ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 4, Fall 1987

ESTIMATING THE NUMBER OF MULTIPLICATIVE PARTITIONS

F.W. DODD AND L.E. MATTICS

Two factorizations of a positive integer n are considered to be essentially the same if they differ only in the order of the factors. The essentially different factorizations of n are called the multiplicative partitions of n, and $f(n)$ denotes the number of such partitions (for example, $f(1) = 1$ and $f(12) = 4$).

In this paper we approximate the function $(3/2)f(n)$ by a multiplicative function $g(n)$. This approximation is then used to conclude that $f(n) \leq n(\log n)^{-\alpha}$ for each fixed $\alpha > 0$ and all sufficiently large n; and that $f(n) \neq O(n^{\beta})$ for $\beta < 1$. A further improvement in the approximation of $f(n)$ enables us to deduce that $f(n) \leq n/\log n$ for all $n > 1, n \neq 144$.

We owe the initial impetus for this paper to two conjectures made by Hughes and Shallit in [**3**]. They conjectured that $f(n) \leq n$ for all n and that $f(n) \leq n/\log n$ for all $n > 1, n \neq 144$. In [**4**] we established the first conjecture and the second conjecture (hereafter referred to as the Hughes–Shallit conjecture) is established in §4.

We remark that if $a_1 \geq a_2 \geq \cdots \geq a_r \geq 1$ and p_i is the i^{th} prime, then $f(2^{a_1}, 3^{a_2} \ldots p_r^{a_r})$ is the number of additive partitions of the "multi–partite number" $(a_1, a_2, \ldots, a_r)$ where addition is defined component–wise (see Chapter 12 of [**1**] for further details). Thus, our results may be used to estimate the number of additive partitions of $(a_1, a_2, \ldots, a_r)$. We further note that $f(2^{a_1}) = p(a_1)$, the number of additive partitions of a_1; and that $f(2 \cdot 3 \cdots p_r) = B_r$, the r^{th} Bell number (see [**9**]).

The related problem of the asymptotics of $c(n)$, the number of multiplicative partitions of n in which the order of factors is taken into account, has been studied by a number of people including Erdös, Hille, Kalmar, Ikehara, Sen, and Sklar. A comprehensive bibliography is given in [**8**]. We are grateful to the referee for calling our attention to this reference.

Received by the editors on November 25, 1985, and in revised form on February 10, 1986.

1. Notations and preliminaries. For the remainder of the article we will use the notations and conventions described in this section. The symbols $a_1, a_2, \ldots, a_r$, n denote nonnegative integers with a_r and n positive; and p_r denotes the r^{th} prime (i.e., $p_1 = 2, p_2 = 3$, etc.). Then x and the arithmetic functions θ, h, and F are defined by $x = (a_1 + a_2 + \cdots + a_r)/r$, $\theta(r) = \log(p_1 p_2 \ldots p_r), h(r) = 1/p_1 + 1/p_2 + \cdots + 1/p_r$, and $F(n) = \sum_{d/n} f(d)$. The arithmetic function K is defined by $K(r) = 2/3$ for $r = 1, 2, 3$ and by

$$K(r) = (2/3) \prod_{i=4}^{r} (4i + 7)/(4p_i - 1)$$

for $r \geq 4$.

The prime factorization of $n > 1$ will always be assumed to be given in the form $n = q_1^{a_1} q_2^{a_2} \ldots q_r^{a_r}$ where $q_1, q_2, \ldots, q_r$ are distinct primes and $a_1 \geq a_2 \geq \cdots \geq a_r \geq 1$. The positive integer $p_1^a p_2^a \cdots p_r^{a_r}$ obtained in the obvious manner from n is then denoted by m. Clearly, $f(n) = f(m)$. When $r \leq 5$ we frequently use a, b, c, d, e to denote a_1, a_2, a_3, a_4, a_5, respectively. Finally, if the monotonicity of the a_i is not assumed (i.e., only that $a_i \geq 0, a_r \geq 1$), then we denote $p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$ by m'.

The multiplicative function g is defined as follows: $g(2^a) = (7/4)^a$; $g(3^a) = (11/4)^a; g(p_r^a) = (r + 7/4)^a, r \geq 3$. The multiplicative function G is then defined by $G(n) = \sum_{d/n} g(d)$.

In the course of the article it will be necessary to evaluate $f(n)$ for certain values of $n < 10^q$ having no more than 188 positive divisors. In order to evaluate these $f(n)$ we developed a program for the Commodore 64. With this program we evaluated $f(m)$ for all $m < 332640$ in addition to the other required values of $f(n)$. The program is based on formula (1.1) below which is derived in the article by Hughes and Shallit. In this formula, $H(a, n)$ is the number of multiplicative partitions of n having no factor greater than a. Clearly, $f(n) = H(n, n)$.

$$H(a, n) = \sum_{\substack{d/n \\ d \leq a}} H(d, n/d). \tag{1.1}$$

The two propositions and corollary which follow will be used often in the remainder of our work. It will be obvious when we use these results and so we will make no further reference to them.

PROPOSITION 1.1. *If* $a_r = 1$ *then* $f(m') = F(m'/p_r)$.

PROOF. Since every multiplicative partition of m' has precisely one factor divisible by p_r, it follows that

$$f(m') = \sum_{d|m'/p_r} f((m'/p_r)/d) = F(m'/p_r).$$

PROPOSITION 1.2.

$$f(m') \leq f(m' \cdot p_{r+1}/p_r) - f(m'/p_r) + f(m'/p_r^{a_r}).$$

PROOF. If $a_r = 1$ equality holds. Thus, suppose $a_r > 1$ and set $y = m' \cdot p_{r+1}/p_r$. For any factorization $d_1 d_2 \cdots d_s$ of m', let d_1 be a largest factor divisible by p_r, so that replacing d_1 by $d_1 \cdot p_{r+1}/p_r$ yields a factorization of y. Thus, we see that essentially different factorizations of m' yield essentially different factorizations of y, and that no essentially different factorization of y with p_{r+1} as a factor comes (in this manner) from a factorization of m' unless the factorization of y has $a_r - 1$ factors p_r. Consequently, as there are $f(m'/p_r) - f(m'/p_r^{a_r})$ multiplicative partitions of y having p_{r+1} as a factor and not having $a_r - 1$ factors p_r, the proposition follows.

The following corollary is an immediate consequence of the preceding two propositions.

COROLLARY 1.3. *We have* $f(m') \leq f(m' \cdot p_{r+1}/p_r)$ *with equality when* $a_r = 1$.

2. Comparisons between f and g. The main result of this section is Proposition 2.4 which establishes an upper bound, namely $(2/3)g(m)$, for $f(n)$ when $n > 1$. This result will prove to be instrumental in settling the Hughes–Shallit conjecture as well as in the investigation of the growth properties of $f(n)$.

In the proof of the following proposition it is convenient to denote the number of multiplicative partitions of n with exactly i factors by $R_i(n)$. Since factoring 2 out of each factor of a multiplicative partition of 2^a with exactly i factors yields a multiplicative partition of 2^{a-i} we have $R_i(2^a) \leq f(2^{a-i})$. Also, it is easily established that $R_2(2^a) = [a/2]$ where [ ] is the greatest integer function.

PROPOSITION 2.1. *We have $f(2^a) \leq (3/2)^a$ for all $a \geq 0$.*

PROOF. From readily available tables of $p(a)$ we see that $f(2^a) = p(a) \leq (3/2)^a$ for $a \leq 8$. Assume, by way of induction, that the proposition is true for all nonnegative integers less than a where $a \geq 9$. Then

$$\begin{aligned} f(2^a) &= \sum_{i=1}^{a} R_i(2^a) = 1 + [a/2] + \sum_{i=3}^{a} R_i(2^a) \\ &\leq 1 + [a/2] + \sum_{i=3}^{a} f(2^{a-i}) \\ &\leq 1 + [a/2] + \sum_{i=3}^{a} (3/2)^{a-i} \\ &= [a/2] - 1 + (8/9)(3/2)^a \leq (3/2)^a. \end{aligned}$$

PROPOSITION 2.2. *If $r = 2$ then $f(m') \leq (1/2)g(m')$.*

PROOF. We have $m' = 2^a \cdot 3^b$ where $a \geq 0$ and $b \geq 1$. We prove the result by induction on b. For $b = 1$ and $a \geq 6$ we have

$$f(2^a \cdot 3) = F(2^a) = \sum_{i=0}^{a} f(2^i) \leq \sum_{i=0}^{a} (3/2)^i \leq 3(3/2)^a \leq (1/2)g(2^a \cdot 3)$$

and for $b = 1, 0 \leq a \leq 5$, the result is verified by computer evaluation of $f(2^a \cdot 3)$.

Now assume that $f(2^i \cdot 3^j) \leq (1/2)g(2^i \cdot 3^j)$ for all nonnegative i and $0 < j \leq b$. Set $A = (1/2)g(2^a \cdot 3^{b+1})$ and $B = F(2^a \cdot 3^b) - F(2^a) -$

$f(2^a \cdot 3^b)$. Then

$$\begin{aligned} f(2^a \cdot 3^{b+1}) &\le f(2^a \cdot 3^b \cdot 5) - f(2^a \cdot 3^b) + f(2^a) \\ &= F(2^a \cdot 3^b) - f(2^a \cdot 3^b) + f(2^a) \\ &= F(2^a) + B + f(2^a) = f(2^a \cdot 3) + B + f(2^a). \end{aligned}$$

Since

$$B = \sum_{i=0}^{a} \sum_{j=1}^{b} f(2^i \cdot 3^j) - f(2^a \cdot 3^b)$$

and $f(2^a \cdot 3^b)$ is a term of the double sum, then

$$\begin{aligned} B &\le (1/2) \sum_{i=0}^{a} \sum_{j=1}^{b} g(2^i \cdot 3^j) - (1/2) g(2^a \cdot 3^b) \\ &= (1/2) G(2^a) \sum_{j=1}^{b} (11/4)^j - (1/2) g(2^a \cdot 3^b) \\ &\le (1/2)(11/3)\{(11/4)^b - 1\} g(2^a) - (1/2) g(2^a \cdot 3^b) \\ &= \{32/33 - (4/3)(4/11)^b\} \cdot A. \end{aligned}$$

Also,

$$f(2^a \cdot 3) \le (1/2) g(2^a \cdot 3) = (4/11)^b \cdot A,$$

and

$$f(2^a) \le (3/2)^a = (8/11)(6/7)^a (4/11)^b \cdot A.$$

Combining these inequalities we obtain

$$\begin{aligned} f(2^a \cdot 3^{b+1}) &\le \{\frac{32}{33} - (\frac{4}{3})(\frac{4}{11})^b + (\frac{4}{11})^b + (\frac{8}{11})(\frac{6}{7})^a (\frac{4}{11})^b\} \cdot A \\ &= \{(\frac{4}{11})^b ((\frac{8}{11})(\frac{6}{7})^a - \frac{1}{3}) + \frac{32}{33}\} (\frac{1}{2}) g(2^a \cdot 3^{b+1}). \end{aligned}$$

From this inequality we conclude that $f(2^a \cdot 3^{b+1}) \le \frac{1}{2} g(2^a \cdot 3^{b+1})$ when (i) $b \ge 3$; (ii) $b = 1$ and $a \ge 4$; (iii) $b = 2$ and $a \ge 2$. The induction is completed by computer evaluation of $f(2^a \cdot 3^{b+1})$ for the few remaining cases not covered by (i)-(iii).

LEMMA 2.3. *If $r \ge 3$ then $G(m/p_r) \le (44/45) g(m)$.*

PROOF. Summing the appropriate geometric series we obtain $G(2^{a_1}) \leq (7/3)g(2^{a_1}), G(3^{a_2}) \leq (11/7)g(3^{a_2})$, and

$$G(p_r^{a_r}/p_r) \leq (4/(4r+3))g(p_r^{a_r}).$$

Moreover, if $r > 3$ then

$$G(p_i^{a_i}) \leq ((4i+7)/(4i+3))g(p_i^{a_i})$$

for $3 \leq i \leq r-1$. Consequently,

$$G(m/p_r) \leq (7/3)(11/7)(4/15)g(m) = (44/45)g(m).$$

PROPOSITION 2.4. *If $n > 1$ then $f(n) \leq (2/3)g(m)$.*

PROOF. If $r = 2$ then $f(n) = f(m) \leq (2/3)g(m)$ by Proposition 2.2. The proposition is also true for $r = 1$ since $f(2) = 1 \leq (2/3)g(2), f(2^2) = 2 \leq (2/3)g(2^2)$, and, by Proposition 2.1, $f(2^a) \leq (2/3)g(2^a)$ when $a \geq 3$.

Now assume, by way induction, that the proposition is true for all positive integers less than n and greater than 1, and that n has 3 or more distinct prime factors. Then since

$$f(n) = f(m) \leq f(m \cdot p_{r+1}/p_r) = F(m/p_r) = \sum_{d|m/p_r} f(d),$$

it follows from Lemma 2.3 and the induction hypothesis that

$$\begin{aligned} f(n) &\leq 1/3 + (2/3) \sum_{d|m/p_r} g(d) = 1/3 + (2/3)G(m/p_r) \\ &\leq 1/3 + (2/3)(44/45)g(m). \end{aligned}$$

Since $r \geq 3, g(m) \geq (7/4)(11/4)(19/4) \geq 45/2$; and so,

$$1/3 + (2/3)(44/45)g(m) \leq (2/3)g(m).$$

Consequently, $f(n) \leq (2/3)g(m)$, and we are done.

3. Growth properties of f. We now focus attention on the growth of f. We will need the crude estimate

$$\theta(r) \leq 16r \log r, \ \ r > 1. \tag{3.1}$$

This estimate follows from the inequalities $\theta(r) \leq (2 \log 2) p_r$, and $p_r \leq 8r \log r / \log 2$, $r > 1$. The first inequality is well–known [**2**, p. 341]; and the second inequality is easily obtained by a slight refinement of the proof of Theorem 8.2 in [**6**, p. 220]. The following obvious lemma will be frequently used (often without reference) in the remainder of our work.

LEMMA 3.1. *If $u_1, u_2, \ldots, u_r$ are monotonically increasing positive real numbers and $a_1 \geq a_2 \geq \cdots \geq a_r \geq 1$, then*

$$u_1^{a_1} u_2^{a_2} \ldots u_r^{a_r} \leq (u_1 u_2 \ldots u_r)^x.$$

PROPOSITION 3.2. *We have $f(m)/m \leq K(r) \cdot \exp(-x \cdot h(r)/4)$. Moreover, if $\alpha > 0$ and $e = \exp(1)$ then $f(m) \leq m/(\log m)^\alpha$ when $K(r) \cdot (\theta(r))^\alpha \leq (h(r) \cdot e/(4\alpha))^\alpha$.*

PROOF. A simple computation shows that $g(m)/m = A \cdot B$ where

$$A = \prod_{i=1}^{r} (1 - 1/(4p_i))^{a_i},$$

$B = 1$ for $r = 1, 2, 3$, and

$$B = (23/27)^{a_4} \ldots ((4r + 7)/(4p_r - 1))^{a_r}$$

for $r \geq 4$. By Lemma 3.1,

$$A \leq \prod_{i=1}^{r} (1 - 1/(4p_i))^x$$

and since $(1 - 1/(4p_i))^x \leq \exp(-x/(4p_i))$, then $A \leq \exp(-x \cdot h(r)/4)$. Thus, by Proposition 2.4,

$$\begin{aligned} f(m)/m \leq (2/3) g(m)/m &\leq (2/3) \cdot B \cdot \exp(-x \cdot h(r)/4) \\ &\leq K(r) \cdot \exp(-x \cdot h(r)/4). \end{aligned}$$

Now suppose that $K(r) \cdot (\theta(r))^\alpha \le (h(r) \cdot e/(4\alpha))^\alpha$. Since $(h(r) \cdot e/(4\alpha))^\alpha \le x^{-\alpha} \exp(x \cdot h(r)/4)$, then $K(r) \cdot (\theta(r))^\alpha \le x^{-\alpha} \exp(x \cdot h(r)/4)$. Hence

$$f(m)/m \le K(r) \cdot \exp(-x \cdot h(r)/4) \le (x \cdot \theta(r))^{-\alpha} \le (\log m)^{-\alpha}.$$

With the aid of the preceding proposition and an asymptotic formula for Bell numbers due to L. Moser and M. Wyman [**5**], we now characterize the growth of $f(n)$.

THEOREM 3.3. (i) *For any real number α, $f(n) \le n/(\log n)^\alpha$ for all sufficiently large n.* (ii) *If $0 < \beta < 1$ then $f(n) \ne O(n^\beta)$ as $n \to \infty$.*

PROOF. Proof of (i). We can assume that $\alpha > 0$. Let δ be the maximum value of $(2/3)(2^8)(16r \log r)^\alpha/2^r$; ε the positive minimum of $x^{-\alpha} \exp(x/8)$ for $x \ge 1$; r_0 a positive integer such that $(2/3)(2^8)(16r \log r)^\alpha/2^r \le \varepsilon$ for all $r > r_0$; and $x_0 \ge 1$ a number such that $x^{-\alpha} \exp(x/8) \ge \delta$ for all $x > x_0$. Since $(4s+7)/(4p_s - 1) \le 1/2$ for $s \ge 9$, it follows that $K(r) \le (2/3)(2^8)(1/2)^r$. Therefore, by Proposition 3.2,

$$f(m)/m \le K(r) \cdot \exp(-x \cdot h(r)/4) \le (2/3)(2^8)(1/2)^r \exp(-x/8).$$

Thus, by estimate (3.1) and the definitions of ε and r_0, when $r > r_0$,

$$\begin{aligned} f(m)/m &\le (2/3)(2^8)(1/2)^r (1/(\varepsilon \cdot x^\alpha)) \\ &\le 1/(16 \cdot x \cdot r \log r)^\alpha \\ &\le 1/(x \cdot \theta(r))^\alpha \le 1/(\log m)^\alpha. \end{aligned}$$

Similarly, when $1 < r \le r_0$, and $x > x_0$,

$$f(m)/m \le (2/3)(2^8)(1/2)^r (1/(\delta \cdot x^\alpha)) \le 1/(\log m)^\alpha.$$

Also, when $r = 1$ (i.e., $m = 2^a$), it follows from Proposition 2.1 that for all sufficiently large a,

$$f(m)/m \le (3/4)^a \le 1/(\log m)^\alpha.$$

From the three preceding inequalities involving $f(m)/m$, we have that $f(m) \le m/(\log m)^\alpha$ for all sufficiently large m, say all $m > m_0$. Let

M be the maximum value of the $f(m)$ satisfying $m \leq m_0$. Then if n is sufficiently large and the corresponding m is at most m_0, we have

$$f(n) = f(m) \leq M \leq n/(\log n)^\alpha.$$

On the other hand, if n is sufficiently large and the corresponding m is greater than m_0, we have

$$f(n) = f(m) \leq m/(\log m)^\alpha \leq n/(\log n)^\alpha$$

and the proof of (i) is complete.

Proof of (ii). For each positive integer r let $N_r = p_1 \cdot p_2 \cdot p_r$, so that $f(N_r)$ is just the r^{th} Bell number. Choose c so that $\beta < c < 1$. Since $\theta(r)/r \log r$ has limit 1 as $r \to \infty$ [**2**, Chapter 22], then for all sufficiently large r we have $N_r^\beta < \exp(cr \log r)$. Moser and Wyman give the following asymptotic formula for $f(N_r)$ in [**5**]:

$$f(N_r) \sim (R+1)^{-\frac{1}{2}}\{1 - \frac{R^2(2R^2 + 7R + 10)}{24r(R+1)^3}\} \cdot \exp(r(R + R^{-1} - 1) - 1)$$

where R is the unique solution of $Re^R = r$. Since $c < 1$ it is an easy matter to show that $f(N_r)/\exp(cr \log r) \to \infty$ as $r \to \infty$. Thus, $f(N_r)/N_r^\beta \to \infty$ as $r \to \infty$ and the proof of (ii) is complete.

4. The Hughes–Shallit conjecture. Theorem 3.3(i) reduces the resolution of the Hughes–Shallit conjecture to checking only a finite number of cases. In this section we resolve the conjecture in the affirmative by improving our estimates of f to the extent that it is necessary to only compute $f(n)$ for 131 values of n, all of which are within the range of the computer program mentioned in §1. The following proposition reduces the problem to a consideration of $f(m)$.

PROPOSITION 4.1. *The Hughes–Shallit conjecture is true if* $f(m) \leq m/\log m$ *for all* $m \neq 144$.

PROOF. If $n > 1$ is a prime, then $f(n) = 1 \leq n/\log n$. Now suppose $f(m) \leq m/\log m$ for all $m \neq 144$ and that n is composite. If $n \neq q_1^4 \cdot q_2^2$ then $m \neq 144 = 2^4 \cdot 3^2$ and $n \geq m \geq 4$ so that

$$f(n) = f(m) \leq m/\log m \leq n/\log n.$$

If $n = q_1^4 \cdot q_2^2 \neq 144$, then $n \geq 3^4 \cdot 2^2 = 324$ and so,

$$f(n) = f(144) = 29 < 324/\log 324 \leq n/\log n.$$

Our attack is now obvious, i.e., to verify that $f(m) \leq m/\log m$ for all $m \neq 144$. The first giant step in this endeavor is accomplished in Proposition 4.3 using our previous estimates of f. First, however, we establish the following lemma which will be frequently used in this section.

LEMMA 4.2. *If* $T(r,x) = (3/2) \cdot K(r) \cdot \theta(r) \cdot x \cdot \exp(-x \cdot h(r)/4)$, *then* $g(m) \cdot \log m/m \leq T(r,x)$.

PROOF. We have $\log m \leq x \cdot \theta(r)$ and the proof of Proposition 3.2 showed that $g(m)/m \leq (3/2) \cdot K(r) \cdot \exp(-x \cdot h(r)/4)$.

PROPOSITION 4.3. *If* $r \leq 2$ *or* ≥ 8 *then* $f(m) \leq m/\log m$ *except when* $m = 144$.

PROOF. (for $r \geq 8$). We apply Proposition 3.2 with $\alpha = 1$. The following table verifies the result for $8 \leq r \leq 12$.

r	$K(r) \cdot \theta(r)$	$h(r) \cdot e/4$
8	0.95	0.99
9	0.53	1.02
10	0.26	1.04
11	0.12	1.06
12	0.05	1.08

Thus, we may suppose that $r \geq 13$. For $s \geq 9$, $(4s+7)/(4p_s-1) \leq 1/2$, and so

$$K(r) \leq (\frac{2}{3})\{(\frac{23}{27})(\frac{27}{43})(\frac{31}{51})(\frac{35}{67})(\frac{39}{75})(2^8)\}(\frac{1}{2})^r < (15.1)(\frac{1}{2})^r.$$

Thus, using (3.1), we have

$$\begin{aligned} K(r)\cdot\theta(r) &< 16(15.1)(1/2)^r(r\log r)\\ &\le 16(15.1)(1/2)^{13}(13\log 13)\\ &< 0.984 < e\cdot h(13)/4 \le e\cdot h(r)/4. \end{aligned}$$

Consequently, $f(m) \le m\log m$ when $r \ge 13$.

Proof for $r = 1$. The result follows from Proposition 2.1 since $f(2^a) \le (3.2)^a \le 2^a \log 2^a$ for all $a \ge 1$.

Proof for $r = 2$. Set $m = 2^a \cdot 3^b, a \ge b \ge 1$, and $L(m) = (1/2)g(m)\cdot \log m/m$. Since $f(m) \le (1/2)g(m)$, by Proposition 2.2, then $f(m)\cdot\log m/m \le L(m)$, so that $f(m) \le m\log m$ when $L(m) \le 1$. If $x = (a+b)/2 \ge 11$ then, by Lemma 4.2, $L(m) \le (1/2)T(2,x) \le 1$, and so we need only further consider $a+b \le 21$. For $a+b \le 21$ we computed (via a simple computer program) $L(m)$ and found that $L(m) \le 1$ with the exception of 50 values of m. The largest of these 50 values was $m = 2^{10}\cdot 3^9$ and this m was also the one with the most positive divisors; i.e., all 50 values were within the range of the computer program for $f(m)$. Computing $f(m)$ for each of these 50 values we found that $f(m) \le m/\log m$ except when $m = 144$.

We need improved estimates for $f(m)$ to deduce that $f(m) \le m/\log m$ for $3 \le r \le 7$. The estimates we have in mind are given in Proposition 4.5. The following lemma will be used in the proof of Proposition 4.5(1).

LEMMA 4.4. *If $r = 3$ and $a_3 = 1$, then*

$$f(m') \le (11/24)(16/19)g(m').$$

PROOF. Set $a = a_1, b = a_2$ so that $m' = 2^a\cdot 3^b\cdot 5$. The lemma is true for $a \ge 0$ and $b = 0$ since

$$f(2^a\cdot 3^0\cdot 5) = f(2^a\cdot 3) \le (\frac{1}{2})g(2^a\cdot 3) \le (\frac{11}{24})(\frac{16}{19})g(2^a\cdot 3^0\cdot 5).$$

Thus, we may assume that $b \ge 1$. We proceed by induction on b. The result is true for $b = 1$ since

$$\begin{aligned}
f(2^a \cdot 3 \cdot 5) = F(2^a \cdot 3) &= F(2^a) + \sum_{i=0}^{a} f(2^i \cdot 3) \\
&= f(2^a \cdot 3) + \sum_{i=0}^{a} f(2^j \cdot 3) \\
&\leq (\frac{1}{2})g(2^a \cdot 3) + (\frac{1}{2})(\frac{7}{3})g(2^a \cdot 3) \\
&= (\frac{10}{24})(\frac{16}{19})g(2^a \cdot 3 \cdot 5) < (\frac{11}{24})(\frac{16}{19})g(2^a \cdot 3 \cdot 5).
\end{aligned}$$

If the lemma is true for the positive integer b, then

$$\begin{aligned}
f(2^a \cdot 3^{b+1} \cdot 5) = F(2^a \cdot 3^{b+1}) &= F(2^a \cdot 3^b) + \sum_{i=0}^{a} f(2^i \cdot 3^{b+1}) \\
&= f(2^a \cdot 3^b \cdot 5) + \sum_{i=0}^{a} f(2^i \cdot 3^{b+1}) \\
&\leq (\frac{11}{24})(\frac{16}{19})g(2^a \cdot 3^b \cdot 5) + (\frac{1}{2}) \sum_{i=0}^{a} g(2^i \cdot 3^{b+1}) \\
&\leq (\frac{11}{24})(\frac{16}{19})g(2^a \cdot 3^b \cdot 5) + (\frac{1}{2})(\frac{7}{3})g(2^a \cdot 3^{b+1}) \\
&= (\frac{11}{24})(\frac{16}{19})g(2^a \cdot 3^{b+1} \cdot 5)
\end{aligned}$$

completing the induction and the proof.

PROPOSITION 4.5.
(1) *If* $r = 3$ *then* $f(m') \leq (11/24)(16/19)^{a_3} 9(m')$.
(2) *If* $r = 4$ *then* $f(m') \leq (242/567)(16/19)^{a_3}(21/23)^{a_4} g(m')$.
(3) *If* $r = 5$ *then* $f(m') \leq (42592/111537)(16/19)^{a_3}(21/23)^{a_4} g(m')$.
(4) *If* $r = 6, a_6 = 1$, *then* $f(m') \leq (7/20)(16/19)^{a_3}(21/23)^{a_4} g(m')$.
(5) *If* $r = 7, a_6 = a_7 = 1$, *then* $f(m') \leq (1/3)(16/19)^{a_3}(21/23)^{a_4} g(m')$.

PROOF. Proof of (1). Set $a_1 = a, a_2 = b, a_3 = c$. We use induction on c. Lemma 4.4 shows the proposition is true for $c = 1$. Now assume that

$$f(2^i \cdot 3^j \cdot 5^k) \leq (11/24)(16/19)^k g(2^i \cdot 3^j \cdot 5^k)$$

for all nonnegative i, j and $0 < k \le c$. Set

$$A = (11/24)(16/19)^{c+1} g(2^a \cdot 3^b \cdot 5^{c+1})$$

and

$$B = F(2^a \cdot 3^b \cdot 5^c) - F(2^a \cdot 3^b) - f(2^a \cdot 3^b \cdot 5^c).$$

Then

$$\begin{aligned} f(2^a \cdot 3^b \cdot 5^{c+1}) &\le f(2^a \cdot 3^b \cdot 5^c \cdot 7) - f(2^a \cdot 3^b \cdot 5^c) + f(2^a \cdot 3^b) \\ &= F(2^a \cdot 3^b \cdot 5^c) - f(2^a \cdot 3^b \cdot 5^c) + f(2^a \cdot 3^b) \\ &= F(2^a \cdot 3^b) + B + f(2^a \cdot 3^b) \\ &= f(2^a \cdot 3^b \cdot 5) + B + f(2^a \cdot 3^b). \end{aligned}$$

Since

$$B = \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{k=1}^{c} f(2^i \cdot 3^j \cdot 5^k) - f(2^a \cdot 3^b \cdot 5^c)$$

and $f(2^a \cdot 3^b \cdot 5^c)$ is a term of the triple sum, then

$$\begin{aligned} B &\le \left(\frac{11}{24}\right) \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{k=1}^{c} \left(\frac{16}{19}\right)^k g(2^i \cdot 3^j \cdot 5^k) - \left(\frac{11}{24}\right)\left(\frac{16}{19}\right)^c g(2^a \cdot 3^b \cdot 5^c) \\ &= \left(\frac{11}{24}\right) G(2^a \cdot 3^b) \sum_{k=1}^{c} 4^k - \left(\frac{11}{24}\right)\left(\frac{16}{19}\right)^c g(2^a \cdot 3^b \cdot 5^c) \\ &\le \left(\frac{11}{24}\right)\left(\frac{44}{9}\right)(4^c - 1) g(2^a \cdot 3^b) - \left(\frac{11}{24}\right)\left(\frac{16}{19}\right)^c g(2^a \cdot 3^b \cdot 5^c) \\ &= \left\{\frac{35}{36} - \left(\frac{11}{9}\right)\left(\frac{1}{4}\right)^c\right\} \cdot A. \end{aligned}$$

Also,

$$f(2^a \cdot 3^b \cdot 5) \le \left(\frac{11}{24}\right)\left(\frac{16}{19}\right) g(2^a \cdot 3^b \cdot 5) = \left(\frac{1}{4}\right)^c \cdot A;$$

and if $b > 0$, then

$$f(2^a \cdot 3^b) \le \left(\frac{1}{2}\right) g(2^a \cdot 3^b) = \left(\frac{3}{11}\right)\left(\frac{1}{4}\right)^c \cdot A.$$

Combining these inequalities we obtain

$$\begin{aligned} f(2^a\cdot 3^b\cdot 5^{c+1}) &\le \left\{\left(\frac{1}{4}\right)^c+\frac{35}{36}-\left(\frac{11}{9}\right)\left(\frac{1}{4}\right)^c+\left(\frac{3}{11}\right)\left(\frac{1}{4}\right)^c\right\}\cdot A\\ &=\left\{\frac{35}{36}+\left(\frac{5}{99}\right)\left(\frac{1}{4}\right)^c\right\}\cdot A\\ &\le A=\left(\frac{11}{24}\right)\left(\frac{16}{19}\right)^{c+1} g(2^a\cdot 3^b\cdot 5^{c+1})\end{aligned}$$

when $b>0$. Finally, when $b=0$ we have

$$\begin{aligned} f(2^a\cdot 3^0\cdot 5^{c+1}) = f(2^a\cdot 3^{c+1}) &\le (\frac{1}{2})g(2^a\cdot 3^{c+1})\\ &\le (\frac{11}{24})(\frac{16}{19})^{c+1}g(2^a\cdot 3^0\cdot 5^{c+1})\end{aligned}$$

and the induction is complete.

The proofs of (2) and (3) are almost identical to the proof of (1). For example, to prove (2) we first use (1) to establish the result for $a_4=1$. We then use (1) and induction as in (1) to finish the proof. To prove (4) we use (3) and proceed as in Lemma 4.4. Finally, to prove (5) we use (4) and proceed as in the $b=1$ case of Lemma 4.4.

PROPOSITION 4.6. *If* $3\le r\le 5$ *then* $f(m)\le m/\log m$.

PROOF. The proof is similar to the $r=2$ case. For $m=2^a\cdot 3^b\cdot 5^c$, $m=2^a\cdot 3^b\cdot 5^c\cdot 7^d$, and $m=2^a\cdot 3^b\cdot 5^c\cdot 7^d\cdot 11^e$ respectively, let

$$L(3,m)=(\frac{11}{24})(\frac{16}{19})^c g(m)\cdot \log m/m,$$

$$L(4,m)=(\frac{242}{567})(\frac{16}{19})^c(\frac{21}{23})^d g(m)\cdot \log m/m,$$

and

$$L(5,m)=(\frac{42592}{111537})(\frac{16}{19})^c(\frac{21}{23})^d g(m)\cdot \log m/m.$$

It then follows from (1)-(3) of Proposition 4.5 that $f(m)\le m/\log m$ when the corresponding L–value does not exceed 1. Using Lemma 4.2

we have the following:

$$L(3,m) \leq (\frac{11}{24})(\frac{16}{19})T(3,x) \leq 1$$

when $x = (a+b+c)/3 \geq 10$;

$$L(4,m) \leq (\frac{242}{567})(\frac{16}{19})(\frac{21}{23})T(4,x) \leq 1$$

when $x = (a+b+c+d)/4 \geq 8.8$; and

$$L(5,m) \leq (\frac{42592}{111537})(\frac{16}{19})(\frac{21}{23})T(5,x) \leq 1$$

when $x = (a+b+c+d+e)/5 \geq 6.6$. Thus, for $r = 3, 4$ and 5 respectively, we need only further consider values of m given by $a+b+c \leq 29, a+b+c+d \leq 35$, and $a+b+c+d+e \leq 32$. For each of these values of m we computed the corresponding L–value. We observed that none of the L–values in the $r = 5$ case exceeded 1; there were exactly 22 values of m in the $r = 4$ case having L–value greater than 1; and there were exactly 59 values of m in the $r = 3$ case having L–value greater than 1. The largest of these 81 values of m was $m = 2^8 \cdot 3^8 \cdot 5$, whereas the one with the largest number of positive divisors was $m = 2^7 \cdot 3^6 \cdot 5^2$. Thus, all 81 values of m were within the range of the computer program for $f(m)$. By actual computation of $f(m)$ we found that $f(m) \leq m/\log m$ for each of the 81 values of m.

The only cases left to consider are when $r = 6$ and $r = 7$. The following lemma simplifies our work in these two remaining cases.

LEMMA 4.7.
(1) *If* $r = 6$ *and* $f(m) > m/\log m$, *then* $a_6 = 1$.
(2) *If* $r = 7$ *and* $f(m) > m/\log m$, *then* $a_6 = a_7 = 1$.

PROOF. Suppose that $a_6 \geq 2$ so that $a_i \geq 2$ for $1 \leq i \leq 6$. A slight modification of the proof of Proposition 3.2 then gives

$$\begin{aligned}(\frac{2}{3})g(m) \cdot \log m/m &\leq (\frac{23}{27})(\frac{27}{43})(\frac{31}{51})K(r) \cdot \exp(-x \cdot h(r)/4) \cdot \log m \\ &\leq (\frac{2}{3})(\frac{23}{43})(\frac{31}{51})T(r,x)\end{aligned}$$

for $r = 6$ and $r = 7$. A computation shows that the right side of the inequality is less than 1 when $x \geq 1$ and $6 \leq r \leq 7$. Thus, by Proposition 2.4, $f(m) \leq m/\log m$ when $a_6 \geq 2$ and $6 \leq r \leq 7$.

PROPOSITION 4.8. *If* $6 \leq r \leq 7$ *then* $f(m) \leq m/\log m$.

PROOF. In view of Lemma 4.7 we may suppose that $a_6 = 1$ when $r = 6$, and $a_6 = a_7 = 1$ when $r = 7$. Then, by Proposition 4.5(4)-(5) and Lemma 4.2,

$$f(m) \cdot \log m/m \leq (\frac{7}{20})(\frac{16}{19})(\frac{21}{23})T(r, x),$$

and a computation shows that for $6 \leq r \leq 7$ and $x \geq 1$ the right side of the ineqaulity is less than 1.

The resolution of the Hughes–Shallit conjecture is now at hand.

THEOREM 4.9. *The Hughes–Shallit conjecture is true.*

PROOF. Immediate from Propositions 4.1, 4.3, 4.6, and 4.8.

Adendum-Asymptotic Behavior of $f(n)$. After this article was accepted for publication it was brought to our attention by Professor Carl Pomerance that he in collaboration with Paul Erdös and E.R. Canfield in [**7**] obtained asymptotic results superior to those of Theorem 3.3. Indeed, in their article, they show that $f(n) = n \cdot L(n)^{-1+k(n)}$, where $L(n) = \exp[\log n \cdot \log\log\log n/\log n]$ and $k(n)$ is a function with $\lim_{n\to\infty} k(n) = 0$. It is an easy exercise to deduce our Theorem 3.3 from this result. Their result, however, cannot be used to solve the Hughes-Shallit conjecture since the behavior of $k(n)$ is known only to asymptotically.

REFERENCES

1. G.E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications 2, Gian–Carlo Rota, Editor, Addison–Wesley, Reading, MA, 1976.

2. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford University Press, Oxford, 1979.

3. J.F. Hughes and J.O. Shallit, *On the number of multiplicative partitions*, Amer. Math. Monthly, **90** (1983), 468-471.

4. L.E. Mattics and F.W. Dodd, *A bound for the number of multiplicative partitions*, Amer. Math. Monthly (to appear in Notes).

5. L. Moser and M. Wyman, *An asymptotic formula for the Bell numbers*, Trans. Roy. Soc. Canada Sect. III (3), **49** (1955), 49-54.

6. I. Niven and H.S. Zuckerman, *An Introduction to the Theory of Numbers*, 4th ed., Wiley, New York, 1980.

7. C. Pomerance, P. Erdös and E. Canfield, *On a problem of Oppenheim concerning "Factorisatio Numerorum"*, J. of Number Theory **17** (1983), 1-28.

8. A. Sklar, *On the factorization of squarefree integers*, Proc. Amer. Math. Soc. **3** (1952), 701-705.

9. G.T. Williams, *Numbers generated by the function* e^{e^x-1}, Amer. Math. Monthly, **52** (1945), 323-327.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SOUTH ALABAMA, MOBILE, ALABAMA 36688