (a_{r-1}) + (a_r - c_1 - c_2) \\
&\vdots \\
&\sim (a_1 + c_1) + \cdots + (a_{r-1} + c_{r-1}) + (a_r - c_1 - \cdots - c_{r-1}) \\
&= (a_1 + c_1) + \cdots + (a_r + c_r),
\end{aligned}
$$

as desired.

As a corollary, we obtain a lower bound on the cardinality of $E[\eta]$:

COROLLARY 1. *If $\eta = (a_1) = \cdots + (a_r)$ is an effective 0-cycle of degree $r \geq 2$ on A, then $E[\eta]$ is at least a countably infinite set.*

PROOF. Since the abelian variety A is a divisible group [**7**, p. 62], we can find, for each prime p, a point $c_p \in A$ of order p. By Theorem 2, $E[\eta]$ contains the 0-cycles $\eta_p = (a_1 + c_p) + (a_2 - c_p) + (a_3) + \cdots + (a_r)$ for all p. We shall show that this subset of 0-cycles is countably infinite by showing that, for all sufficiently large primes p and q, η_p and η_q are distinct; the corollary follows at once.

Indeed, if η_p and η_q are distinct for all distinct prime pairs p, q, we are done, so suppose that $\eta_p = \eta_q$ for some pair of primes $p \neq q$. It follows that $(a_1+c_p)+(a_2-c_p) = (a_1+c_q)+(a_2-c_q)$, whence $a_1+c_p+c_q = a_2$; in other words, a_1 and a_2 differ by a point of finite order $N = pq$. But now, given distinct primes $p', q' > N$ we cannot have that $\eta_{p'} = \eta_{q'}$, since otherwise the previous argument would show that a_1 and a_2 differ by a point of finite order $p'q' > N$, a contradiction.

4. A generic upper bound for the cardinality of $E[\eta]$ over C. Let X be a smooth projective variety /**C**, and let η be an effective 0-cycle of degree r on X. From [**9**; Lemma 1, p. 574], it follows that $E[\eta]$ is a *c-closed* subset of the r-fold symmetric product $X(r)$, that is, $E[\eta]$ is a countable union of closed subsets of $X(r)$. Thus $E[\eta]$ has a well-defined dimension, the maximum of the dimensions of its (countably many) irreducible components. As η varies over$X(r)$, the dimension of $E[\eta]$ is minimized on a *c-open* (= complement of c-closed) subset of $X(r)$ [**9**, Remark 5, p. 576]; we say that the dimension of $E[\eta]$ is minimized for *c-generic* $\eta \in X(r)$.

The following result is essentially a special case of [**9**; Theorem 3, p. 584] (see also [**4**. pp. 133-135]).

(3) *Let X be a nonsingular projective variety defined over* **C**. *If* $\dim(X) = n \geq 2$ *and* $p_g(X) > 0$ *(i.e., if X supports a nonzero global holomorphic n-form), then $E[\eta]$ has dimension* 0 *for c-generic $\eta \in X(r)$, and therefore consists of a countable set of points of $X(r)$.*

In particular, the hypotheses of (3) are true when $X = A$ is an abelian variety /**C** of dimension $n \geq 2$; combining this with Corollary 1, we immediately obtain

COROLLARY 2. *Let A be an abelian variety /***C** *of dimension $n \geq 2$. For c-generic $\eta \in A(r), r \geq 2$, we have that $E[\eta]$ is a countably infinite set.*

EXAMPLES.

i) Samuel [**11**] (see [**4**, p. 135]) has given an example of a surface X/**C** such that a) not all points on X are rationally equivalent to one another, and b) for at least one point $p \in X$, the locus $E[p] = \{q \in X | q \sim p\}$ contains a countably infinite family of rational curves on X - in particular, $E[p]$ is c-closed but not closed. The countably infinite sets $E[\eta]$ identified in Corollary 2 provide another example of this phenomenon.

ii) Let A be the Jacobian variety of a curve C of genus $g \geq 2/\mathbf{C}$, and let $f : C \to A$ be a canonical embedding. Consider the following commutative diagram, in which the horizontal arrow is induced by f in the obvious way, the vertical arrow is the addition map, and the diagonal arrow is the Abel-Jacobi map:

$$\begin{tikzcd} C(r) \arrow[r] \arrow[dr, "f_r"'] & A(r) \arrow[d, "+"] \\ & A \end{tikzcd}$$

For all $r \geq 2$, (3) implies that $E[\eta]$ has dimension 0 in the r-fold symmetric product $A(r)$, and is therefore a countably infinite set of points, for c-generic $\eta \in A(r)$. However, for $r > g$, every fiber of f_r is a projective space of positive dimension, by the Abel-Jacobi theorem. Therefore, if $\eta \in C(r) \subseteq A(r), r > g$, we have that $E[\eta]$ contains at least one component of positive dimension. Thus, the conclusion of Corollary 2 need not hold for special $\eta \in A(r)$.

5. Further consequences of Theorem 2.

COROLLARY 3. *Let $c \in A$ be a point of finite order r on the abelian variety A. Then the cycle class $[(c) - (0)] \in I$ is r-torsion.*

PROOF. By Theorem 2, $r(c) = (c) + \cdots + (c) \sim (0) + \cdots + (0) = r(0)$, whence $r[(c) - (0)] = 0$.

Corollary 3 enables us to generalize Theorem 1 to cycles of higher dimension on A. Let $\mathrm{CH}_s(A)_{alg} \subseteq \mathrm{CH}_s(A)$ denote the subgroup of cycle classes consisting of cycles *algebraically equivalent* to 0 (e.g., [**5**, p. 56]).

COROLLARY 4. *If $[\gamma] \in CH_s(A)_{alg}$, and $c \in A$ has finite order, then γ is rationally equivalent to its translate $\gamma_c = (c) * \gamma$.*

PROOF. Since $\mathrm{CH}_s(A)_{alg}$ is a divisible group [**1**, Lemma 1.3, p. 219], and $[(c) - (0)]$ is torsion (say of order r), we have that $[\gamma_c - \gamma] =$

$[(c)-(0)]*[\gamma]=[(c)-(0)]*[r\gamma']=r[(c)-(0)]*[\gamma']=0*[\gamma']=0$, whence $\gamma_c \sim \gamma$.

REMARK. It is well-known that if γ is a *divisor* on A which is algebraically equivalent to 0, then $\gamma \sim \gamma_a$ for any $a \in A$, finite order or not. (Proof: If D is ample on A, then $\gamma \sim D_b - D$ for some $b \in A$([**7**, Theorem 1, p. 77] or [**1**. p. 216]), hence $\gamma_a - \gamma = ((a)-(0)) * \gamma = ((a)-(0)) * ((b)-(0)) * D = 0$ by the Theorem of the Square [**7**. pp. 59-60].) Corollary 4 is therefore of interest for cycles of codimension > 1 on A.

We end this paper by noting a connection between Corollary 3 and the following theorem of Roitman [**10**, Theorem 3.1, Consequence III, p. 565] (see also [**2**, Theorem 4.1, p. 119]):

THEOREM (ROITMAN). *Let X be a smooth projective variety over an algebraically closed field k. Let $\mathrm{CH}_0(X)_{tors}$ denote the torsion subgroup of the Chow group $\mathrm{CH}_0(X)$, and let $\mathrm{ALB}(X)_{\mathrm{tors}}$ be the torsion subgroup of the Albanese variety $\mathrm{ALB}(X)$ of X. Then, except possibly for p-torsion in characteristic $p > 0$, the natural map $\mathrm{CH}_0(X)_{\mathrm{tors}} \to \mathrm{ALB}(X)_{\mathrm{tors}}$ is an isomorphism.*

In particular, when $X = A$ is an abelian variety in characteristic 0, we have that the natural addition map $\mathrm{CH}_0(A) \to A$, defined by $[\gamma = n_1(a_1) + \cdots + n_s)a_s] \to n_1a_1 + \cdots + n_sa_s \in A$, restricts to an isomorphism $\mathrm{CH}_0(A)_{\mathrm{tors}} \to A_{\mathrm{tors}}$. Corollary 5 gives us the inverse of this map.

COROLLARY 5. *Let A be an abelian variety over an algebraically closed field of characteristic 0. The inverse of the addition isomorphism $\mathrm{CH}_0(A)_{\mathrm{tors}} \to A_{\mathrm{tors}}$ is given by $c \mapsto [(c)-(0)]$.*

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