ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 1, Winter 1988

# THE RATIONAL CUBOID AND A QUARTIC SURFACE

ANDREW BREMNER

**1.** The problem of solving in integers the system of Diophantine equations

$$X^2 + Y^2 = R^2$$
$$Y^2 + Z^2 = S^2 \qquad (1)$$
$$Z^2 + X^2 = T^2$$

has attracted much historical interest, see for example Dickson [**4**; Chapter XIX, references 1-29]. The numerical solution $(X, Y, Z) = (44, 117, 240)$ was observed as early as 1719, and Euler in 1772 provided the parametric solution

$$X = 8\lambda(\lambda - 1)(\lambda + 1)(\lambda^2 + 1)$$
$$Y = (\lambda - 1)(\lambda + 1)(\lambda^2 - 4\lambda + 1)(\lambda^2 + 4\lambda + 1) \qquad (2)$$
$$Z = 2\lambda(\lambda^2 - 3)(3\lambda^2 - 1)$$

although this was apparently discovered by Sanderson in 1740. Kraitchik [**6,7**] discusses the problem extensively and brings together many ad hoc methods for producing further parametric solutions. Of course the system (1) corresponds to a rectangular parallelepiped of which the edges and face diagonals are all integral. The further requirement that the cuboid diagonal be integral is given by the equation

$$X^2 + Y^2 + Z^2 = \text{square}, \qquad (3)$$

and a great deal of effort has been spent in trying to decide the solvability or otherwise of equations (1) and (3). See Lal and Blundon [**8**], Leech [**9**] and Korec [**5**].

The approach here is to consider (1) geometrically, as the intersection $V$ of three quadrics in five-dimensional projective space. There is a birational map to a quartic surface $W$, and it is this latter surface that we study. It possesses four isolated double points, and contains several pencils of elliptic curves. All the straight lines and conics

Received by the editors on October 1, 1985.

lying on $W$ are determined (over the complex numbers rather than just the rationals), together with some cubic and quartic curves of genus zero. In particular, the divisor group of $W$ is seen to have for field of definition, an extension of $\mathbf{Q}(\sqrt{1+\sqrt{2}}, \sqrt{-1}, \sqrt{3}, \sqrt{5}, \sqrt{7})$. Many rational curves on $W$ of degrees four and greater are also produced, thereby leading to new parametric solutions to the equations at (1).

**2.** Parametrizing the quadrics at (1) in the classical Pythagorean manner shows that, without loss of generality, we may put

$$\begin{aligned} X &= k(2ab) = \ell(c^2 - d^2);\ R = \ell(c^2 + d^2) \\ Y &= m(2ef) = \ell(2cd);\ S = m(e^2 + f^2) \\ Z &= k(a^2 - b^2) = m(e^2 - f^2);\ T = k(a^2 + b^2). \end{aligned} \tag{4}$$

Here, $(a,b),(c,d)$ may be considered as the generators of two of the Pythagorean triangles, with $(e,f)$ the generators of the consequent third triangle. Eliminating $k, l, m$ gives

$$(a^2 - b^2)(c^2 - d^2) \cdot 2ef = (e^2 - f^2) \cdot 2ab \cdot 2cd, \tag{5}$$

a relation dependent only upon the ratios $a/b, c/d, e/f$. Now (1) and (4) imply

$$\frac{a}{b} = \frac{Z+T}{X};\ \frac{c}{d} = \frac{X+R}{Y};\ \frac{e}{f} = \frac{Z+S}{Y},$$

so that, from (5),

$$(x,y,z,t) = ((Z+S)(Z+T), X(Z+S), (X+R)(Z+T), Y(Z+T)) \tag{6}$$

is a point on the surface $W$:

$$(x^2 - y^2)(z^2 - t^2) = (x^2 - t^2)2yz. \tag{7}$$

Conversely, from a point on $W$ we obtain a point of $V$ via the map

$$\begin{aligned} (X,Y,Z,R,S,T) =&(x(z^2 - t^2), 2xzt, z(x^2 - t^2), \\ & x(z^2 + t^2), z(x^2 + t^2), (x^2 - t^2)z + (z^2 - t^2)y). \end{aligned} \tag{8}$$

The maps at (6) and (8) are birational inverses.

The surface $W$ possesses the obvious rational symmetries

$$\begin{pmatrix} x & y & z & t \\ x & y & z & -t \end{pmatrix}, \begin{pmatrix} x & y & z & t \\ -x & y & z & t \end{pmatrix}, \begin{pmatrix} x & y & z & t \\ t & z & -y & x \end{pmatrix};$$

and there are further rational involutions of $W$ corresponding to the symmetry in the choice of generators $(a,b),(c,d),(e,f)$ at (4). For instance, taking the generators as $(c,d),(a,b),(e,f)$ gives the involution $(x,y,z,t) \to (1/t, 1/z, 1/y, 1/x)$. Taking the generators as $(a+b, a-b), (c+d, c-d), (e+f, e-f)$ gives the involution

$$(x,y,z,t) \to (x+t, \frac{(x-y)(x+t)}{x+y}, \frac{(x-t)(z+t)}{z-t}, x-t).$$

The above involutions and symmetries commute with each other and together generate a group of order 64. They induce on $V$ the symmetries obtained by changing signs of the coordinates, together with the involution

$$(X,Y,Z,R,S,T) \to (YZ, ZX, XY, RZ, SX, TY). \tag{9}$$

The remaining symmetries of $V$ given by

$$\begin{pmatrix} X & Y & Z & R & S & T \\ Y & X & Z & R & T & S \end{pmatrix}, \quad \begin{pmatrix} X & Y & Z & R & S & T \\ Y & Z & X & S & T & R \end{pmatrix}$$

correspond respectively to the involution of $W$

$$(x,y,z,t) \to (x, t, \frac{y(z+t)}{z-t}, y)$$

and the automorphism of order 3

$$(x,y,z,t) \to (x+y, \frac{t(x+y)}{z}, \frac{(x-y)(x+t)}{x-t}, x-y).$$

The image of a point of $V$ under the involution (9) is classically called the derived point, so that, up to symmetries of $V$, rational cuboids occur naturally in pairs. For example, the derived cuboid corresponding to the Euler cuboid at (2) gives a parametrization of degree 8:

$$\begin{aligned} X &= (\lambda^2 - 4\lambda + 1)(\lambda^2 + 4\lambda + 1)(\lambda^2 - 3)(3\lambda^2 - 1) \\ Y &= 8\lambda(\lambda^2 + 1)(\lambda^2 - 3)(3\lambda^2 - 1) \\ Z &= 4(\lambda - 1)(\lambda + 1)(\lambda^2 + 1)(\lambda^2 - 4\lambda + 1)(\lambda^2 + 4\lambda + 1). \end{aligned}$$

The surface $W$ is singular; indeed, the singularities of $W$ are precisely the four isolated double points

$$(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1).$$

Accordingly, there is not a satisfactory intersection theory of divisors on $W$, and the methods of Swinnerton-Dyer [**10**] as exemplified in Bremner [**1,2,3**] for computing curves of genus 0 on the surface, cannot be applied directly. Instead, we content ourselves here with determining on $W$ a large supply of curves of small degree.

**3.** We show first that there are precisely 22 straight lines on $W$, of which 14 are defined over $\mathbf{Q}$ and 8 defined over $\mathbf{Q}(\sqrt{2})$. Certainly, $(x-y)$ has precisely one zero along any straight line $\ell$ not contained in the plane $x=y$; and so from (7), one of the following functions has neither zero nor pole along $\ell$, and hence is a constant:

$$\frac{x-y}{x-t};\quad \frac{x-y}{x+t};\quad \frac{x-y}{y};\quad \frac{x-y}{z}.$$

Suppose the first instance. Now $x-y=\lambda(x-t)$ cuts $W$ in the line $x=y=t$ and a residual pencil $E_1$ of elliptic cubic curves. Thus, $\ell$ occurs as a component in one of the singular fibres of $E_1$. Similar arguments apply in the remaining cases. So it suffices up to symmetry to determine the singular fibres of the elliptic pencils arising as the residual intersection with $W$ of the planes

(10)

(i) $x-y=\lambda(x-t)$

(ii) $x=\lambda y$

(iii) $x-y=\lambda z$.

In the first case, the residual intersection has equation

$$E_1 : \lambda((\lambda-2)x-\lambda t)(z^2-t^2) = 2z[(\lambda-1)x-\lambda t](x+t). \tag{11}$$

Under the mapping,

$$\begin{aligned} \frac{1}{2}ut^2 &= \lambda(\lambda-1)((\lambda-2)x-\lambda t)z \\ vt &= (2(\lambda-1)x-t)u+2\lambda^2(\lambda-1)(\lambda-2)((\lambda-2)x-\lambda t); \end{aligned} \tag{12}$$

then $E_1$ transforms to

$$\mathcal{E}_1 : v^2 = u(u^2+(\lambda^4-4\lambda^3+8\lambda^2-4\lambda+1)u-8\lambda^3(\lambda-1)^2). \tag{13}$$

The birational inverse of the map (12) is given by

$$\begin{aligned}
x &= v + u + 2\lambda^3(\lambda-1)(\lambda-2)\\
y &= (\lambda-1)((2\lambda-1)u - v - 2\lambda^3(\lambda-2))\\
z &= 2(\lambda-1)u(u+\lambda^2(\lambda-2)^2)^2/(\lambda((\lambda-2)v-(2\lambda^2-3\lambda+2)u))\\
t &= 2(\lambda-1)(u+\lambda^2(\lambda-2)^2).
\end{aligned}$$

It is easy to calculate the values of $\lambda$ for which $E_1$ is singular. Thus, mindful of possible singular points of the transformation, we have that $E_1$ is singular in precisely the following instances, where the decomposition into components is given with the notation of Table 1:

$$\begin{aligned}
&\lambda = 0 : L_2 + L_9 + L_{11}\\
&\lambda = 1 : L_3 + L_{18} + L_{22}\\
&\lambda = \infty : L_4 + L_5 + L_8\\
&\lambda = \phi,\\
&\phi^8 - 8\phi^7 + 32\phi^6 - 40\phi^5 + 34\phi^4 - 40\phi^3 + 32\phi^2 - 8\phi + 1 = 0,
\end{aligned}\tag{14}$$

and the component is a plane unicursal cubic.

In the second case at (10), the residual intersection has equation

$$E_2 : t^2((\lambda^2-1)y - 2z) = yz(-2\lambda^2 y + (\lambda^2-1)z)\tag{15}$$

mapping to

$$E_2 : v^2 = u(u+4\lambda^2)(u+(\lambda^2-1)^2)\tag{16}$$

under the maps

$$v = 2(\lambda^2-1)t((\lambda^2-1)y-2z)/y^2, \quad u = -2(\lambda^2-1)z/y.$$

Singular fibres occur (up to conjugacy of the algebraic integers involved) in precisely the following cases:

$$\begin{aligned}
&\lambda = 0 : \text{plane cubic}\\
&\lambda = 1 : L_1 + L_2 + L_{11}\\
&\lambda = -1 : L_3 + L_4 + L_{13}\\
&\lambda = \infty : L_9 + L_{12} + L_{14}\\
&\lambda = 1+\sqrt{2} : L_{21} + \text{conic}\\
&\lambda = -1+\sqrt{2} : L_{15} + \text{conic}.
\end{aligned}\tag{17}$$

In the third case at (10), the residual intersection has equation

$$\begin{aligned} E_3 : t^2(-2(\lambda-1)x+\lambda(\lambda-2)z) \\ = 2x^3-2\lambda x^2z-2\lambda xz^2+\lambda^2z^3 \end{aligned} \tag{18}$$

mapping to

$$\begin{aligned} \mathcal{E}_3 : v^2 =& u^3+\lambda(\lambda-1)(\lambda^3+4\lambda^2-12\lambda+4)u^2 \\ &-\lambda^3(\lambda-1)^2(\lambda-4)(\lambda^4-4\lambda^3+8\lambda-4)u \\ &-\lambda^4(\lambda-1)^3(\lambda^4-4\lambda^3+8\lambda-4)^2 \end{aligned} \tag{19}$$

under the maps

$$\begin{aligned} v &= (-2\lambda^2(\lambda-1)^3(\lambda^4-4\lambda^3+8\lambda-4)t)/(2(\lambda-1)x-\lambda(\lambda-2)z), \\ u &= (\lambda^2(\lambda-1)(\lambda^4-4\lambda^3+8\lambda-4)z)/(2(\lambda-1)x-\lambda(\lambda-2)z). \end{aligned}$$

Singular fibres (up to conjugacy) occur in precisely the following cases:

$$\begin{aligned} &\lambda=0 : L_1+L_2+L_9 \\ &\lambda=\infty : L_{10}(\text{twice})+L_{13} \\ &\lambda=\sqrt{2} : L_{21}+\text{conic} \\ &\lambda=2+\sqrt{2} : L_{16}+\text{conic} \\ &\lambda=\frac{-13+7\sqrt{-7}}{16} : \text{plane cubic.} \end{aligned} \tag{20}$$

It follows that the straight lines in Table 1 exhaust the possibilities for straight lines on $W$.

Table 1

| | | | |
|---|---|---|---|
| $L_1 : x=y=t$ | $L_2 : x=y=-t$ | $L_3 : -x=y=t$ | $L_4 : x=-y=t$ |
| $L_5 : x=z=t$ | $L_6 : x=z=-t$ | $L_7 : -x=z=t$ | $L_8 : x=-z=t$ |
| $L_9 : x=y=0$ | $L_{10} : z=t=0$ | | |
| $L_{11} : x=y, z=0$ | $L_{12} : y=0, z=t$ | $L_{13} : x=-y, z=0$ | $L_{14} : y=0, z=-t$ |
| $L_{15} : x=(1+\sqrt{2})y=z$ | | $L_{16} : y=(-1-\sqrt{2})z=-t$ | |
| $L_{17} : x=(-1-\sqrt{2})y=-z$ | | $L_{18} : y=(-1-\sqrt{2})z=t$ | |
| $L_{19} : x=(1-\sqrt{2})y=z$ | | $L_{20} : y=(-1+\sqrt{2})z=-t$ | |
| $L_{21} = (-1+\sqrt{2})y=-z$ | | $L_{22} : y=(-1+\sqrt{2})z=t$ | |

**4.** There are conics evident amongst the singular fibres of $E_2, E_3$ at (17) and (20). From $E_2$ with $\lambda = 1 + \sqrt{2}$, there results the parametrization

$$(x, y, z, t) = (1, 1 + \sqrt{2}, s^2, s). \tag{21}$$

The conic at $\lambda = -1 + \sqrt{2}$ just gives a symmetry of the conjugate conic to (21).

From $E_3$ with $\lambda = \sqrt{2}$,

$$\begin{aligned}(x, y, z, t) = (&\sqrt{2}\, s^2 - 2s + 1, (-2 + 2\sqrt{2})s - 1, \\ &s^2 - 2s + \sqrt{2}, s^2 - (1 + \sqrt{2})s + 1);\end{aligned} \tag{22}$$

and with $\lambda = 2 + \sqrt{2}$,

$$\begin{aligned}(x, y, z, t) = (&\sqrt{2}\, s^2 + (2 + 2\sqrt{2})s + (1 + \sqrt{2}), 2s - (1 + \sqrt{2}), \\ &(-1 + \sqrt{2})s^2 + (-2 + 2\sqrt{2})s + \sqrt{2}, \\ &(1 - \sqrt{2})s^2 + s + (1 + \sqrt{2})).\end{aligned} \tag{23}$$

Now, in each of these three instances, the plane of the conic meets $W$ residually in a pair of lines. It is straightforward to find all the conics on $W$ with this property, by use of Table 1. Up to symmetry and conjugacy, precisely the following further example arises:

The plane $y - z = t\sqrt{2}$ cuts $W$ in $L_{18}, L_{20}$, and the conic

$$(x, y, z, t) = (s, 1, -\frac{1}{2}(s^2 - 1))\frac{1}{2\sqrt{2}}(s^2 + 1)). \tag{24}$$

To find the remaining conics on $W$, it is necessary to investigate the alternative possibility, that the plane of a conic on $W$ cuts $W$ in two conics. It is easy to verify that two such conics must be distinct.

Following are three such planes with the resulting conics of intersection.

$$y = \text{it}: \quad (x, y, z, t) = (1 + s, 1 - s, s(1 - s), -i(1 - s)) \tag{25}$$

$$(x, y, z, t) = (-1 - s, 1 - s, s(1 - s), -i(1 - s)) \tag{25$'$}$$

$$\begin{aligned}x = (1 + \sqrt{2})t: (x, y, z, t) = (&(1 + \sqrt{2})(s - 1), (1 + \sqrt{2})(s + 1) \\ &- s(s - 1), s - 1)\end{aligned} \tag{26}$$

$$\begin{aligned}(x, y, z, t) = (&(1 + \sqrt{2})(s - 1), (-1 - \sqrt{2})(s + 1) \\ &s(s - 1), s - 1).\end{aligned} \tag{26$'$}$$

(Note that (25′) and (26′) are symmetries of (25) and (26) respectively.)

$$x + \text{it} = \frac{1+i}{\sqrt{2}}(y + iz):$$

$$(x, y, z, t) = (s, 1, (\frac{1+i}{2})s^2 - i\sqrt{2}s +$$

$$(27) \quad (\frac{-1+i}{2}), \frac{i}{\sqrt{2}}s^2 + s - \frac{i}{\sqrt{2}})$$

$$(x, y, z, t) = ((1 - 2i + i\sqrt{2})s^2 + (-2 + 2i - i\sqrt{2})s + (1 - i), s^2 - 2is - (1 - i),$$

$$(28) \quad s((1 - \sqrt{2})s + \sqrt{2}(1 + i)), s^2 - (2 + i\sqrt{2})s + (1 + i + i\sqrt{2})).$$

The following proposition shows that all the conics on $W$ have now been determined.

PROPOSITION. *Up to conjugacy and symmetry, there are precisely eight conics on $W$, given parametrically at* (21)-(28).

PROOF. Let $C$ be a conic on $W$. If any of the following functions $f_1$ on $W$

$$\frac{x-y}{x-t}, \quad \frac{x-y}{x+t}, \quad \frac{x-y}{y}, \quad \frac{x-y}{z}$$

has no zero or pole along $C$, then as in §3, $C$ arises as a component in a singular fibre of $E_1, E_2$ or $E_3$ and so is determined as above. Now consider a function $f_2 = q_1/q_2$, where $q_1$ is a quadratic factor of $(x^2 - y^2)(z^2 - t^2)$ and $q_2$ is a quadratic factor of $(x^2 - t^2)yz$; and suppose $f_2$ has no zero or pole along $C$. Take, for example, $f_2 = (x^2 - y^2/(x^2 - t^2)$. The family of conics $x^2 - y^2 = \lambda(x^2 - t^2)$ cuts $W$ in the four lines $L_1, L_2, L_3, L_4$ and the pencil of quartic curves

$$E_4: \quad \begin{aligned} x^2 - y^2 &= \lambda(x^2 - t^2) \\ \lambda(z^2 - t^2) &= 2yz \end{aligned}$$

so that $C$ is a component of one of the singular elements of the pencil $E_4$. Similarly, in dealing with the other possibilities for the quadratic numerator and denominator of $f_2$, it suffices by symmetry to determine the singular elements in the following pencils of quartic curves.

$$E_5 \quad \begin{aligned} x^2 - y^2 &= \lambda(x - t)y \\ \lambda(z^2 - t^2) &= 2(x + t)z \end{aligned} \qquad E_6 \quad \begin{aligned} x^2 - y^2 &= \lambda(x - t)z \\ \lambda(z^2 - t^2) &= 2(x + t)y \end{aligned}$$

$$E_7 \quad \begin{aligned} (x - y)(z - t) &= \lambda(x^2 - t^2) \\ \lambda(x + y)(z + t) &= 2yz \end{aligned} \qquad E_8 \quad \begin{aligned} (x - y)(z - t) &= \lambda(x - t)y \\ \lambda(x + y)(z + t) &= 2(x + t)z \end{aligned}$$

$$E_9 \quad \begin{aligned} &(x-y)(z-t) = \lambda(x-t)z \\ &\lambda(x+y)(z+t) = 2(x+t)y. \end{aligned}$$

The singular decompositions are listed in the following section. There remains to consider only the case when no $f_1$ and no $f_2$ is constant along $\mathcal{C}$.

Suppose first that $x, y$ both vanish at a point $P$ of $\mathcal{C}$. Then $(x^2 - y^2)(z^2 - t^2)$ has at least a double zero at $P$, whence so has $(x^2 - t^2)yz$. Thus, either

(i) $z$ has a zero at $P$,

(ii) $t$ has a zero at $P$, or

(iii) $y$ has a double zero at $P$. Normalize the parametrization of $\mathcal{C}$ so that $P$ is the point at infinity. Then we may assume without loss of generality that the parametrization of $\mathcal{C}$ takes the following form in the cases (i)-(iii) respectively:

(i) $(x, y, z, t) = (\ell_1(\mu), \mu, \ell_2(\mu), q_1(\mu))$,

(ii) $(x, y, z, t) = (\ell_1(\mu), \mu, q_1(\mu), \ell_2(\mu))$, and

(iii) $(x, y, z, t) = (\mu, 1, q_1(\mu), q_2(\mu))$,

where $\ell_1, \ell_2$ and $q_1, q_2$ represent linear and quadratic polynomials respectively of $\mu$. In cases (i) and (ii) it is straightforward by direct substitution into the equations of $W$ to verify that the only curves $\mathcal{C}$ that arise are symmetries of (25) and (26). To resolve (iii) in this manner is more tedious, so it is preferable in this instance to observe that with $x = \mu, y = 1$, then $4t^2 = (1-\mu^2)2z - (\mu^4 - 6\mu^2 + 1) - ((\mu^2 - 1)(\mu^2 + 2\mu - 1)(\mu^2 - 2\mu - 1))/(2z - (\mu^2 - 1))$ so that the linear factors of $2z - (\mu^2 - 1)$ in $\mathbf{C}[\mu]$ can be found amongst $\mu \pm 1, \mu \pm 1 \pm \sqrt{2}$. Case by case consideration is not now difficult, and only conics equivalent to those at (21), (24) and (27) arise.

Suppose, secondly, that $x, y$ do not vanish together at any point of $\mathcal{C}$. By symmetry, it may be assumed the same is true for $z, t$. Introduce the notation $(x+y) = P_1 + P_2$ to denote that $(x+y)$ vanishes at the points $P_1, P_2$ of $\mathcal{C}$, and let $(x-y) = P_3 + P_4, (z+t) = P_5 + P_6, (z-t) = P_7 + P_8$, where $P_1, P_2, P_3, P_4$ are distinctive points, as are $P_5, P_6, P_7, P_8$. Then $(yz(x^2 - t^2)) = P_1 + \cdots + P_8$ so that without loss of generality the various assumptions imply that $(y) = P_5 + P_7, (z) = P_1 + P_3$ and either

(a) $(x+t) = P_2 + P_6, (x-t) = P_4 + P_8$ or

(b) $(x+t) = P_2 + P_8, (x-t) = P_4 + P_6$.

In either instance, $(y-t) = ((y+x)-(x+t)) = ((y-x)+(x-t))$ vanishes at both $P_2$ and $P_4$, so that $(y-t) = P_2 + P_4$. Thus, $(x^2-y^2)/((y-t)z)$ has no zero or pole on $\mathcal{C}$. Arguing as before, the quadrics $x^2 - y^2 = \lambda(y-t)z$ cut $W$ in the lines $L_1, L_3, L_{11}, L_{13}$ together with the quartic pencil

$$E_{10}: \quad \begin{aligned} x^2 - y^2 &= \lambda(y-t)z \\ \lambda(z^2-t^2) &= 2y(y+t+\lambda z), \end{aligned}$$

and $\mathcal{C}$ occurs as a component in a singular fibre of $E_{10}$.

In the following section are listed the singular elements of the pencils $E_4, \ldots, E_{10}$; up to symmetry and conjugacy, the conics that arise all occur at (21)-(28), as required.

**5.** Given two quadrics $Q_1, Q_2$ in $\mathbf{P}^3$, then their curve $\Gamma$ of intersection, possessing a distinguished point $P_0$ over some field $k$, is an elliptic curve over $k$ with Jacobian

$$\Gamma' : \det(\Lambda Q_1 + M Q_2) = N^2. \tag{29}$$

The simplest way of achieving a birational map between $\Gamma$ and $\Gamma'$ is as follows: given a point $P$ of $\Gamma$, associate to it that quadric $Q$ of the pencil $\Lambda Q_1 + M Q_2$ such that the tangent linear space to $Q$ at $P$ contains $P_0$.

In practice, to find the singular elements of the pencils $E_4, \ldots, E_{10}$ it is generally easier to find the singular elements of the corresponding Jacobian (29), taking into account also the cases where the mapping between $\Gamma$ and $\Gamma'$ fails to be biregular. For example, consider

$$E_7: \quad \begin{aligned} (x-y)(z-t) &= \lambda(x^2-t^2) \\ \lambda(x+y)(z+t) &= 2yz \end{aligned}$$

with distinguished point $P_0 = (0,1,0,0)$. The above construction gives (30)

$$\begin{aligned} \Gamma' : \lambda^2\Lambda^4 &+ (-2\lambda^3 + 4\lambda^2)\Lambda^3 M + (\lambda^4 - 4\lambda^3 + 8\lambda^2 - 4\lambda + 1)\Lambda^2 M^2 \\ &+ (4\lambda^2 - 2\lambda)\Lambda M^3 + \Lambda^2 M^4 = N^2, \end{aligned}$$

with $\Gamma \to \Gamma'$ given by

$$(\Lambda, M, N) = ((\lambda-2)z+\lambda t, z-t, 2(\lambda-1)(\lambda x((\lambda-2)z+\lambda t)-(z-t)((\lambda-1)z+\lambda t)))$$

with inverse

$$\begin{aligned}(x,y,z,t) = ((\lambda-1)\Lambda((2\lambda-1)\Lambda M + \lambda M^2 + N),\\ -\lambda(\Lambda+M)((2\lambda-1)\Lambda M + \lambda M^2 + N),\\ \lambda(\lambda-1)\Lambda^2(\Lambda+\lambda M), \lambda(\lambda-1)\Lambda^2(\Lambda-(\lambda-2)M)).\end{aligned}$$

Now the discriminant of the quartic at (30) is a constant multiple of $\lambda^6(\lambda-1)^4(\lambda^8-8\lambda^7+32\lambda^6-40\lambda^5+34\lambda^4-40\lambda^3+32\lambda^2-8\lambda+1)$, and hence the singular values of $\lambda$ are $0, 1, \infty, \phi$ (as at (14)). The singular decompositions are accordingly

$\lambda = 0 : L_9 + L_{10} + L_{11} + L_{12}$
$\lambda = \infty : L_3 + L_4 + L_6 + L_8$
$\lambda = 1 :$ two conics, both equivalent to (31)
(which will be denoted abusively
by (31)+(31))
$\lambda = \phi :$ unicursal quartic.

We list Jacobians and singular decompositions for the remaining pencils without giving further details.

$E_4 : \lambda(\lambda-1)\Lambda M(\Lambda - M)(\lambda\Lambda + M) = N^2$
$\lambda = 0 : 2L_9 + L_{11} + L_{13}$
$\lambda = 1 : L_{16} + L_{18} + L_{20} + L_{22}$
$\lambda = \infty : L_5 + L_6 + L_7 + L_8$
$\lambda = -1 :$ (24)+(24)

$E_5 : \Lambda M(-2\lambda^3\Lambda^2 + (\lambda^4 + 8\lambda^2 + 4)\Lambda M - 2\lambda M^2) = N^2$
$\lambda = 0 : L_2 + L_3 + L_{11} + L_{13}$
$\lambda = \infty : L_5 + L_8 + L_{12} + L_{14}$
$\lambda = i\sqrt{2} :$ (27)+(27)
$\lambda = i(2+\sqrt{2}) :$ unicursal quartic

$E_6 : 4\lambda^2\Lambda^4 - 2\lambda^3\Lambda^3 M + 8\lambda^2\Lambda^2 M^2 - 2\lambda\Lambda M^3 + \lambda^2 M^4 = N^2$
$\lambda = 0 : L_2 + L_3 + 2L_9$
$\lambda = \infty : L_1 + L_8 + 2L_{10}$
$\lambda = i\sqrt{2} :$ (28) + (28)
$\lambda = \dfrac{14+5\sqrt{10}}{3\sqrt{3}} :$ unicursal quartic

$$E_8 : \lambda^2\Lambda^4 + (2\lambda^3 - 12\lambda)\Lambda^3 M + (\lambda^4 + 8\lambda^2 + 4)\Lambda^2 M^2 + (-12\lambda^3 + 8\lambda)\Lambda M^3 + 4\lambda^2 M^4 = N^2$$

$\lambda = 0 : L_2 + L_7 + L_{10} + L_{11}$

$\lambda\infty : L_4 + L_8 + L_9 + L_{14}$

$\lambda = \sqrt{2} : L_{15} + L_{22} + (26)$

$\lambda = i(5 + 3\sqrt{3}) :$ unicursal quartic

$$E_9 : \lambda^2\Lambda^4 + (2\lambda^3 - 16\lambda^2 + 12\lambda)\Lambda^3 M + (\lambda^4 - 16\lambda^3 + 56\lambda^2 - 32\lambda + 4)\Lambda^2 M^2 + (12\lambda^3 - 32\lambda^2 + 8\lambda)\Lambda M^3 + 4\lambda^2 M^4 = N^2$$

$\lambda = 0 : L_2 + L_7 + L_9 + L_{12}$

$\lambda = \infty : L_4 + L_8 + L_{10} + L_{13}$

$\lambda = 1 + i : (25) + (25)$

$\lambda = 2 + \sqrt{2} : L_{18} + L_{19} + (26)$

$\lambda = 10 + 7\sqrt{2} :$ unicursal quartic

$$E_{10} : 2\lambda\Lambda(2\lambda\Lambda^3 + (\lambda^2 + 4\lambda)\Lambda^2 M + (2\lambda^2 + 4\lambda)\Lambda M^2 + (\lambda^2 + 2\lambda - 1)M^3) = N^2$$

$\lambda = 0 : L_2 + L_4 + 2L_9$

$\lambda = \infty : 2L_{10} + L_{18} + L_{22}$

$\lambda = \psi :$ unicursal quartic (where $2\psi^4 - 10\psi^3 - 18\psi^2 + 28\psi - 27 = 0$).

**6.** The decompositions of $E_1, E_2$ and $E_3$ at (14), (17) and (20) each reveal one plane unicursal cubic, which are given parametrically as follows.

From (14), with $\lambda = \phi = ((1+i)/\sqrt{2})(1 + \sqrt{2} + \sqrt{2}\sqrt{1+\sqrt{2}})$ :

(31)
$$\begin{aligned} x &= s(s^2 + (0,0,1,-1;0,-2,1,1)s + (3,-3,0,-1;2,0,0,-2) \\ y &= s((0,-1,-\tfrac{1}{2},-\tfrac{1}{2};-1,-1,0,0)s^2 + (-2,0,-2,0;0,0,-2,0)s \\ &\quad + (-3,3,0,1;-2,0,0,2)) \\ z &= (-1,-1,-1,-2;-4,-2,2,0)s + (-4,-4,-2,2;4,-4,-6,2) \\ t &= s(2s + (1,-1,2,-1;2,-4,0,2)). \end{aligned}$$

(Here, the notation $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4)$ denotes the element of $\mathbf{Q}(\theta, i), \theta^2 = 1 + \sqrt{2}$, given by $a_1 + ia_2 + (a_3 + ia_4)\sqrt{2} + (a_5 + ia_6 + (a_7 + ia_8)\sqrt{2})\theta)$. From (17), with $\lambda = 0$:

$$(x, y, z, t) = (0, 2s, s^3 - s, s^2 - 1). \tag{32}$$

From (20), with $\lambda = (-13 + 7\sqrt{-7})/10$:

$$
(33) \qquad
\begin{aligned}
x &= \left(\frac{-29 - 7\sqrt{-7}}{16}\right)s^2 + \left(\frac{-3 + \sqrt{-7}}{8}\right) \\
y &= s^2 - 1 \\
z &= \left(\frac{121 + 203\sqrt{-7}}{256}\right)s^2 + \left(\frac{-1 - 3\sqrt{-7}}{16}\right) \\
t &= \left(\frac{-231 + 171\sqrt{-7}}{256}\right)s^3 + \left(\frac{21 - \sqrt{-7}}{16}\right).
\end{aligned}
$$

Now given any plane parametrized cubic on $W$, the residual intersection with $W$ of the corresponding plane is a straight line. It follows by symmetry that, to determine all such cubics, it is necessary to consider further only the pencil of planes through the line $L_{15}$, say.

The planes $(1 - \sqrt{2})x + y + \lambda(x - z) = 0$ cut out on $W$ the line $L_{15}$ and the pencil of curves

$$
\begin{aligned}
E_{11}: t^2((\lambda^2 + (2 - 2\sqrt{2})\lambda + (2 - 2\sqrt{2}))y + \lambda^2(1 + \sqrt{2})z) \\
= z(2y^2 + (\lambda^2 - 2\lambda(1 + \sqrt{2}))yz + (1 + \sqrt{2})\lambda^2 z^2).
\end{aligned}
$$

And $E_{11}$ is singular (up to conjugacy) exactly when $\lambda = 0, 1, \infty, \sqrt{2}, -1 + \sqrt{2}, -2 + \sqrt{2}, 2 + 2\sqrt{2} + 2\sqrt{2}\theta (\theta^2 = 1 + \sqrt{2})$.

Only in the latter case is the intersection irreducible, and we obtain the curve

$$
(34) \qquad
\begin{aligned}
x &= -(13 + 9\sqrt{2} + (8 + 6\sqrt{2})\theta)s(s^2 \\
&\quad - (8 + 22\sqrt{2} + (-28 + 2\sqrt{2})\theta)) \\
y &= s(s^2 - (6 + 2\sqrt{2} - 4\sqrt{2}\theta)) \\
z &= (-1 + 3\sqrt{2} + (12 - 10\sqrt{2})\theta)s \\
t &= (1 - \sqrt{2})(s^2 + (16 + 6\sqrt{2} + (-20 + 3\sqrt{2})\theta)).
\end{aligned}
$$

There is now the following result.

THEOREM. *Up to conjugacy and symmetry, there are precisely four parametrizable plane cubics on $W$, given at* (31)-(34).

**7.** Of the quartics that are revealed in the decompositions of $E_1, \ldots, E_{10}$ only that from $E_5$ seems to have a relatively simple parametrization, namely

$$\begin{aligned}
x &= (1+\sqrt{2})((1-2\sqrt{2})s^2-1)((3+2\sqrt{2})s^2-4s+1)\\
y &= i(1+\sqrt{2})((1-2\sqrt{2})s^2-1)^2\\
z &= i((1+2\sqrt{2})s^2-1)^2\\
t &= ((1+2\sqrt{2})s^2-1)((3-2\sqrt{2})s^2-4s+1).
\end{aligned}$$

**8.** By using the map at (8), there result several curves of small degree on the original surface (1).

Up to symmetry and conjugacy, the straight lines pull back to the singular point $(X,Y,Z,R,S,T)=(1,0,0,1,0,1)$, the rational conic

(35) $$(X,Y,Z,R,S,T)=(s^2-1,2s,0,s^2+1,2s,s^2-1),$$

and the conic defined over $Q(\sqrt{2})$ with

(36) $$(X,Y,Z,R,S,T)=(s^2-1,2s,2s,s^2+1,2s\sqrt{2},s^2+1).$$

The derived curve corresponding to (36) is again a conic equivalent to (36); it also arises from (21) and (26). The conic (25) leads to the mutually derived conic and quartic solutions, with

(37) $$(X,Y,Z,R,S,T)=(2s,i(s^2+1),-(s^2-1),i(s^2-1),2is^2,s^2+1),$$

(38) $$\begin{aligned}(X,Y,Z,R,S,T)=(s^4-1,-2is(s^2-1),&\\ -2s(s^2+1),(s^2-1)^2,4s^2,(s^2+1)^2).&\end{aligned}$$

Further quartic curves arise from (24), i.e.,

(39)

$$\begin{aligned}(X,Y,Z,R,S,T)=(s^4-1,2s(s^2+1),\frac{1}{2\sqrt{2}}(s^4-6s^2+1),(s^2+1)^2,&\\ \frac{1}{2\sqrt{2}}(s^4+10s^2+1),\frac{1}{2\sqrt{2}}(3s^4-2s^2+3)),&\end{aligned}$$

with derived solution of degree 6, and from (27), i.e.,

(40)

$$\begin{aligned}(X,Y,Z,R,S,T)=((s^2-1)(s^2+i\sqrt{2}s-1),&2s(s^2+i\sqrt{2}s-1),\\
&\frac{1}{i\sqrt{2}}(s^2-1)(s^2+2i\sqrt{2}s-1),\\
&(s^2+1)(s^2+i\sqrt{2}s-1),\\
&\frac{1}{i\sqrt{2}}(s^4+2i\sqrt{2}s^3-6s^2-2i\sqrt{2}s+1),\\
&\frac{1}{\sqrt{2}}(s^2-1)(s^2+1))\end{aligned}$$

with derived curve

(41)

$$\begin{aligned}(X,Y,Z,R,S,T) = (&2s(s^2+2i\sqrt{2}s-1),(s^2-1)(s^2+2i\sqrt{2}s-1),\\ &2i\sqrt{2}s(s^2+i\sqrt{2}s-1),(s^2+1)(s^2+2i\sqrt{2}s-1),\\ &s^4+2i\sqrt{2}s^3-6s^2-2i\sqrt{2}s+1,2is(s^2+1)).\end{aligned}$$

Finally, from (22), (23) and (28) there arise sextic curves of which each derived curve is also of degree 6.

Observe from (5) that when regarding $a,b,c,d,e,f$ as polynomials in $\mathbf{C}[s]$, then $e$ divides $abcd$ in $\mathbf{C}[s]$. If $a,b,c,d,e,f$ are of first degree, then by interchanging the pairs of generators $(a,b),(c,d)$ if necessary, it follows that $e$ divides $ab$. Further, by taking generators $(b,a)$ for $(a,b)$ if necessary, then without loss of generality, $e$ divides $a$, that is, $a/e\in\mathbf{C}$. But the transformation (6) is just $(x,y,z,t) = (ade,bde,acf,adf) = (ad,bd,\frac{a}{e}cf,\frac{a}{e}df)$and hence corresponds to a point on $W$ belonging to a straight line or conic.

Now parametrized conics and cubics on the original surface $X$ clearly have corresponding generators of degree at most 1, and hence some equivalent conic or cubic corresponds via the transformation (6) to a line or conic on $W$. Accordingly, we have determined all possible quadratic parametrizations (up to conjugacy and symmetry) at (35), (36), (37) and cubic parametrizations (none) to the equations (1) for $X$. It seems plausible that all quartic parametrizations are given by (38), (39), (40) and (41), but this has not been verified.

One should note that, over $\mathbf{C}[s]$, the conic (37) furnishes a solution, albeit rather cheekily, to the 'diagonal' requirement (3), in that $X^2+Y^2+Z^2$ is now zero!

**9.** If the motivation is to find parametric solutions over the rationals $\mathbf{Q}$ to the equations (1), then the elliptic pencils $E_1$ to $E_{11}$ can be of great use. We illustrate this with reference to the pencil

$$\begin{aligned}&E_3 : x-y=\lambda z\\ &\qquad t^2(-2(\lambda-1)x+\lambda(\lambda-2)z) = 2x^3-2\lambda x^2z-2\lambda xz^2+\lambda^2z^3.\end{aligned}$$

Taking $(0,0,0,1)$ as the zero for the the group structure, then it may be verified that the group $G$ of points of $E_3$ which are defined over $\mathbf{Q}(\lambda)$, is torsion-free and has rank 2. Generators may be chosen as

$$P_1 = (\lambda,0,1,1)\quad P_2 = (1,1-\lambda,1,1).$$

Then, by addition on $E_3$,

$$P_1 + P_2 = (1, 1 + \lambda, -1, 1)$$
$$P_1 + 2P_2 = (\lambda, -\lambda, 2, -\lambda)$$
$$P_1 - P_2 = ((\lambda - 1)(\lambda^2 - 2\lambda - 1), -(\lambda - 1)(\lambda^3 + \lambda^2 - \lambda + 1), (\lambda - 1)(\lambda^2 + 2\lambda - 3), \lambda^3 + \lambda^2 + 3\lambda - 1)$$
$$2P_1 + P_2 = ((\lambda + 1)(3\lambda^2 - 2\lambda + 1), -(\lambda + 1)(\lambda^3 + 3\lambda^2 - \lambda - 1), (\lambda + 1)(\lambda^2 + 6\lambda - 3), \lambda^3 - \lambda^2 - 5\lambda + 1)$$
$$2P_1 + 2P_2 = (\lambda(\lambda^3 - 4\lambda - 2), -2\lambda(2\lambda^2 + 2\lambda - 1), \lambda^3 + 4\lambda^2 - 4, \lambda(3\lambda^2 + 8\lambda + 6)$$
$$2P_2 = (\lambda(\lambda^3 - 4\lambda^2 + 4\lambda - 2), -2\lambda(2\lambda - 1), \lambda^3 - 4\lambda^2 + 8\lambda - 4, \lambda(\lambda^2 - 2))$$
$$P_1 + 3P_2 = ((\lambda - 2)(3\lambda^2 - 4\lambda - 4), -(\lambda - 2)(\lambda^3 + \lambda^2 + 4), (\lambda - 2)(\lambda^2 + 4\lambda - 4), \lambda^3 - 6\lambda^2 - 4\lambda - 8).$$

The latter five points of $G$ give rise (either directly, or by taking the derived solution) to the following parametrizations of the equations (1), all of degree 8.

$$X = 4(2\lambda - 1)(\lambda^2 - 2\lambda - 1)(\lambda^3 + \lambda^2 - \lambda + 1)$$
$$Y = (\lambda - 1)^2(\lambda + 3)(\lambda^2 - 2\lambda - 1)(\lambda^3 + \lambda^2 + 3\lambda - 1)$$
$$Z = 2\lambda(\lambda - 1)(\lambda + 1)(\lambda + 3)(2\lambda - 1)(\lambda^2 - \lambda + 2);$$

$$X = 4(2\lambda^2 + 2\lambda - 1)(3\lambda^2 - 2\lambda + 1)(\lambda^3 + 3\lambda^2 - \lambda - 1)$$
$$Y = (\lambda + 1)(\lambda^2 + 6\lambda - 3)(3\lambda^2 - 2\lambda + 1)(\lambda^3 - \lambda^2 - 5\lambda + 1)$$
$$Z = 2\lambda(\lambda - 1)(\lambda^2 + \lambda + 2)(\lambda^2 + 6\lambda - 3)(2\lambda^2 + 2\lambda - 1);$$

$$X = \lambda(\lambda^2 - 4\lambda - 8)(3\lambda^2 + 8\lambda + 6)(\lambda^3 + 4\lambda^2 - 4)$$
$$Y = -2(\lambda + 1)(\lambda + 2)(\lambda^2 + \lambda + 2)(\lambda^2 - 4\lambda - 8)(2\lambda^2 + 2\lambda - 1)$$
$$Z = -4(2\lambda^2 + 2\lambda - 1)(3\lambda^2 + 8\lambda + 6)(\lambda^3 - 4\lambda - 2);$$

$$X = \lambda^2(\lambda - 4)(\lambda^2 - 2)(\lambda^3 - 4\lambda^2 + 8\lambda - 4)$$
$$Y = -2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 4)(2\lambda - 1)(\lambda^2 - \lambda + 2)$$
$$Z = -4(2\lambda - 1)(\lambda^2 - 2)(\lambda^3 - 4\lambda^2 + 4\lambda - 2);$$

$$X = 8\lambda(\lambda - 2)(\lambda + 2)(\lambda - 4)(3\lambda + 2)(\lambda^2 - \lambda + 2)$$
$$Y = (\lambda - 2)^2(3\lambda + 2)(\lambda^2 + 4\lambda - 4)(\lambda^3 - 6\lambda^2 - 4\lambda - 8)$$
$$Z = 4\lambda^2(\lambda - 4)(\lambda^2 + 4\lambda - 4)(\lambda^3 - 2\lambda^2 + 4\lambda + 8).$$

The second and fourth of these appear in different guise (and in the latter case misprinted) on pages 93 and 96 of Kraitchik [**7**] but the others appear to be new.

In conclusion, we point out that it is still unknown whether the Eulerian parametrization (2) is the non-trivial rational parametrization to the equations (1) of smallest degree. Further, is any rational solution of degree 6 equivalent to the Eulerian solution?

The surface $W$ at (7) has a superabundance of rational curves lying upon it (we refrain from bombarding the reader with more examples), all of which pull back to parametrizations of (1) of even degree. The evidence suggests that there will be a rational parametrization of the equations (1) of every even degree greater than or equal to six.

I wish to thank the College of Liberal Arts at Arizona State University for the award of a Summer Research Grant whilst this paper was being written.

## REFERENCES

**1.** A. Bremner, *A geometric approach to equal sums of fifth powers,* J. Number Theory **13** (1981), 337-356.

**2.** ———, *A geometric approach to equal sums of sixth powers,* Proc. London Math. Soc. **43** (1981), 544-581.

**3.** ———,*Pythagorean triangles and a quartic surface,* J. reine angew. Math. **318** (1980), 120-125.

**4.** L.E. Dickson, *History of the Theory of Numbers,* vol. 2, Diophantine Analysis, Carnegie Inst. Washington 1920 (reprinted: Chelsea, New York, 1952, 1966).

**5.** I. Korec, *Nonexistence of a small perfect rational cuboid* II, Acta. Math. Univ. Comenian. **44**(45) (1984), 39-48.

**6.** M. Kraitchik, *Théorie des nombres,* v.3, *Analyse Diophantine et applications aux cuboides rationnels,* Gauthier-Villars, Paris, 1947.

**7.** M. Kraitchik, *Sur les cuboides rationnels,* Proc. Int. Congr. Math., Amsterdam, (1954), 33-34.

**8.** M. Lal and W.J. Blundon, *Solutions of the Diophantine equations* $x^2 + y^2 = \ell^2, y^2 + z^2 = m^2, z^2 + x^2 = n^2$, Math. Comp. **20** (1966), 144-147.

**9.** J. Leech, *The rational cuboid revisited,* Amer. Math. Monthly **84** (1977), 518-533.

**10.** H.P.F. Swinnerton-Dyer, *Applications of algebraic geometry to number theory,* Proc. Symp. Pure Mathematics **20** (Amer. Math. Soc., Providence, R.I., 1971), 1-52.

DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, AZ 85287