ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 1, Winter 1988

COMPARISON THEOREMS FOR FOCAL POINTS OF SYSTEMS OF N-TH ORDER NONSELFADJOINT DIFFERENTIAL EQUATIONS

E.C. TOMASTIK

ABSTRACT. A comparison theorem will be given for focal points of $x^{(n)} - \sum_{\mu=0}^{n-1} P_\mu(t)x^{(\mu)} = 0$, where $n \geq 2, P_\mu$ are $m \times m$ matrices with continuous elements on$[a.b], a \geq 0$, and where no assumptions are made concerning the symmetry of any of the P_μ nor the sign of the elements of P_μ.

A comparison theorem will be given for focal points of a very general class of linear ordinary differential equations, with continuous coefficient matrices. The system is

$$x^{(n)} - \sum_{\mu=0}^{n-1} P_\mu(t)x^{(\mu)} = 0 \tag{1}$$

where $n \geq 2, P_\mu$ are $m \times m$ matrices with continuous elements on $[a, b], a \geq 0$.

No assumptions are made concerning the symmetry of any of the P_μ so that (1) may be nonselfadjoint. If (1) is selfadjoint, the results presented here are new. No assumptions are made concerning the sign of the elements of P_μ, making the results new in the scalar case.

The focal point of (1) will be compared to that of

$$y^{(n)} - (-1)^{n-k} \sum_{\mu=0}^{n-1} Q_\mu(t)y^{(\mu)} = 0, \tag{2}$$

where $k \in \{1, \ldots, n-1\}$ and Q_μ are continuous $m \times m$ matrices on $[a, b]$ satisfying some positivity conditions with respect to a cone.

Received by the editors on June 13, 1986.

Focal points play a critical role in variational theory when (1) is selfadjoint. Comparison theorems for such systems [**5,6**] have long been known. More recently, [**4,7,8**] have established new comparison theorems for focal points for selfadjoint and nonselfadjoint systems.

Comparison theorems for focal points for general n^{th} order scaler equations are found in [**1,2,3,9**], where coefficients of the equation are assumed to be of constant sign except in [**1**] where the coefficient of the lowest order term has no sign restriction.

We assume (I, J) are disjoint sets such that $I \cup J = \{1, \ldots, m\}$. The cone K is defined by

$$K = \{(z_1, \ldots, z_m) : \mu \in I \Rightarrow z_\mu \geq 0, \mu \in J \Rightarrow z_\mu \leq 0\}$$

K° denotes the interior of K.

Throughout we assume: for all $t \in [a, b]$, $v \in K$ and $v \neq 0$ that $Q_\circ(t)v \in K^\circ$; for all $t \in [a, b]$ and $v \in K$

$$Q_\mu(t)v \in K, \quad \mu = 1, \ldots, k,$$
$$(-1)^\mu Q_{k+\mu}(t)v \in K, \quad \mu = 1, \ldots, n-k-1.$$

A point $f_p(\alpha) \in [\alpha, \beta]$ is called the first focal point of α relative to (1) provided there is a nontrivial solution $x(t)$ of (1) satisfying $x^{(\mu)}(\alpha) = 0, \mu = 0, \ldots, k-1$, and $x^{(k+\mu)}(f_p(\alpha)) = 0, \mu = 0, \ldots, n-k-1$, and there is no nontrivial solution $z(t)$ of (1) which satisfies $z^{(\mu)}(\alpha) = 0, \mu = 0, \ldots, k-1$, and $z^{(k+\mu)}(\gamma) = 0, \mu = 0, \ldots, n-k-1$,for $\gamma \in [\alpha, \beta)$.

Instead of dealing with (1) directly, a certain equivalent integral equation using an appropriate Green's function will be considered. The Green's function is

$$g(t, s, \alpha) = \frac{1}{(n-k-1)!(k-1)!} \int_a^\delta (s-\xi)^{n-k-1}(t-\xi)^{k-1} d\xi,$$
$$\delta = \min\{t, s\}.$$

Thus $x(t)$ is a solution of (1) with $x^{\mu)}(\alpha) = 0, \mu = 0, \ldots, k-1$, and $x^{(k+\mu)}(\beta) = 0, \mu = 0, \ldots, n-k-1$, if and only if

$$x(t) = \int_\alpha^\beta g(t, s, \alpha)(-1)^{n-k} \sum_{\mu=0}^{n-1} P_\mu(s) x^{(\mu)}(s) ds.$$

Recall the lemma [**4**]:

LEMMA 1. *Suppose* $g : [\alpha, \beta] \to K$ *is continuous and* $g(t) \in K^\circ$ *for some* $t \in [\alpha, \beta]$. *Then* $\int_\alpha^\beta g(s)ds \in K^\circ$.

This is needed to prove:

THEOREM 1. *If the first focal point of* (2) *is* $f_Q(a) = b$, *then* (2) *has a solution* $y(t)$ *such that:* $y^{(i)}(a) = 0, i = 0, \ldots, k-1; y^{(i)}(b) = 0, i = k, \ldots, n-1$; *for* $i = 0, \ldots, k-1, y^{(i)}(t) \in K^\circ$ *for all* $t \in (a, b)$; *and, for* $i = 0, \ldots, n-k-1, (-1)^i y^{(k+i)}(t) \in K^\circ$ *for all* $t \in (a, b)$.

PROOF. Define the Banach space

$$B = \{v \in c^{n-1}[a, b] : v^{(i)}(a) = 0, i = 0, \ldots, k-1\}$$

equipped with the usual sup norm. Also define the cone

$$\begin{aligned}\tilde{K} =&\{\nu \in B : \nu^{(i)}(t) \in K \text{ on } [a, b] \text{ for } i = 0, \ldots, k \text{ and} \\ &(-1)^i \nu^{k+i}(t) \in K \text{ on } [a, b] \text{ for } i = 1, \ldots, n-k-1\}\end{aligned}$$

with interior

$$\begin{aligned}\tilde{K}^\circ =&\{\nu \in B : \nu^{(i)}(t) \in K^\circ \text{ on } (a, b) \text{ for } i = 0, \ldots, k \text{ and} \\ &(-1)^i \nu^{k+i}(t) \in K^\circ \text{ on } (a, b) \text{ for } i = 1, \ldots, n-k-1\}.\end{aligned}$$

Consider the operator

$$T(\nu) = \int_a^b g(t, s, a) \sum_{\mu=0}^{n-1} Q_\mu(s) \nu^{(\mu)}(s) ds.$$

If $\nu \in \tilde{K}$, then $\sum_{\mu=0}^{n-1} Q_\mu(s)\nu^{(\mu)}(s) \in K$ for all $s \in [a, b]$. For $i = 0, \ldots, k-1$, $\frac{\partial^i g}{\partial t^i} \geq 0$ for $a \leq s, t \leq b$ since

$$\text{(3)} \quad \frac{\partial^i g}{\partial t^i} = \frac{1}{(n-k-1)!(k-1)!} \int_a^\delta (s-\xi)^{n-k-1} \frac{\partial^i}{\partial t^i}(t-\xi)^{k-1} d\xi, \quad \delta = \min\{t, s\}.$$

Since

$$(T(v))^{(i)}(t) = \int_a^b \frac{\partial^i g}{\partial t^i}(t,s) \sum_{\mu=0}^{n-1} Q_\mu(s)\nu^{(\mu)}(s)ds,$$
$$i = 0, 1, \dots, k-1,$$

$(T(v))^{(i)}(t) \in K$ for $t \in [a,b]$ and $i = 0, \dots, k-1$. For $i = 0, \dots, n-k-1$,

$$(-1)^i (T(v))^{(k+i)}(t) = \frac{1}{(n-k-i-1)!} \int_t^b (s-t)^{n-k-i-1} \sum_{\mu=0}^{n-1} Q_\mu(s)v^{(\mu)}(s)ds,$$

$(-1)^i(T(v))^{(k+i)}(t) \in K$ for $t \in [a,b]$ and $i = 0, \dots, n-k-1$. Thus $T : \tilde{K} \to \tilde{K}$.

Now define $u_o = (\delta_i)$, where $\delta_i = 1$ if $i \in I$ and $\delta_i = -1$ if $i \in J$, so that $u_o \in K^0$ and also define $\mu_0(t) = \int_a^b g(t,s,a)u_0 ds$. Thus $\mu_0 \in \tilde{K}^0$. It will be shown that T is μ_0-positive with respect to $\tilde{K}$. To demonstrate this, it will be shown that, given any $v \in \tilde{K}, v \neq 0$, there exist positive constants k_1 and k_2 such that $k_1\mu_0 - T(v) \in \tilde{K}$. This will be done by showing that, for all $t \in [a,b]$,

$$k_1\mu_0^{(i)}(t) \leq (T(v))^{(i)}(t) \leq k_2\mu_0^{(i)}(t), \quad i = 0, \dots, k,$$

$$(4) \qquad k_1(-1)^i\mu_0^{(k+i)}(t) \leq (-1)^i(T(v))^{(k+i)}(t) \leq k_2(-1)^i\mu_0^{(k+i)}(t),$$
$$i = 0, \dots, n-k-1,$$

where the inequalities are with respect to K.

Since $v \neq 0$ and $v \in \tilde{K}$, it follows readily that $v(b) \neq 0$; then $Q_0(b)v(b) \in K^0$ and $\sum_{\mu=0}^{n-1} Q_\mu(b)v^{(\mu)}(b) \in K^0$. Since $(-1)^{n-k}[T(v)]^{(n)}(b) = \sum_{\mu=0}^{n-1} Q_\mu(b)v^{(\mu)}(b), (-1)^{n-k}(T(v))^{(n)}(b) \in K^0$. Of course $(-1)^{n-k}\mu_0^{(n)}(t) = u_0 \in K^0$.

By continuity, there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and $c \in [a,b]$ such that, for $t \in [c,b]$,

$$\varepsilon_1(-1)^{n-k}\mu_0^{(n)}(t) \leq (-1)^{n-k}(T(v))^{(n)}(t) \leq \varepsilon_2(-1)^{n-k}\mu_0^{(n)}(t),$$

where the inequalities are with respect to K, i.e.,

$$\varepsilon_2(-1)^{n-k}\mu_0^{(n)}(t) - (-1)^{n-k}(T(v))^{(n)}(t) \in K$$

and

$$(-1)^{n-k}(T(v))^{(n)}(t) - \varepsilon_1(-1)^{n-k}\mu_0^{(n)}(t) \in K.$$

Integrating and using $\mu_0^{(n-1)}(b) = 0 = (T(v))^{(n-1)}(b)$ yields

$$\begin{aligned}\varepsilon_1(-1)^{n-k-1}\mu_0^{(n-1)}(t) &\leq (-1)^{n-k-1}(T(v))^{(n-1)}(t)\\ &\leq \varepsilon_2(-1)^{n-k-1}\mu_0^{(n-1)}(t)\end{aligned}$$

on $[c,b]$ with respect to K. From $Q_0(b)v(b) \in K^0$ and Lemma 1 it follows that

$$(-1)^{n-k-1}(T(v))^{(n-1)}(t) = \int_t^b \sum_{\mu=0}^{n-1} Q_\mu(s)v^{\mu)}(s)ds \in K^0, \quad t \in [a,b).$$

Also,

$$(-1)^{n-k-1}\mu_0^{(n-1)}(t) = \int_t^b u_0 ds \in K^0, \quad t \in [a,b).$$

Thus, by continuity there exist $\delta_1 > 0, \delta_2 > 0$ such that

$$\begin{aligned}\delta_1(-1)^{n-k-1}\mu_0^{(n-1)} &\leq (-1)^{n-k-1}(T(v))^{(n-1)}(t)\\ &\leq \delta_2(-1)^{n-k-1}\mu_0^{(n-1)}(t)\end{aligned}$$

on $[a,c]$ with respect to K. Let $k_1 = \min\{\varepsilon_1,\delta_1\}, k_2 = \min\{\varepsilon_2,\delta_2\}$. Then $k_1(-1)^{n-k-1}\mu_0^{(n-1)}(t) \leq (-1)^{n-k-1}(T(v))^{(n-1)} \leq k_2(-1)^{n-k-1}$ $\mu_0^{(n-1)}(t)$ on $[a,b]$ with respect to K. Proceeding in this manner, we obtain (4).

The remainder of the proof proceeds as in [**3**] and [**4**].

It is useful at this time to give a result that characterizes the structure of the matrices $Q_\mu = (q_{ij}^\mu)$. The proof follows readily and is not given.

LEMMA 2. *Let $Q_\mu(t) = (q_{ij}^\mu(t))$. Then, for $i,j = 1,\ldots,m$ and $t \in [a,b]$, $q_{ij}^0(t) \neq 0$. Also, for $i,j = 1,\ldots,m$, $\mu = 0,1,\ldots,k$ and*

$t \in [a,b]$, we have $|q_{ij}^{\mu}(t)| = \delta_i \delta_j q_{ij}^{\mu}(t)$. Finally, for $i,j = 1,\dots,m, \nu = 1,\dots,n-k-1$, and $t \in [a,b]$, we have

$$|q_{ij}^{k+\nu}(t)| = (-1)^{\nu} \delta_i \delta_j q_{ij}^{k+\nu}(t).$$

Given any $\alpha \in [a,b), f_p(\alpha)$ and $f_Q(\alpha)$ will be the first focal points of α of (1) and (2) respectively. The main theorem can now be given.

THEOREM 2. *Suppose $\alpha \in [a,b], f_Q(a) = b$ and $|p_{ij}^{\mu}(t)| \leq |q_{ij}^{\mu}(t)|$, for all $t \in [a,b], i,j = 1,\dots,m$ and all $\mu = 0,\dots,n-1$. Furthermore, assume that, for any $i = 1,\dots,m$, there exist $j_i \in \{1,\dots,m\}, \mu_i \in \{0,\dots,k-1\}$ such that $|p_{ij_i}^{\mu_i}(b)| < |q_{ij_i}^{\mu_i}(b)|$.*

Then $f_p(\alpha) > f_Q(\alpha)$.

PROOF. Suppose contrary to the conclusion of Theorem 2, that $x(t)$ is a nontrivial solution of (1) satisfying the boundary conditions $x^{(i)}(\alpha) = 0, i = 0,\dots,k-1$, and $x^{(k+i)}(\beta) = 0, i = 0,\dots,n-k-1$, for some $\alpha, \beta \in [a,b], \alpha < \beta$.

Then of course

$$x(t) = \int_{\alpha}^{\beta} (-1)^{n-k} g(t,s,\alpha) \sum_{\mu=0}^{n-1} P_{\mu}(s) x^{(\mu)}(s) ds.$$

From Theorem 1, there exists a nontrivial solution y of (2) such that $y^{(i)}(a) = 0, i = 0,\dots,k-1, y^{(k+i)}(b) = 0, i = 0,\dots,n-k-1$, and

$$y(t) = \int_{a}^{b} g(t,s,a) \sum_{\mu=0}^{n-1} Q_{\mu}(s) y^{(\mu)}(s) ds.$$

If $y = (y_i)$, it will now be shown, for any $r = 0,\dots,n-1$ and $i = 1,\dots,m$, that $(x_i^{(r)}(t))/(y_i^{(r)}(t))$ is continuous and bounded on (α, β).

From the conclusion of Theorem 1, it follows immediately, for $r = 0,\dots,n-1$ and $i = 1,\dots,m$ that $y_i^{(r)}(t) \neq 0$ on (a,b). Now

$$y^{(k)}(a) = \frac{1}{(n-k-1)!} \int_{a}^{b} (s-a)^{n-k-1} \sum_{\mu=0}^{n-1} Q_{\mu}(s) y^{\mu)}(s) ds.$$

It follows immediately, from Lemma 1 and the fact that $\sum_{\mu=0}^{n-1} Q_\mu(s)y^{(\mu)}$ $(s) \in K^0$ for all $s \in (a,b)$, that $y^{(k)}(a) \in K^0$ and thus $y_i^{(k)}(a) \neq 0$ for $i = 1,\dots,m$. Thus $y_i(t)$ has a zero at $t = \alpha$ of at most k, whereas $x_i(t)$ has a zero at $t = \alpha$ of at least k. It follows that, for $r = 0,\dots,n-1$ and $i = 1,\dots,m$ all the terms $(x_i^{(r)}(t))/(y_i^{(r)}(t))$ are bounded as $t \to \alpha+$.

It follows readily from (3) that, for $r = 0,\dots,k-1, y^{(r)}(b) \in K^0$ and thus $y_i^{(r)}(b) \neq 0$, for any $i = 1,\dots,m$ and $r = 0,\dots,k-1$. Furthermore $(-1)^{n-k}y^{(n)}(b) = \sum_{\mu=0}^{n-1} Q_\mu(b)y^{(\mu)}(b)$, and by the same argument that was used in the proof of Theorem 1, $\sum_{\mu=0}^{n-1} Q_\mu(b)y^{(\mu)}(b) \in K^0$. Thus $(-1)^{n-k}y^{(n)}(b) \in K^0$ and $y_i^{(n)}(b) \neq 0$ for $i = 1,\dots,m$. Thus $y_i^{(k)}(t)$ has a zero at $t = \beta$ of order $t = \beta$ of order at most $n-k$, whereas $x_i^{(k)}(t)$ has a zero at $t = \beta$ of order at least $n-k$. It follows that, for $r = 0,\dots,n-1$ and $i = 1,\dots,m$, all the terms $(x_i^{(r)}(t))/(y_i^{(r)}(t))$ are bounded as $t \to \beta-$. It has thus been shown, for any $r = 0,\dots,n-1$ and $i = 1,\dots,m$, that $(x_i^{(r)}(t))/y_i^{(r)}(t))$ is continuous and bounded on (α,β).

Define

$$||x_i^r|| = \sup\{|x_i^{(r)}(t)|/|y_i^{(r)}(t)| : t \in (\alpha,\beta)\}$$

and

$$||x|| = \max\{||x_i^r|| : i = 1,\dots,m; r = 0,\dots,n-1\}.$$

It is clear, for $a \leq \alpha$ and $r = 0,\dots,n-1$, that $\frac{\partial^r}{\partial t^r}g(t,s,a) \geq \frac{\partial^r}{\partial t^r}g(t,s,\alpha)$. For $t \in (\alpha,\beta), r = 0,\dots,k-1$, and $i = 1,\dots,m$, it readily follows that

$$\begin{aligned}
|x_i^{(r)}(t)| &= \left|\int_\alpha^\beta \frac{\partial^r}{\partial t^r} g(t,s,\alpha) \sum_{\mu=0}^{n-1}\sum_{j=1}^{m} p_{ij}^\mu(s)x_j^{(\mu)}(s)ds\right| \\
&\leq \sum_\mu \sum_j \int_\alpha^\beta \frac{\partial^r}{\partial t^r} g(t,s,\alpha)|p_{ij}^\mu(s)||y_j^{(\mu)}(s)||x_j^\mu(s)||y_j^{(\mu)}|^{-1}ds \\
&\leq \sum_\mu \sum_j \int_a^b \frac{\partial^r}{\partial t^r} g(t,s,a)|p_{ij}^\mu(s)||y_j^{(\mu)}(s)ds||x|| \\
&< \sum_\mu \sum_j \int_a^b \frac{\partial^r}{\partial t^r} g(t,s,a)|q_{ij}^\mu(s)||y_j^{(\mu)}(s)|ds||x||,
\end{aligned}$$

where the very last inequality is a strict inequality because of the hypothesis relating p_{ij}^{μ} and q_{ij}^{μ}, the fact that none of the $y_j^{(\mu)}$ vanish on (a,b), and the fact that $g(t,s,a)$ does not vanish for $s \in (a,b)$. Thus, for $r = 0,1,\ldots,k-1, i = 1,\ldots,m$ and $t \in (\alpha,\beta)$, we have

$$\frac{|x_i^{(r)}(t)|}{|y_i^{(r)}(t)|} < \frac{1}{|y_i^{(r)}(t)|}\sum_{\mu}\sum_{j} \int_a^b \frac{\partial^r}{\partial t^r} g(t,s,a)|q_{ij}^{\mu}(s)||y_j^{(\mu)}(s)|ds||x|| \tag{5}$$

Of course if $[\alpha,\beta] \subset (a,b]$, then (5) extends to a strict inequality on $[\alpha,\beta]$ for $r = 0,\ldots,k-1$. If $\alpha = a$, there remains a problem. However, it will be shown that if $\alpha = a$, (5) holds as a strict ineqaulity when $t = \alpha+$. To show this one needs to show that the following strict inequality given by

$$\sum_{\mu}\sum_{j}\int_a^b \frac{\partial^r}{\partial t^r} g(t,s,a)|p_{ij}^{\mu}(s)||y_j^{(\mu)}(s)|ds/|y_i^{(r)}(t)| < \sum_{\mu}\sum_{j}\int_a^b \frac{\partial^r}{\partial t^r} g(t,s,a)|q_{ij}^{\mu}(s)||y_j^{(\mu)}(s)|ds/|y_i^{(r)}(t)| \tag{6}$$

remains a strict inequality as $t \to a+$. This can be seen by realizing that the limit as $t \to a+$ of the left-had side of (6) is just

$$\begin{aligned}&\sum_{\mu}\sum_{j}\frac{\partial^k}{\partial t^k} g(a,s,a)|p_{ij}^{\mu}(s)||y_j^{(\mu)}(s)|ds/|y_i^{(k)}(a)| \\ &= \sum_{\mu}\sum_{j}\frac{1}{(n-k-1)!}\int_a^b (s-a)^{n-k-1}|p_{ij}^{\mu}(s)||y_j^{(\mu)}(s)|ds/|y_i^{(k)}(a)|,\end{aligned} \tag{7}$$

and the limit as $t \to a+$ of the right-hand side of (6) is just

$$\begin{aligned}&\sum_{\mu}\sum_{j}\int_a^b \frac{\partial^k}{\partial t^k} g(a,s,a)|q_{ij}^{\mu}(s)||y_j^{(\mu)}(s)|ds/|y_i^{(k)}(a)| \\ &= \sum_{\mu}\sum_{j}\frac{1}{(n-k-1)!}\int_a^b (s-a)^{n-k-1}|q_{ij}^{\mu}(s)| \\ &\cdot |y_j^{(\mu)}(s)|ds/|y^{(k)}(a)|.\end{aligned} \tag{8}$$

By familiar arguments it follows that (7) is strictly less than (8). It has therefore been established, for $r = 0, \dots, k-1$ and $i = 1, \dots, m$ that

$$\begin{aligned} \|x_i^r\| < \sup_{t\in(\alpha,\beta)} \sum_\mu \sum_j \int_a^b \frac{\partial^r}{\partial t^r} g(t,s,a)|q_{ij}^\mu(s)| \\ \cdot |y_j^{(\mu)}(s)|ds\|x\|/|y_i^{(r)}(t)|. \end{aligned} \tag{9}$$

For $r = 0, \dots, k-1, |y_i^{(r)}(t)| = y_i^{(r)}(t)\delta_i$, and, for $r = 0, \dots, n-k-1, |y_i^{(k+r)}(t)| = (-1)^r y_i^{(k+r)}(t)\delta_i$. Using Lemma 2 it follows that, for $\mu = 0, \dots, k-1$,

$$\begin{aligned} |q_{ij}^\mu(s)||y_j^{(\mu)}(s)| &= \delta_i\delta_j q_{ij}^\mu(s) y_j^{(\mu)}(s)\delta_j \\ &= \delta_i q_{ij}^\mu(s) y_j^{\mu)}(s). \end{aligned}$$

Also, for $\mu = 0, \dots, n-k-1$,

$$\begin{aligned} |q_{ij}^{k+\mu}(s)||y_j^{k+\mu}(s)| &= (-1)^u \mu \delta_i \delta_j q_{ij}^{k+\mu}(s)(-1)^\mu y_j^{(k+\mu)}(s)\delta_j \\ &= \delta_i q_{ij}^{k+\mu}(s) y_j^{(k+\mu)}(s). \end{aligned}$$

Thus, for $r = 0, \dots, k-1$, the right-hand side of (9) is

$$\begin{aligned} &\sup_{t\in(\alpha,\beta)} \sum_\mu \sum_j \int_a^b \frac{\partial^r}{\partial t^r} g(t,s,a) q_{ij}^\mu(s) y_j^{(\mu)}(s)ds\|x\|/y_i^{(r)}(t) \\ &= \sup_{t\in(\alpha,\beta)} y_i^{(r)}(t)\|x\|/y_i^{(r)}(t) \\ &= \|x\|. \end{aligned}$$

Thus, for $r = 0, \dots, k-1$ and $i = 1, \dots, m$,

$$\|x_i^r\| < \|x\|.$$

For $r = 0, \dots, n-k-1$ and $i = 1, \dots, m$, it readily follows that

$$\begin{aligned} \frac{|x_i^{(k+r)}(t)|}{|y_i^{(k+r)}(t)|} &\leq \sum_{\mu=0}^{n-1} \sum_{j=1}^m \int_t^b \frac{(s-t)^{n-k-r-1}}{(n-k-r-1)!} |p_{ij}^\mu(s)||y_j^{(\mu)}(s)|ds\|x\|/|y_i^{k+r}(t)| \\ &< \sum_\mu \sum_j \int_t^b \frac{(s-t)^{n-k-r-1}}{(n-k-r-1)!} |q_{ij}^\mu(s)||y_j^{(\mu)}(s)|ds\|x\|/|y_i^{(k+r)}(t)| \end{aligned}$$

using familiar arguments. The above strict inequality extends to a strict inequality on $[\alpha,\beta]$ if$[\alpha,\beta]\subset[a,b)$. If $\beta=b$, there remains a problem. Using the fact that $|y_i^{(k+r)}|^{(n-k-r)}=(-1)^{n-k-r}|y_i^{(n)}|$, it can be seen that, as $t\to b-$ the right-hand side of the above strict inequality goes to the limit

$$||x||\sum_{\mu=0}^{n-1}|q_{ij}^{\mu}(b)||y_i^{(\mu)}(b)|/|y_i^{(n)}(b)|$$
$$=||x||\sum_{\mu=0}^{k-1}\sum_{j=1}^{m}|q_{ij}^{\mu}(b)||y_j^{(\mu)}(b)|/|y_i^{(n)}(b)|$$

whereas the left-hand side goes to the same term with the $q_{ij}^{\mu}(b)$ replaced with $p_{ij}^{\mu}(b)$. The proof then proceeds as in the case when $t\to a+$ was considered, and one obtains that for $r=0,\dots,n-k-1, i=1,\dots,m$,

$$||x_i^{(k+r)}||<||x||.$$

It has therefore been established that, for $r=0,\dots,n-1$ and $i=1,\dots,m$

$$||x_i^r||<||x||.$$

Thus $||x||<||x||$ and from the contradiction, the truth of Theorem 2 is implied.

References

1. G. Butler and L. Erbe, *Nonlinear integral Riccati systems and comparison theorems for linear differential equations,* SIAM Math. Anal. **14** (1983), 463-473.

2. U. Elias, *Focal points for a linear differential equation whose coefficients are of constant sign,* Trans. Amer. Math. Soc. **149** (1979), 187-202.

3. M. Keener and C. Travis, *Focal points and positive cones for a class of n-th order differential equations,* Trans. Amer. Math. Soc. **237** (1978), 331-351.

4. ———, *Sturmian theory for a class of nonselfadjoint differential systems,* Ann. Mat. Pure Appl. **CXXIII** (1980), 247-266.

5. M. Morse, *A generalization of the Sturm separation and comparison theorem: in n-space,* Math. Ann. **103** (1980), 53-69.

6. W.T. Reid, *Ordinary Differential Equations,* John Wiley, New York, 1971.

7. E. Tomastik, *Comparison theorems for second order nonselfadjoint differential systems,* SIAM J. Math. Anal. **14** (1983), 6-65.

8. ———, *Comparison theorems for certain differential systems of arbitrary order,* SIAM, J. Math. Anal. **17** (1986), 30-37.

9. C. Travis, *Comparison of eigenvalues for linear differential equations and order* $2n$, Trans. Amer. Math. Soc. **177** (1973), 363-374.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06268