ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 4, Fall 1987

COMPACT WEIGHTED COMPOSITION OPERATORS ON SOBOLEV RELATED SPACES

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ABSTRACT. If m is a positive integer and $1 \leq p \leq \infty$, let $W_{m,p}$ denote the set of functions f on the unit interval $[0,1]$ for which $f, f', \ldots, f^{(m-1)}$ are absolutely continuous and $f^{(m)} \in L^p$. With $\|f\|_{W_{m,p}} = \left(\sum_{s=0}^m \|f^{(s)}\|_p^p\right)^{1/p}$, $1 \leq p < \infty$, $W_{m,p}$ is a Banach space. We show that if $u \in W_{m,\infty}, \varphi : [0,1] \to [0,1], \varphi \in W_{m,\infty} \cap C^1$, and there exists a positive integer N for which $\varphi^{-1}([a,b])$ can be expressed as a union of N intervals for all $a, b \in [0,1]$, then the weighted composition operator $uC_\varphi : f(x) \to u(x)f(\varphi(x))$ is a bounded linear operator on $W_{m,p}$ which is compact if and only if $u\varphi' = 0$. Further, if uC_φ is compact on $W_{m,p}$, then the spectrum $\sigma(uC_\varphi) = \{\lambda | \lambda^n = u(c) \ldots u(\varphi_{n-1}(c))$ for some positive integer n and some fixed point c of φ of order $n\} \cup \{0\}$.

If m is a positive integer and $1 \leq p \leq \infty$ let $W_{m,p}$ denote the set of functions f on $[0,1]$ for which f and the derivatives $f', f'', \ldots, f^{(m-1)}$ lie in AC, the space of absolutely continuous functions on $[0,1]$, and $f^{(m)} \in L^p(0,1) \equiv L^p$. For $1 \leq p < \infty, W_{m,p}$ is a Banach space under the norm $\|f\|_{W_{m,p}} = \left(\sum_{s=0}^m \|f^{(s)}\|_p^p\right)^{1/p}$. These spaces are closely related to Sobolev spaces on $[0,1]$ (see [**1,2,3**]). A weighted composition operator on $W_{m,p}$ is a map from $W_{m,p}$ to itself of the form $f(x) \to u(x)f(\varphi(x))$, where $u : [0,1] \to \mathbf{C}$ and $\varphi : [0,1] \to [0,1]$. We denote such a map by uC_φ.

In [**1**] Antonevich considered weighted composition operators on $W_{m,p}$, where $u, \varphi \in C^m[0,1]$ and φ is a bijection of $[0,1]$ onto itself and determined their spectra. In this note we study other weighted composition operators on $W_{m,p}$ and characterize those operators which are compact. We show that if $u \in W_{m,\infty}, \varphi \in W_{m,\infty} \cap C^1$ and if $uC_\varphi : W_{m,p} \to W_{m,p}$, then uC_φ is compact if and only if $u\varphi' = 0$. Further, if we let φ_n denote the n^{th} iterate of φ and $\sigma(uC_\varphi)$ the spectrum of uC_φ, then if uC_φ is compact on $W_{m,p}$, we have that $\sigma(uC_\varphi)\backslash\{0\} = \{\lambda | \lambda^n = u(c) \ldots u(\varphi_{n-1}(c))$ for some positive integer

1980 Mathematics Subject Classification: 47B37, 47B05.
Received by the editors on October 25, 1984.

n and some fixed point c of φ of order $n\}$

Our first step is to determine maps u and φ which induce weighted composition operators on $W_{m,p}$. In doing so the following result of Josephy [**4**] will be useful. If N is a positive integer, let $J_N = \{E \subset [0,1] | E$ can be expressed as a union of N intervals $\}$ (where the intervals may be open or closed at either end and singletons are allowed as degenerate closed intervals). A function $f : [0,1] \to [0,1]$ is said to be of N–bounded variation if $f^{-1}([a,b]) \in J_N$ for all $[a,b] \subset [0,1]$. As usual, BV will denote the Banach space of functions of bounded variations on $[0,1]$.

THEOREM (JOSEPHY [4]). *For $g : [0,1] \to [0,1]$, the composition $f \circ g$ belongs to* BV *for all $f \in BV$ if and only if g is of N–bounded variation for some positive integer N.*

Combining this theorem with the fact [**6**, p. 250] that a continuous function f of bounded variation is absolutely continuous if and only if f maps each set of measure 0 into a set of measure 0, we have the following.

THEOREM 1. *If $\varphi : [0,1] \to [0,1]$ is absolutely continuous and of N–bounded variation for some positive integer N, then $f \circ \varphi \in AC$ for all $f \in AC$.*

LEMMA 2. *Let $g \in L^1, \varphi : [0,1] \to [0,1], \varphi \in C^1$ and φ be of N–bounded variation for some positive integer N. Then $\int_0^1 |g(\varphi(x))\varphi'(x)| dx \leq N \int_0^1 |g|$.*

PROOF. Since φ' is continuous, $\{x | \varphi'(x) \neq 0\}$ is open and can thus be expressed as a union of disjoint relatively open subintervals $\bigcup(a_i, b_i)$. The image $\varphi(a_i, b_i)$ of (a_i, b_i) is again an interval since φ is continuous. Write $\varphi(\bigcup(a_i, b_i)) = \bigcup \varphi(a_i, b_i) = \bigcup(A_k, B_k)$, where $\{(A_k, B_k)\}$ is again a disjoint union of relatively open intervals. Clearly, for each k, $(A_k, B_k) = \bigcup\{\varphi(a_i, b_i) | \varphi(a_i, b_i) \subset (A_k, B_k)\}$. Since, for each

$x \in \bigcup \varphi(a_i, b_i), \varphi^{-1}(\{x\})$ has at most N elements, it follows that

$$N\int_0^1 |g| \geq N\sum_k \int_{A_k}^{B_k} |g| \geq \sum_k \sum_i \int_{\varphi(a_i,b_i)} |g|,$$

where the inner sum is on all i for which $\varphi(a_i, b_i) \subset (A_k, B_k)$. By a change of variables, $\int_{\varphi(a_i,b_i)} |g| = \int_{(a_i,b_i)} |g(\varphi(x))||\varphi'(x)|dx = \int_{a_i}^{b_i} |g(\varphi(x))||\varphi'(x)|dx$, and since $\varphi'(x) = 0$ on $[0,1]\setminus \cup (a_i, b_i)$, we have

$$N\int_0^1 |g| \geq \sum_i \int_{a_i}^{b_i} |g(\varphi(x))||\varphi'(x)|dx = \int_0^1 |g(\varphi(x))||\varphi'(x)|dx$$

as required.

Now suppose $\varphi : [0,1] \to [0,1], \varphi \in C^1$ and φ is of N–bounded variation for some positive integer N. Let s be a non–negative integer. If $f^{(s)} \in L^p$, then $|f^{(s)}|^p \in L^1$ and, by Lemma 2,

$$N\int_0^1 |f^{(s)}|^p \geq \int_0^1 |f^{(s)}(\varphi(x))|^p|\varphi'(x)|dx.$$

Therefore, for such f and φ,

$$\begin{aligned}\int_0^1 |f^{(s)}(\varphi(x))\varphi'(x)|^P dx &\leq \Big(\int_0^1 |f^{(s)}(\varphi(x))|^p|\varphi'(x)|dx\Big)(||\varphi'||_\infty^{p-1}) \\ &\leq N||\varphi'||_\infty^{p-1}\int_0^1 |f^{(s)}|^p.\end{aligned}$$

Hence $||(f^{(s-1)} \circ \varphi)'||_p^p \leq N||\varphi'||_\infty^{p-1}||f^{(s)}||_p^p$ or

(1) $\quad ||(f^{(s-1)} \circ \varphi)'||_p \leq N^{1/p}||\varphi'||_\infty^{1/q}||f^{(s)}||_p$, where $1/p + 1/q = 1$.

In particular,
(2)
if $f \in W_{m,p}$, then $f^{(m)} \in L^p$ and $||f^{(m-1)} \circ \varphi)'||_p \leq N^{1/p}||\varphi'||_\infty^{1/q}||f^{(m)}||_p$.

Also, letting $p = s = 1$ in (1) we have that if $f \in$ AC, then $\mathrm{Var}\,(f \circ \varphi) = \int_0^1 |(f \circ \varphi)'| \leq N\int_0^1 |f'| = N\mathrm{Var}\,f$. Examples of the form $\varphi(x) = \sin^2 n\pi x$ show that N is the best bound.

THEOREM 3. *Let m be a positive integer and $1 \leq p < \infty$. If $\varphi : [0,1] \to [0,1], \varphi \in W_{m,\infty} \cap C^1$, φ is of N–bounded variation for some positive integer N and $u \in W_{m,\infty}$, then the map $uC_\varphi : f(x) \to u(x)f(\varphi(x))$ is a bounded linear map on $W_{m,p}$.*

PROOF. Suppose φ and u satisfy the hypotheses. We remark that the added assumption that $\varphi \in C^1$ is needed only when $m = 1$. Since φ is absolutely continuous and of N–bounded variation, $C_\varphi : f(x) \to f(\varphi(x))$ maps AC into itself by Theorem 1. Thus if $f \in W_{m,p}$, then also $f', f'', \dots, f^{(m-1)} \in$AC and consequently $f \circ \varphi, f' \circ \varphi, \dots, f^{(m-1)} \circ \varphi \in$ AC.

We next show that if $f \in W_{m,p}$, then $(f \circ \varphi)'$, $(f \circ \varphi)''$, $\dots, (f \circ \varphi)^{(m-1)} \in$ AC. We separate the cases $m = 1,2,3$ from the rest. For $m \geq 1$, we have just seen that $f \circ \varphi \in$ AC. If $m \geq 2$, then $(f \circ \varphi)' = f'(\varphi)\varphi' \in$ AC since $f' \circ \varphi \in$ AC and $\varphi' \in$ AC, and if $m \geq 3$, then $(f \circ \varphi)'' = f''(\varphi)(\varphi')^2 + f'(\varphi)\varphi'' \in$ AC since $f'' \circ \varphi, \varphi', f' \circ \varphi, \varphi'' \in$ AC. Further, for $m > 3$, it follows by induction that for $s = 3, \dots, m-1$,

$$(f \circ \varphi)^{(s)} = f'(\varphi)\varphi^{(s)} + \sum_{k=2}^{s-1} f^{(k)}(\varphi)P_{k,s}(\varphi', \varphi'', \dots, \varphi^{(s-1)}) + f^{(s)}(\varphi)(\varphi')^s, \tag{3}$$

where $P_{k,s}(t_1, \dots, t_{s-1})$ is a polynomial function for $s \geq 3, k = 2, \dots, s-1$. Since each term on the right hand side of equation (3) is also a combination of absolutely continuous functions, we can conclude that $(f \circ \varphi)^{(s)} \in$ AC for $s = 3, \dots, m-1, m > 3$.

We now show that $(f \circ \varphi)^{(m)} \in L^p$ for each m. First, for $m \geq 3$, we have

$$\begin{aligned}(f \circ \varphi)^{(m)}(x) =& f'(\varphi(x))\varphi^{(m)}(x) \\ &+ \sum_{k=2}^{m-1} f^{(k)}(\varphi(x))P_{k,m}(\varphi', \dots, \varphi^{(m-1)})(x) \\ &+ f^{(m)}(\varphi(x))\varphi'(x)^m \text{ a.e. when } \varphi(x) \neq 0,\end{aligned} \tag{4}$$

and

$$(f\circ\varphi)^{(m)}(x) = f'(\varphi(x))\varphi^{(m)}(x) + \sum_{k=2}^{m-1} f^{(k)}(\varphi(x))P_{k,m}(\varphi',\dots,\varphi^{(m-1)})(x)$$
$$\text{a.e. when } \varphi'(x)=0.$$

We observe that each term in the right hand sides of these equations is in L^p–the first term in both equations is in L^p since $f'\circ\varphi\in \mathrm{AC}$ and $\varphi^{(m)}\in L^\infty$, and the last term in the first equation is in L^p since $f^{(m)}(\varphi(x))\varphi'(x) = (f^{(m-1)}\circ\varphi)'(x)\in L^p$ by (2). When $m=1$ and $m=2$, $(f\circ\varphi)^{(m)}\in L^p$ for similar reasons.

Thus, if $f\in W_{m,p}$, then $(f\circ\varphi)^{(s)}\in \mathrm{AC}, s=0,\dots,m-1$, and $(f\circ\varphi)^{(m)}\in L^p$. That is, the map $C_\varphi : f(x)\to f(\varphi(x))$ is a linear map of $W_{m,p}$ into itself. It is easy to show using the closed graph theorem, for example, that C_φ is bounded.

Finally, if $u\in W_{m,\infty}$, then $uf\in W_{m,p}$ for all $f\in W_{m,p}$. Indeed if $f\in W_{m,p}$, then $uf\in AC$. Also, $(uf)^{(s)} = \sum_{k=0}^{s}\binom{s}{k}u^{(k)}f^{(s-k)}$ and if $s=0,1,\dots,m-1$ each term in the right hand sum is absolutely continuous, while if $s=m$, then

$$(uf)^{(m)} = u^{(m)}f + \binom{m}{1}u^{(m-1)}f' + \binom{m}{2}u^{(m-2)}f'' + \cdots + \binom{m}{m-1}u'f^{(m-1)} + uf^{(m)}$$

which is a sum of functions in L^p.

Therefore, if $\varphi:[0,1]\to[0,1], \varphi\in W_{m,\infty}\cap C^1, \varphi$ is of N–bounded variation for some positive integer N, and if $u\in W_{m,\infty}$, then $uC_\varphi : f(x)\to u(x)f(\varphi(x))$ is a linear operator on $W_{m,p}$ which is clearly bounded.

Before continuing it will be convenient for later use to write equations (3) and (4) in matrix form as follows:

$$\begin{bmatrix} f\circ\varphi \\ (f\circ\varphi)' \\ (f\circ\varphi)'' \\ (f\circ\varphi)''' \\ \cdots \\ (f\circ\varphi)^{(m)} \end{bmatrix} = \tag{5}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varphi' & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varphi'' & (\varphi')^2 & 0 & 0 & \cdots & 0 \\ 0 & \varphi''' & P_{2,3} & (\varphi')^3 & 0 & \cdots & 0 \\ \cdots \\ 0 & \varphi^{(m)} & P_{2,m} & P_{3,m} & P_{4,m} & \cdots & (\varphi')^m \end{bmatrix} \begin{bmatrix} f\circ\varphi \\ f'\circ\varphi \\ f''\circ\varphi \\ f'''\circ\varphi \\ \cdots \\ f^{(m)}\circ\varphi \end{bmatrix}.$$

We now turn to the main result that with these conditions the map uC_φ is compact on $W_{m,p}$ if and only if $u\varphi' = 0$. The first step is the following lemma. See Singh [**7**] for a related result.

LEMMA 4. *Let* $1 \leq p < \infty$. *Let* $u \in L^\infty, \varphi \in AC, \varphi : [0,1] \to [0,1]$ *and suppose* $uC_\varphi : f(x) \to u(x)f(\varphi(x))$ *is a bounded linear operator on* L^p. *If* $\{x|\phi'(x)$ *exists and* $u(x)\phi'(x) \neq 0\}$ *has positive measure, then* uC_φ *is not a compact operator on* L^p.

PROOF. Let $X = [0,1]$. For each measurable subset $e \subset X$, let $m(e)$ denote the measure of e. Then it is well known [**6**, p. 261] that, for almost all $x \in e$,

$$\lim_{h\to 0} \frac{m(e \cap (x-h, x+h))}{2h} = 1$$

An x for which this limit equals 1 is called a point of density of e. Also since $\varphi \in \mathrm{AC}, \varphi'(x)$ exists for almost all $x \in X$.

Now assume uC_φ is a compact operator on L^p and suppose $\{x|\varphi'(x)$ exists and $u(x)\varphi'(x) \neq 0\}$ has positive measure. Then there exists $\delta > 0$ so that $E = \{x||u(x)| \geq \delta, \varphi'(x)$ exists and $u(x)\varphi'(x) \neq 0\}$ has positive measure. Let $x_o \in E$ be a point of density of E. For each positive integer n let $E_n = (x_o - 1/n, x_o + 1/n)$ and let $f_n = \psi\varphi(E_n)/(m(\varphi(E_n)))^{1/p}$, where ψ_F denotes the characteristic function of F. Then

$$||f_n||_p = \left(\int_X \left|\frac{\psi_{\varphi(E_n)}}{m(\varphi(E_n))^{1/p}}\right|^p\right)^{1/p} = 1.$$

Since uC_φ is compact on L^p there exists $g \in L^p$ and a subsequence $\{f_{n_k}\}$ with $uC_\varphi f_{n_k} \to g$ in L^p. Therefore

$$\left(\int_X \left|u(x)\frac{\psi_{\varphi(E_{n_k})}(\varphi(x))}{m(\varphi(E_{n_k}))^{1/p}} - g(x)\right|^p dx\right) \to 0,$$

and so

$$(*)\qquad \left(\int_{\varphi^{-1}(\varphi(E_{n_k}))} \left|\frac{u(x)}{m(\varphi(E_{n_k}))^{1/p}} - g(x)\right|^p dx\right) \to 0$$

and

$$(**)\qquad \left(\int_{X\setminus\varphi^{-1}(\varphi(E_{n_k}))} |g(x)|^p dx\right) \to 0.$$

Since $E_{n_k} \downarrow \{x_o\}$, $(**)$ implies $\int_{X\setminus\varphi^{-1}(\varphi\{(x_o)\})} |g(x)|^p dx = 0$ or $g(x) = 0$ a.e. when $\varphi(x) \neq \varphi(x_o)$.

Then $(*)$ implies

$$\int_{\varphi^{-1}(\varphi(E_{n_k}\setminus\{x_o\}))} \left|\frac{u(x)}{m(\varphi(E_{n_k}))^{1/p}}\right|^p dx \to 0$$

and, since $E_{n_k}\setminus\{x_o\} \subset \varphi^{-1}(\varphi(E_{n_k}\setminus\{x_o\}))$,

$$\int_{E_{n_k}\setminus\{x_o\}} \left|\frac{u(x)}{m(\dot{\varphi}(E_{n_k}))^{1/p}}\right|^p dx \to 0.$$

Therefore

$$(***)\qquad \int_{(E_{n_k}\setminus\{x_o\})\cap E} \left|\frac{u(x)}{m(\varphi(E_{n_k}))^{1/p}}\right|^p dx \to 0.$$

But on E, $|u(x)| \geq \delta$. Consequently

$$\int_{E_{n_k}\setminus\{x_o\})\cap E} \left|\frac{u(x)}{m(\varphi(E_{n_k}))^{1/p}}\right|^p dx \geq \delta^p \left(\frac{m((E_{n_k}\setminus\{x_o\}) \cap E)}{m(\varphi(E_{n_k}))}\right)$$

which together with $(***)$ gives

$$\frac{m((E_{n_k}\setminus\{x_o\}) \cap E)}{m(\varphi(E_{n_k}))} \to 0.$$

But $x_o \in E$ is a point of density of E, so that $\lim_{h\to 0} \frac{m((x_o-h,x_o+h)\cap E)}{2h} = 1$. Hence $\lim_{h\to 0} \frac{m((x_o-h,x_o+h)\setminus\{x_o\})\cap E)}{2h} = 1$, and since $E_{n_k}\setminus\{x_o\} = (x_o - \frac{1}{n_k}, x_o + \frac{1}{n_k})\setminus\{x_o\}$, we have

$$(****)\qquad \lim_{k\to\infty} \frac{m((E_{n_k}\setminus\{x_o\}) \cap E)}{\frac{2}{n_k}} = 1.$$

Further, since $\varphi'(x_0)$ exists, $\lim_{x\to x_o} |(\varphi(x) - \varphi(x_o))/(x - x_o) - \varphi'(x_o)| = 0$. Fix $\varepsilon > 0$. There exists $h > 0$ so that $|\varphi(x) - \varphi(x_o)| < (|\varphi'(x_o)| + \varepsilon)\,|x - x_o|$ when $|x - x_o| < h$. Therefore, if $1/n_k < h$ and $y_1, y_2 \in E_{n_k}$, then $|\varphi(y_1) - \varphi(x_o)| < (|\varphi'(x_o)| + \varepsilon)\,|y_1 - x_o|$, and $|\varphi(y_2) - \varphi(x_0)| \leq (|\varphi'(x_0)| + \varepsilon)|y_2 - x_0|$ and thus $|\varphi(y_1) - \varphi(y_2)| < (|\varphi'(x_o)| + \varepsilon)\,(|y_1 - x_o| + |y_2 - x_o|) < 2(1/(n_k)\,(|\varphi'(x_o)| + \varepsilon)$. Hence, if $1/n_k < h$, then $m(\varphi(E_{n_k})) < (2/n_k)\,(|\varphi'(x_o)| + \varepsilon)$ or $1/(m(\varphi(E_{n_k})))$ $n_k/(2(|\varphi'(x_o) + \varepsilon))$. Therefore

$$\frac{m((E_{n_k}\backslash\{x_o\}) \cap E)}{m(\varphi(E_{n_k}))} > \frac{m((E_{n_k}\backslash\{x_o\}) \cap E)}{2/n_k(|\varphi'(x_o)| + \varepsilon)}.$$

Thus

$$0 = \lim_{k\to\infty} \frac{m((E_{n_k}\backslash\{x_o\}) \cap E)}{m(\varphi(E_{n_k}))} > \lim_{k\to\infty} \frac{m((E_{n_k}\backslash\{x_o\}) \cap E)}{\frac{2}{n_k}(|\varphi'(x_o)| + \varepsilon)} = \frac{1}{|\varphi'(x_o)| + \epsilon}$$

by $(****)$. But $1/(|\varphi'(x_o)| + \varepsilon) > 0$.

This contradiction shows that the assumption that uC_φ is a compact operator on L^p is false. That is, if $\{x|\varphi'(x)$ exists and $u(x)\varphi'(x) \neq 0\}$ has positive measure, then the weighted composition operator uC_φ on L^p is not compact.

We now have all the ingredients to prove the main theorem.

THEOREM 5. *Suppose m is a positive integer, $1 \leq p < \infty, u \in W_{m,\infty}, \varphi : [0,1] \to [0,1], \varphi \in W_{m,\infty} \cap C^1$ and φ is of N-bounded variation for some positive integer N. Then the weighted composition operator $uC_\varphi : f(x) \to u(x)f(\varphi(x))$ is compact on $W_{m,p}$ if and only if $u\varphi' = 0$.*

PROOF. Assume uC_φ is compact on $W_{m,p}$. We will show that $u^{m+1}(\varphi')^{\frac{1}{2}m(m+1)}C_\varphi$ is then compact on L^p from which it follows from Lemma 4 that $u\varphi' = 0$. To this end, let $f_n \in L^p$ with $||f_n||_p \leq 1$ and let $F_n(x) = \int_0^x \int_0^{t_1} \cdots \int_0^{t_{m-1}} f_n(t)dtdt_{m-1}\ldots dt_1$. Then $F_n(x), F_n'(x), \ldots, F_n^{(m-1)}(x)$ are absolutely continuous and, almost everywhere, $F_n^{(m)}(x) = f_n(x) \in L^p$. Thus each $F_n \in W_{m,p}$. Also $||F_n||_{W_{m,p}} \leq (m+1)^{1/p}$ for each n.

Since uC_φ is compact on $W_{m,p}$, there exists a subsequence $\{F_{n_k}\}$ and an element $G \in W_{m,p}$ with $u(x)F_{n_k}(\varphi(x)) \to G(x)$ in $W_{m,p}$. That is, $(uF_{n_k}(\varphi))^{(s)} \to G^{(s)}, s = 0,1,\ldots,m$ in L^p-norm. Expanding, we obtain

$$(A) \qquad \sum_{j=0}^{s} \binom{s}{j} u^{(j)} (F_{n_k}(\varphi))^{(s-j)} \to G^{(s)}, s = 0,1,\ldots,m \text{ in } L^p$$

We note that, formally, $(F_{n_k}(\varphi)^{(j)} \to (\frac{G}{u})^{(j)}$ when $u(x) \neq 0$. Also, if we define $G_j(x)$ by $G_j(x) = u^{j+1}(\frac{G}{u})^{(j)}(x)$ when $u(x) \neq 0$ and $G_j(x) = 0$ when $u(x) = 0$, then $G_j(x) \in L^p, j = 0,1,\cdots,m$.

In matrix form, equations (A) become

(B)

$$\begin{bmatrix} u & 0 & 0 & 0 & \cdots & 0 \\ u' & u & 0 & 0 & \cdots & \\ u'' & \binom{2}{1}u' & u & 0 & \cdots & 0 \\ & & \cdots & & & \\ u^{(m)} & \binom{m}{1}u^{(m-1)} & \binom{m}{2}u^{(m-2)} & \binom{m}{3}u^{(m-3)} & \cdots & u \end{bmatrix} \begin{bmatrix} (F_{n_k} \circ \varphi) \\ (F_{n_k} \circ \varphi)' \\ (F_{n_k} \circ \varphi)'' \\ \cdots \\ (F_{n_k} \circ \varphi)^{(m)} \end{bmatrix}$$

$$\to \begin{bmatrix} G \\ G' \\ G'' \\ \cdots \\ G^{(m)} \end{bmatrix}$$

which is equivalent by row operations to

$$(B') \quad \begin{bmatrix} u & 0 & 0 & 0 & \cdots & 0 \\ 0 & u^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & u^3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & u^4 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & u^{m+1} \end{bmatrix} \begin{bmatrix} (F_{n_k} \circ \varphi) \\ (F_{n_k} \circ \varphi)' \\ (F_{n_k} \circ \varphi)'' \\ (F_{n_k} \circ \varphi)''' \\ \cdots \\ (F_{n_k} \circ \varphi)^{(m)} \end{bmatrix} \to \begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \\ \cdots \\ G_m \end{bmatrix},$$

where the G_j's are the functions defined above.

Using the system (5) which appears before Lemma 4, (B') becomes

$$\begin{bmatrix} u & 0 & 0 & 0 & \cdots & 0 \\ 0 & u^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & u^3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & u^4 & \cdots & 0 \\ \cdots \\ 0 & 0 & 0 & 0 & \cdots & u^{m+1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \varphi' & 0 & 0 & \cdots & 0 \\ 0 & \varphi'' & (\varphi')^2 & 0 & \cdots & 0 \\ 0 & \varphi''' & P_{2,3} & (\varphi')^3 & \cdots & 0 \\ \cdots \\ 0 & \varphi^{(m)} & P_{2,m} & P_{3,m} & \cdots & (\varphi')^m \end{bmatrix}$$

$$\begin{bmatrix} F_{n_k} \circ \varphi \\ F'_{n_k} \circ \varphi \\ F''_{n_k} \circ \varphi \\ F'''_{n_k} \circ \varphi \\ \cdots \\ F^{(m)}_{n_k} \circ \varphi \end{bmatrix} \to \begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \\ \cdots \\ G_m \end{bmatrix}$$

or

$$\begin{bmatrix} u & 0 & 0 & 0 & \cdots & 0 \\ 0 & u^2\varphi' & 0 & 0 & \cdots & 0 \\ 0 & u^3\varphi'' & u^3(\varphi')^2 & 0 & \cdots & 0 \\ 0 & u^4\varphi''' & u^4 P_{2,3} & u^4(\varphi')^3 & \cdots & 0 \\ \cdots \\ 0 & u^{m+1}\varphi^{(m)} & u^{m+1}P_{2,m} & u^{m+1}P_{3,m} & \cdots & u^{m+1}(\varphi')^m \end{bmatrix}$$

$$\begin{bmatrix} F_{n_k} \circ \varphi \\ F'_{n_k} \circ \varphi \\ F''_{n_k} \circ \varphi \\ F'''_{n_k} \circ \varphi \\ \cdots \\ F^{(m)}_{n_k} \circ \varphi \end{bmatrix} \to \begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \\ \cdots \\ G_m \end{bmatrix}$$

which is now equivalent to

$$\begin{bmatrix} u & 0 & 0 & 0 & \cdots & 0 \\ 0 & u^2\varphi' & 0 & 0 & \cdots & 0 \\ 0 & 0 & u^3(\varphi')^3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & u^4(\varphi')^6 & \cdots & \\ \cdots \\ 0 & 0 & 0 & 0 & \cdots & u^{m+1}(\varphi')^{\frac{1}{2}m(m+1)} \end{bmatrix}$$

$$\begin{bmatrix} F_{n_k} \circ \varphi \\ F'_{n_k} \circ \varphi \\ F''_{n_k} \circ \varphi \\ F'''_{n_k} \circ \varphi \\ \cdots \\ F^{(m)}_{n_k} \circ \varphi \end{bmatrix} \to \begin{bmatrix} G_0 \\ G_1^* \\ G_2^* \\ G_3^* \\ \cdots \\ G_m^* \end{bmatrix},$$

where each G_s^* on the right side is a combination of the $G_j, 0 \le j \le s$, multiplied by combinations of $\varphi^{(i)}$ and u^i, and thus G_s^* are in L^p.

In particular, $u^{m+1}(\varphi')^{\frac{1}{2}m(m+1)}F_{n_k}^{(m)}(\varphi) \to G_m^*$ in L^p norm. But $F_{n_k}^{(m)}(y) = f_{n_k}(y)$ a.e. and so we have that $u^{m+1}(\varphi')^{\frac{1}{2}m(m+1)}f_{n_k}(\varphi) \to G_m^*$ in L^p. That is, given an arbitrary bounded sequence $\{f_n\}$ in L^p, we can find an element G_m^* in L^p with $u^{m+1}(\varphi')^{\frac{1}{2}m(m+1)}f_{n_k}(\varphi) \to G_m^*$. Thus the operator $u^{m+1}(\varphi')^{\frac{1}{2}m(m+1)}C_\varphi$ is compact on L^p. By Lemma 4, we have $u^{m+1}(\varphi')^{\frac{1}{2}m(m+1)} = 0$ a.e. Since u and φ' are continuous, $u\varphi' = 0$.

Before proving the converse we note that if $h \in W_{m,p}$ and $||h||_{W_{m,p}} \le 1$, then $||h||_\infty \le 2$. Indeed, for such $h \in W_{m,p}, ||h||_p \le 1$ and $||h'||_p \le 1$. By Hölder's inequality $L^p \subset L^1, ||h||_1 \le ||h||_p \le 1$ and $||h'||_1 \le ||h'||_p \le 1$; hence $\operatorname{Var} h = \int_0^1 |h'| \le 1$. Now if $||h||_\infty > 2$, then $|h(x_0)| > 2$ for some x_0. But $\operatorname{Var} h \le 1$ implies $|h(x)| > 1$ for all x since $|h(x_o)| > 2$ and $|h(x)| \le 1$ implies $1 < |h(x_o)| - |h(x)| \le |h(x_o) - h(x)| \le \operatorname{Var} h$. However if $|h(x)| > 1$ for all x, then $\int_0^1 |h| > 1$, contradicting $||h||_1 \le 1$.

Now assume $u\varphi' = 0$. Since $\varphi \in C^1, \varphi$ is constant on each subinterval on which $u(x) \ne 0$. Moreover, $(u\varphi')' = u'\varphi' + u\varphi'' = 0$. Then, since φ is a constant on each subinterval where $u(x) \ne 0$, it follows that $u\varphi'' = 0$ and hence $u'\varphi' = 0$. Thus φ is a constant on each subinterval on which $u'(x) \ne 0$. Continuing, we have that $u\varphi' = u'\varphi' = \cdots = u^{(m-1)}\varphi' = 0$.

Let $E = \bigcup_{s=0}^{m-1}\{x | u^{(s)}(x) \ne 0\}$. Then E is an open subset of $[0,1]$ and thus $E = \cup_i (a_i, b_i)$, a union of disjoint open intervals (where one of the intervals may be $[0, b_i)$ and another $(a_i, 1]$.) Let $\varphi(x) = c_i$ on (a_i, b_i).

To show that uC_φ is compact on $W_{m,p}$, let $f_n \in W_{m,p}$ with $||f_n||_{W_{m,p}} \le 1$. We will prove that there exists an element $g \in W_{m,p}$ and a subsequence $\{f_{n_k}\}$ with $uC_\varphi f_{n_k} \to g$ in $W_{m,p}$.

We construct the subsequence $\{f_{n_k}\}$ as follows. On the interval $(a_1, b_1), \{f_n(\varphi(x))\} = \{f_n(c_1)\}$ is a bounded sequence of complex numbers, and so there is a subsequence $\{f_{1,n}\}$ of $\{f_n\}$ and a number $A_1 \in \mathbf{C}$ with $f_{1,n}(c_1) \to A_1$. For (a_2, b_2), we find similarly $A_2 \in \mathbf{C}$ and a subsequence $\{f_{2,n}\}$ of $\{f_{1,n}\}$ with $f_{2,n}(c_2) \to A_2$. Continuing in this way, by induction we obtain, for each positive integer j, a complex number A_j and a subsequence $\{f_{j,n}\}$ of $\{f_{j-1,n}\}$ with $f_{j,n}(c_j) \to A_j$. We then define $f_{n_k} = f_{k,k}$ for each positive integer k and note that this construction implies that $f_{n_k}(c_j) \to A_j$ for all j.

Let $g(x) = A_j u(x)$ when $x \in (a_j, b_j)$ and $g(x) = 0$ when $x \notin E = \bigcup(a_i, b_i)$. That is, if $u(x), u'(x), \ldots, u^{(m-1)}(x)$ do not all vanish, we let $g(x) = A_j u(x)$ when $x \in (a_j, b_j)$, while if $u(x) = u'(x) = \cdots = u^{(m-1)}(x) = 0$ we let $g(x) = 0$.

The following then hold

(i) If $x \in E$, then $x \in (a_j, b_j)$ for some j, so that $g(x) = A_j u(x)$ and hence $g^{(s)}(x) = A_j u^{(s)}(x), s = 0, 1, \ldots, m-1$. Clearly $|g^{(s)}(x)| \leq 2|u^{(s)}(x)|$ for $x \in E, s = 0, 1, \ldots, m-1$, since $\|f_{n_k}\|_\infty \leq 2$.

(ii) If $x \notin E$, then $g(x) = g'(x) = \cdots = g^{(m-1)}(x) = 0$. Indeed, if $x \notin E$, then $g(x) = 0$ by definition. Also, for $s = 1, 2, \ldots, m-1$, if $g(x) = \cdots = g^{(s-1)}(x) = 0$ for all $x \notin E$ and if $x_o \notin E$, then

$$\left| \lim_{t \to x_o} \frac{g^{(s-1)}(t) - g^{(s-1)}(x_o)}{t - x_o} \right| \leq \overline{\lim_{t \to x_o}} \left| \frac{g^{(s-1)}(t)}{t - x_o} \right|$$
$$\leq \overline{\lim_{t \to x_o}} 2 \left| \frac{u^{(s-1)}(t)}{t - x_o} \right| = 2|u^{(s)}(x_o)| = 0.$$

Therefore, $g^{(s)}(x_o)$ exists and equals 0. Hence $g, g', \ldots, g^{(m-1)}$ vanish off E.

(iii) If $x_o \notin E$ and $u^{(m)}(x_o) = 0$, a proof similar to (ii) shows that $g^{(m)}(x_o) = 0$.

The preceding two statements assert that if $x \notin E$, then $u(x) = u'(x) = \cdots = u^{(m-1)}(x) = 0, g(x) = g'(x) = \cdots = g^{(m-1)}(x) = 0$ and if $x \notin E$ and $u^{(m)}(x) = 0$, then $g^{(m)}(x) = 0$.

(iv) $\{x \notin E | u^{(m)}(x)$ exists and $u^{(m)}(x) \neq 0\}$ is countable. For suppose $x_o \notin E$ and $u^{(m)}(x_o) \neq 0$. Then

$$\lim_{x \to x_o} \frac{u^{(m-1)}(x) - u^{(m-1)}(x_o)}{x - x_o} = u^{(m)}(x_o) \neq 0.$$

Since $u^{(m-1)}(x_o) = 0$, there exists $\delta > 0$ so that $|u^{(m-1)}(x)| > \frac{1}{2}|u^{(m)}(x_o)||x - x_o|$ for $0 < |x - x_o| < \delta$. Therefore $|u^{(m-1)}(x)| > 0$ for $x_o - \delta < x < x_o$ and $x_o < x < x_o + \delta$, and so $(x_o - \delta, x_o) \subset \cup(a_i, b_i)$ and $(x_o, x_o + \delta) \subset \cup(a_i, b_i)$. Since $\{(a_i, b_i)\}$ are disjoint, x_o is one of the b_i's and one of the a_i's. Hence $\{x \notin E | u^{(m)}(x)$ exists and $u^{(m)}(x) \neq 0\} \subset \{a_1, a_2, \ldots, b_1, b_2, \ldots\}$ which is clearly countable.

(v)

$$\begin{aligned}[0,1] \backslash E = \{x \notin E | u^{(m)}(x) = g^{(m)}(x) = 0\} \\ \cup \{x \notin E | u^{(m)}(x) \text{ does not exist } \} \\ \cup \{x \notin E | u^{(m)}(x) \neq 0\}.\end{aligned}$$

The last two sets on the right hand side have measure 0.

With these facts we now show that $g \in W_{m,p}$ and that $uC_\varphi f_{n_k} \to g$ in $W_{m,p}$.

First we show that $g, g', \ldots, g^{(m-1)} \in \mathrm{AC}$. To this end fix an integer s between 0 and $m-1$. Let $\varepsilon > 0$. Since $u^{(s)} \in \mathrm{AC}$, there exists $\delta > 0$ so that if $\{(x_k, y_k)\}_{k=1}^n$ is a finite collection of non–overlapping intervals with $\sum_{k=1}^n (y_k - x_k) < \delta$, then $\sum_{k=1}^n |u^{(s)}(y_k) - u^{(s)}(x_k)| < \varepsilon/2$.

There are two types of intervals (x_k, y_k). One where x_k and y_k belong to the same subinterval of E and a second where x_k and y_k do not lie in the same subinterval of E. In the case $[x_k, y_k] \subset (a_j, b_j) \subset E$, let $z_k = \frac{1}{2}(x_k + y_k)$, while if x_k and y_k do not lie in the same subinterval of E, let z_k be any point in $[x_k, y_k]$ which lies in the complement of E. Then in both cases

$$|g^{(s)}(y_k) - g^{(s)}(x_k)| \le |g^{(s)}(y_k) - g^{(s)}(z_k)| + |g^{(s)}(z_k) - g^{(s)}(x_k)|.$$

If $[x_k, y_k] \subset (a_j, b_j)$ for some j, then

$$\begin{aligned} |g^{(s)}(y_k) - g^{(s)}(x_k)| &\le |A_j| \Big(|u^{(s)}(y_k) - u^{(s)}(z_k)| + |u^{(s)}(z_k) - u^{(s)}(x_k)| \Big) \\ &\le 2 \Big(|u^{(s)}(y_k) - u^{(s)}(z_k)| + |u^{(s)}(z_k) - u^{(s)}(x_k)| \Big), \end{aligned}$$

and in the second case

$$\begin{aligned} |g^{(s)}(y_k) - g^{(s)}(x_k)| &\le |g^{(s)}(y_k) - g^{(s)}(z_k)| + |g^{(s)}(z_k) - g^{(s)}(x_k)| \\ &\le 2 \Big(|u^{(s)}(y_k) - u^{(s)}(z_k)| + |u^{(s)}(z_k) - u^{(s)}(x_k)| \Big) \end{aligned}$$

since $g^{(s)}(z_k) = u^{(s)}(z_k) = 0$.

Therefore if $\sum_{k=1}^n (y_k - x_k) < \delta$, then certainly the finite collection of non–overlapping intervals $\{(x_k, z_k)\} \cup \{z_k, y_k)\}$ that has just been constructed satisfies $\sum_{k=1}^n \Big((y_k - z_k) + (z_k - x_k) \Big) < \delta$ so

$$\begin{aligned} &\sum_{k=1}^n |g^{(s)}(y_k) - g^{(s)}(x_k)| \\ &\qquad \le 2 \sum_{k=1}^n (|u^{(s)}(y_k) - u^{(s)}(z_k)| + |u^{(s)}(z_k) - u^{(s)}(x_k)|) < 2\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have that $g^{(s)} \in \mathrm{AC}$ for $s = 0, 1, \ldots, m-1$.

Next, for $s = 0, 1, \ldots, m-1$, we write

$$\int_0^1 |(uC_\varphi f_{n_k})^{(s)} - g^{(s)}|^p = \int_E |(uC_\varphi f_{n_k})^{(s)} - g^{(s)}|^p + \int_{[0,1]\setminus E} |(uC_\varphi f_{n_k})^{(s)} - g^{(s)}|.$$

On $[0,1]\setminus E, (uC_\varphi f_{n_k})^{(s)}(x) = \sum_{r=0}^{s} \binom{s}{r} u^{(r)}(x)(f_{n_k}(\varphi(x)))^{(s-r)} = 0$ since $u^{(r)}(x) = 0$ when $x \notin E$ and $r = 0, 1, \ldots, m-1$. Moreover, for $x \notin E, g(x) = g'(x) = \cdots = g^{(m-1)}(x) = 0$ by (ii). Therefore

$$\int_0^1 |(uC_\varphi f_{n_k})^{(s)} - g^{(s)}|^p = \int_E |(uC_\varphi f_{n_k})^{(s)} - g^{(s)}|^p = \sum_i \int_{a_i}^{b_i} |u^{(s)}(x) f_{n_k}(c_i) - A_i u^{(s)}(x)|^p dx.$$

Let $\varepsilon > 0$. Choose N_1 so large that $\sum_{i>N_1} \int_{a_i}^{b_i} |u^{(s)}(x)|^p dx < \varepsilon^p/8^p$, $s = 0, 1, \ldots, m-1$. Then choose N_2 so that

$$|f_{n_k}(c_i) - A_i| < \frac{\varepsilon}{2 \max_{0\le s\le m-1} ||u^{(s)}||_p}, \quad k \ge N_2, i = 1, \ldots, N_1.$$

Then

$$\begin{aligned}\int_0^1 |(uC_\varphi g_{n_k})^{(s)} - g^{(s)}|^p &= \sum_{i=1}^{N_1} \int_{a_i}^{b_i} |(u^{(s)}(x) f_{n_k}(c_i) - u^{(s)}(x) A_i|^p \\ &\quad + \sum_{i>N_1} \int_{a_i}^{b_i} |u^{(s)}(x) f_{n_k}(c_i) - u^{(s)}(x) A_i|^p \\ &\le \sum_{i=1}^{N_1} \int_{a_i}^{b_i} |u^{(s)}(x)|^p |f_{n_k}(c_i) - A_i|^p dx \\ &\quad + \sum_{i>N_1} \int_{a_i}^{b_i} |u^{(s)}(x)|^p 4^p dx.\end{aligned}$$

Hence

$$\begin{aligned}&\Big(\int_0^1 (uC_\varphi f_{n_k})^{(s)} - g^{(s)}|^p\Big)^{1/p} \\ &\le \Big(\sum_{i=1}^{N_1} \int_{a_i}^{b_i} |u^{(s)}(x)|^p \frac{\varepsilon^p dx}{2^p \max_{0\le s\le m-1} ||u^{(s)}||_p^p} + \frac{4^p \varepsilon^p}{8^p}\Big)^{1/p} \\ &< (2\frac{\varepsilon^p}{2^p})^{1/p} = 2^{1/p}\frac{\varepsilon}{2}, \ k \ge N_2.\end{aligned}$$

Thus $(uC_\varphi f_{n_k})^{(s)} \to g^{(s)}$ in $L^p, s = 0, 1, \ldots, m-1$.

Finally, essentially the same proof works to show that $g^{(m)} \in L^p$ and $(uC_\varphi f_{n_k})^{(m)} \to g^{(m)}$ in L^p. The key observation is that (v) implies $m([0,1]\backslash E) = m(\{x \notin E | u^{(m)}(x) = g^{(m)}(x) = 0\})$.

Thus we have shown that if $u\varphi' = 0$ and $f_n \in W_{m,p}$ with $||f_n||_{W_{m,p}} \leq 1$, then there exists a subsequence $\{f_{n_k}\}$ and an element $g \in W_{m,p}$ with $uC_\varphi f_{n_k} \to g$ in $W_{m,p}$. That is, $u\varphi' = 0$ implies uC_φ is a compact operator on $W_{m,p}$.

Before commenting on the spectra of weighted composition operators we recall several definitions. If X is a set and $\varphi : X \to X$, then φ_n denotes the n$^{\text{th}}$ iterate of φ, i.e., $\varphi_o(x) = x$ and $\varphi_n(x) = \varphi(\varphi_{n-1}(x))$ for $n > 0, x \in X$. Also if $\varphi : X \to X$, then a point c in X is called a fixed point of φ of order n if n is a positive integer, $\varphi_n(c) = c$ and $\varphi_k(c) \neq c, k = 1, \ldots, n-1$.

In [**5**] it was shown that if X is a compact Hausdorff space, $u, \varphi \in C(X), \varphi : X \to X$, then a necessary and sufficient condition that $T : f(x) \to u(x)f(\varphi(x))$ be a compact operator on $C(X)$ is that for each connected component C of $\{x | u(x) \neq 0\}$ there exists an open set $V \supset C$ such that φ is constant on V. Further, for such a compact operator T, $\sigma(T)\backslash\{0\} = \{\lambda | \lambda^n = u(c) \ldots u(\varphi_{n-1}(c))$ for some positive integer n and some fixed point c of φ of order $n\}$.

The techniques that were used in proving the results in [**5**] about the spectra can be carried over essentially unchanged to our situation. Specifically, using these techniques one can prove the following theorem.

THEOREM 6. *Suppose m is a positive integer, $1 \leq p < \infty, u \in W_{m,\infty}, \varphi : [0,1] \to [0,1], \varphi \in W_{m,\infty} \cap C^1$ and is of N–bounded variation for some positive integer N. If the weighted composition operator uC_φ is compact on $W_{m,p}$, then $\sigma(uC_\varphi) = \{\lambda | \lambda^n = u(c) \ldots u(\varphi_{n-1}(c))$ for some positive integer n and some fixed point c of φ of order $n\} \cup \{0\}$.*

REFERENCES

1. A.B. Antonevich, *On the spectrum of a weighted shift operator on the space $W_p^\ell(X)$* Sov. Math. Dokl. **25** (1983), 772-774.

2. A. Brown and C. Pearcy, *Introduction to Operator Theory 1: Elements of Functional Analysis,* Springer–Verlag, New York 1977, 371.

3. S. Goldberg, *Unbounded Linear Operators,* McGraw–Hill, New York 1966.

4. M. Josephy, *Composing functions of bounded variation,* Proc. Amer. Math. Soc. **33** (1981), 354-356.

5. H. Kamowitz, *Compact weighted endomorphisms of $C(X)$,* Proc. Amer. Math.

Soc. **83** (1983), 517-521.

6. I.P. Natanson, *Theory of Functions of a Real Variable,* Vol. I, Frederick Ungar, New York 1955.

7. R.K. Singh, *Compact and quasinormal composition operators,* Proc. Amer. Math. Soc. **45** (1974), 80-82.

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