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DISCRETE CUBIC SPLINE INTERPOLATION OVER A NONUNIFORM MESH

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1. Introduction. Let us consider a mesh P on $[a,b]$ which is defined by

$$P : a = x_o < x_1 < \cdots < x_n = b.$$

For $i = 1, 2, \cdots, n$, p_i shall denote the length of the mesh interval $[x_{i-1}, x_i]$. Let $p = \max_{1 \leq i \leq n} p_i$ and $p' = \min_{1 \leq i \leq n} p_i$. P is said to be a uniform mesh if p_i is a constant for all i. Throughout, h will represent a given positive real number. Consider a real function $s(x,h)$ defined over $[a,b]$ which is such that its restriction s_i on $[x_{i-1}, x_i]$ is a polynomial of degree 3 or less for $i = 1, 2, \cdots, n$. Then $s(x,h)$ defines a discrete cubic spline if

$$(1.1) \qquad (s_{i+1} - s_i)(x_i + jh) = 0, j = -1, 0, 1; i = 1, 2, \cdots, n-1.$$

Discrete splines have been introduced by Mangasarian and Schumaker [**5**] in connection with certain studies of minimization problems involving differences. Discrete cubic splines which interpolate given functional values at one point lying in each mesh interval of a uniform mesh have been studied in [**1**]. The case in which these points of interpolation coincide with the mesh points of a nonuniform mesh was studied earlier by Lyche [**3**], [**4**]. The object of the present paper is to study the existence, uniqueness and convergence properties of a discrete cubic spline interpolant of a nonuniform mesh which takes prescribed values at one point of each mesh interval. In comparison to the results proved in [**1**], the use of nonuniform mesh in our results permits a wider choice for the points of interpolation. This will be demonstrated by employing a certain nonuniform geometric mesh. The results obtained in this paper include in particular some earlier results due to Lyche [**4**] and Dikshit and Powar [**1**]. For the corresponding results on continuous

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splines reference may be made to [**2**] and [**6**]. It may also be mentioned that certain interesting studies concerning spline interpolation over a geometric mesh have been made by Micchelli ([**7**], p. 241).

For an equivalent definition of a discrete cubic spline, we introduce the difference operators [4].

$$D_h^{\{O\}}f(x) = f(x); D_h^{\{1\}}f(x) = (f(x+h) - f(x-h))/2h,$$
$$D_h^{\{2\}}f(x) = (f(x+h) - 2f(x) + f(x-h))/h^2.$$

It may be observed that the condition (1.1) has the following equivalent form [4]

(1.2) $$D_h^{\{j\}}s_i(x_i,h) = D_h^{\{j\}}s_{i+1}(x_i,h), j = 0,1,2; i = 1,2,\ldots,n-1.$$

We would also use polynomials $x^{\{j\}}$ given by

$$x^{\{j\}} = x^j, j = 0,1,2; x^{\{3\}} = x(x^2 - h^2).$$

We shall denote by $D(3,P,h)$ the class of all discrete cubic splines over the mesh P. A discrete cubic spline s which is $b-a$ periodic, is said to be a periodic discrete cubic spline. The class of all such splines is denoted by $D_1(3,P,h)$. It may be observed that as $h \to 0$ a discrete cubic spline reduces to a continuous cubic spline.

Writing $t_i = x_{i-1} + \theta p_i$ with $0 \leq \theta \leq 1$ and considering a given function $f(x)$, we introduce the interpolatory condition:

(1.3) $$s(t_i,h) = f(t_i),\ i = 1,2,\cdots,n,$$

and pose the following:

PROBLEM A. Given $h > 0$, for what restrictions on θ and p does there exist a unique $s(x,h) \in D_1(3,P,h)$ which satisfies the condition (1.3)?

2. Existence and uniqueness. Setting $M_i = M_i(h) = D_h^{\{2\}}s(x_i,h)$ and observing that $D_h^{\{2\}}s(x,h)$ is linear, we notice that for the interval $[x_{i-1},x_i]$

(2.1) $$p_iD_h^{\{2\}}s(x,h) = (x_i - x)M_{i-1} + (x - x_{i-1})M_i.$$

Summing the above equation twice we have,

(2.2)

$$6p_i s(x,h) = (x_i - x)^{\{3\}} M_{i-1} + (x - x_{i-1})^{\{3\}} M_i + 6p_i(x - t_i)c_i + 6p_i d_i$$

where c_i, d_i are appropriate constants. For any sequence $< \alpha_n >$, we set $\delta\alpha_n = (1-\theta)\alpha_n + \theta\alpha_{n+1}$ and observe that in view of the condition (1.2),

$$2\Delta c_i = M_i(p_i + p_{i+1}); M_i \Delta p_i^2 = 6[\delta(p_i c_i) - \Delta d_i] \tag{2.3}$$

where Δ is the usual forward difference operator given by $\Delta\alpha_i = \alpha_{i+1} - \alpha_i$. For any function g of p_i, p_{i+1}, p_{i+2} and θ we denote by g^* the function obtained from g by (i) interchanging θ and $\theta^* = (1-\theta)$ and (ii) interchanging p_i and p_{i+2}.

Thus using (1.3), (2.2) and (2.3) we have

$$R_i M_{i+2} + T_i M_{i+1} + T_i^* M_i + R_i^* M_{i-1} = 6F_i, \tag{2.4}$$

where

$$\begin{aligned} R_i &= (\theta^3 p_{i+2}^2 - \theta h^2)\delta p_i; F_i = (\delta p_i \Delta f(t_{i+1}) - \delta p_{i+1} \Delta f(t_i)); \\ T_i &= \delta p_i(\theta^2(3-\theta)p_{i+2}^2 + (1-\theta^3)p_{i+1}^2 + 3\theta p_{i+1}p_{i+2} \\ &\quad - (1-2\theta)h^2) - \delta p_{i+1}(\theta^3 p_{i+1}^2 - \theta h^2) \end{aligned}$$

We are now set to answer Problem A in the following.

THEOREM 1. *Suppose that f is a given $b-a$ periodic function and $h \le p'$. Then there exists a unique $s(x,h) \in D_1(3,P,h)$ which satisfies* (1.3) *if either* (i) $0 \le \theta \le 1/3$ *and $\langle p_i \rangle_{i=1}^n$ is nonincreasing or if* (ii) $2/3 \le \theta \le 1$ *and $\langle p_i \rangle_{i=1}^n$ is nondecreasing.*

PROOF. In order to prove Theorem 1 it is clearly sufficient to show that the system of equations (2.4) for $i = 1, 2, \cdots, n$ has a unique solution. We observe that since $h \le p'$, T_i and T_i^* are nonnegative for $0 \le \theta \le 1$. Also we notice that

$$|R_i| + |R_i^*| < \theta(\theta^2 p_{i+2}^2 + h^2)\delta p_i + \theta^*(\theta^{*2} p_i^2 + h^2)\delta p_{i+1}.$$

Thus, in the coefficient matrix of (2.4), the excess of the positive value of T_i^* over the sum of the positive values of T_i, R_i and R_i^* is not less than

$$y_i(\theta, h) = (r_{i+1} + 2\theta^* h^2)\delta p_i + (r_i + 2\theta^3 p_{i+1}^2)\delta p_{i+1}$$

where

$$r_i = \theta^{*2}(1+2\theta)p_i^2 - 3\theta^2 p_{i+1}^2 - 2\theta h^2.$$

Considering first the case in which $< p_i >_{i=1}^n$ is nonincreasing and $0 \leq \theta \leq 1/3$ we observe that $y_i(\theta, h) > 0$. In the other case in which $< p_i >_{i=1}^n$ is nondecreasing and $2/3 \leq \theta \leq 1$, we see that the excess of the positive value of T_i over the sum of the positive values of R_i, R_i^* and T_i^* in (2.4) is not less than $y_i^*(\theta, h)$ which is clearly positive. Thus, using the diagonal dominance Theorem we see that the coefficient matrix of the system of equations (2.4) is invertible. This proves Theorem 1.

3. Norm of differences between splines. In this section we give an estimate for the difference between two spline interpolants $s(x, h)$ of Theoren 1 with $h = u, v$. For convenience we write

$$y(h) = \max_i[\{y_i(\theta, h)\}^{-1}, \{y_i^*(\theta, h)\}^{-1}]. \tag{3.1}$$

Setting

$$M_i(u, v) = M_i(u) - M_i(v), \overline{M}_i(u, v) = u^2 M_i(u) - v^2 M_i(v)$$

we denote the single column matrices $(M_i(u, v))$ or $(\overline{M}_i(u, v))$ by $M(u, v)$ or $\overline{M}(u, v)$. F denotes the single column matrix (F_i). Unless stated otherwise $||.||$ will denote the sup norm throughout the present paper.

We shall first prove the following Lemma.

LEMMA 3.1. *Let $s(x, h)$ be the unique discrete periodic cubic spline interpolant of f under the assumptions of Theorem 1. Then we have*

$$||M(u, v)|| \leq 24p|u^2 - v^2|y(u)y(v)||F|| \tag{3.2}$$

and

$$||\overline{M}(u, v)|| \leq 6k_1|u^2 - v^2|y(u)||F||, \tag{3.3}$$

where $y(h)$ for $h = u, v$ is given by (3.1) *and* $k_1 = 1 + 24pv^2y(v)$.

PROOF. It may be observed that the system of equations (2.4) may be written as

$$A(h)M(h) = 6F \tag{3.4}$$

where $A(h)$ is the coefficient matrix and $M(h) = (M_i(h))$. From (3.4) it is clear that

$$A(u)M(u,v) = [A(v) - A(u)]M(v). \tag{3.5}$$

However, as already shown in the proof of Theorem 1, $A(h)$ is invertible. Denoting the inverse of $A(h)$ by $A^{-1}(h)$ we notice that the row max norm: $||A^{-1}(h)||$ satisfies the following inequality,

$$||A^{-1}(h)|| \leq y(h). \tag{3.6}$$

It may also be easily seen that

$$||M(v)|| \leq 6y(v)||F|| \tag{3.7}$$

and

$$||A(v) - A(u)|| \leq 4p|v^2 - u^2|. \tag{3.8}$$

We thus prove (3.2) by combining (3.6)-(3.8) with (3.5). By a parallel reasoning we prove (3.3).

We are now set to prove the following.

THEOREM 2. *Suppose $s(x,h)$ is the unique periodic discrete cubic spline interpolant of f under the assumptions of Theorem 1. Then for $u, v > o$*

(3.9), $$||s(x,u) - s(x,v)|| \leq Ky(u)|v^2 - u^2|||F||$$

where K is some positive function depending on θ, p, p' and v and $y(u)$ is given by (3.1).

PROOF. We determine c_i, d_i from equations (1.3), (2.2) and (2.3) and observe that for $x \in [x_{i-1}, x_i]$

$$\begin{aligned}
&6s(x,h)\\
&= [\{(x_i - x)^{\{3\}}/p_i\} - R_i^*(1 - A_i)/\delta p_{i+1}]M_{i-1} + [\{(x - x_{i-1})^{\{3\}}/p_i\}\\
&- A_i\{\theta^2(3-\theta)p_{i+1}^2 + (1-\theta^3)p_i^2 + 3\theta p_i p_{i+1} - (1-2\theta)h^2\} - B_{i-1}]M_i\\
&- A_i B_i M_{i+1} + 6A_i \Delta f(t_i) + 6f(t_i),
\end{aligned}$$

where $A_i = (x - t_i)/\delta p_i$ and $B_i = R_{i-1}/\delta p_{i-1}$. Now taking $h = u, v$ and writing $x = x_{i-1} + p_i t$ with $0 \le t \le 1$, we have

(3.10)

$$\begin{aligned}
&6|s(x,u) - s(x,v)|\\
&\quad \le |\theta - t|[\{(1-t)^2 + \theta^*(2-\theta-t) + \theta^2 + t^2 + \theta t\}p^2\\
&\quad + (\theta^{*3} + (1+\theta)^3))p^3/p']||M(u,v)|| + |\theta - t|(4\theta p/p')||\overline{M}(u,v)||\\
&\quad \le (3p^2 + 8p^3/p')||M(u,v)|| + (4p/p')||\overline{M}(u,v)||.
\end{aligned}$$

Thus, applying Lemma 3.1 to (3.10) we complete the Proof of Theorem 2.

4. Error bounds. For a given $h > 0$, we introduce the set

$$R_{ha} = \{a + jh : j \text{ is an integer }\}$$

and define a discrete interval as follows

$$[a,b]_h = [a,b] \bigcap R_{ha}.$$

For a function f and three distinct points x_1, x_2, x_3 in its domain, the first and second divided differences are defined by

$$\begin{aligned}
[x_1, x_2]f &= \{f(x_1) - f(x_2)\}/(x_1 - x_2) \text{ and } [x_1, x_2, x_3]f\\
&= \{[x_1, x_2]f - [x_2, x_3]f\}/(x_1 - x_3), \text{ respectively.}
\end{aligned}$$

For convenience, we write $f^{\{2\}}$ for $D_h^{\{2\}} f$ and $w(f,p)$ for the modulus of continuity of f. The discrete norm of a function f over the interval $[a,b]_h$ is defined by

$$||f||' = \max_{x \in [a,b]_h} |f(x)|.$$

Without assuming any smoothness condition on the data f, we shall obtain in the following the bounds for the error function over the discrete interval $[a,b]_h$.

THEOREM 3. *Let $s(x,h)$ be the unique periodic discrete cubic spline interpolant of f under the assumptions of Theorem 1. Then over the discrete interval $[a,b]_h$*

$$(4.1) \qquad ||e^{\{r\}}||' \leq p^{2-r}K(r)J(h)w(f^{\{2\}},p), r = 0,1,2,$$

where $J(h)$ is a positive function of h and $K(0) = 1/8, K(1) = 1/2$ and $K(2) = 1$.

In order to prove Theorem 3 we shall need the following results due to Lyche [**3**; Lemma 5.3 and Corollary 5.2 respectively].

LEMMA 4.1. *Let c, d be given real numbers such that $c < d$ and $d \in R_{hc}$ for some $h > 0$ and let $g : [c-h, d+h]_h \to R$ be a given function. For the operators L and U defined by*

$$(d-c)(Lg)(x) = (x-c)g(d) + (d-x)g(c); Ug(x) = g(x) - (Lg)(x),$$

we have,

$$(4.2) \qquad ||Ug||' \leq w(g;p),$$

$$(4.2)' \qquad ||Ug||' \leq (p^2/8)||g^{\{2\}}||',$$

$$(4.2)'' \qquad ||Ug^{\{1\}}||' \leq (p/2)||g^{\{2\}}||',$$

where the discrete norm is taken over the interval $[c,d]_h$.

LEMMA 4.2. *Let $a = < a_j >_{j=1}^{n}$ and $b = < b_j >_{j=1}^{m}$ be given sequences of nonnegative real numbers such that $\sum_{j=1}^{n} a_j = \sum_{j=1}^{m} b_j$. Then for* *any real valued function f defined on a discrete interval $[\alpha,\beta]_h$ we have*

$$|\sum_{j=1}^{n} a_j[x_j^0,x_j^1,x_j^2]f-\sum_{j=1}^{m} b_j[y_j^0,y_j^1,y_j^2]f| \le w(f^{\{2\}},|\beta-\alpha-2h|)\sum_{j=1}^{n} a_j/2, \tag{4.3}$$

where all the distinct points $x_j^0,x_j^1,x_j^2,y_j^0,y_j^1,y_j^2\in[\alpha,\beta]_h$.

PROOF OF THEOREM 3. Since $x_i\in[a,b]_h$ we may take $c=x_{i-1},d=x_i$ in Lemma 4.1. Now taking $g=e^{\{2\}}$ we see that in view of (2.1), $Ue^{\{2\}}=Uf^{\{2\}}$. It directly follows from the definition of Lg that over the discrete interval $[x_{i-1}-h,x_i+h]_h$

$$||Le^{\{2\}}||'\le\max_i|e_i^{\{2\}}|. \tag{4.4}$$

Since $e^{\{2\}}=Ue^{\{2\}}+Le^{\{2\}}$, we see from (4.4) that for $x\in[x_{i-1}-h,x_i+h]_h$

$$||e^{\{2\}}||'\le||Uf^{\{2\}}||'+||(e_i)^{\{2\}}||'. \tag{4.5}$$

We notice that the equation (3.4) may be written as

$$A(h)(e_i^{\{2\}})=A(h)(f_i^{\{2\}})-6(F_i)=(H_i), \tag{4.6}$$

say. Then, we observe that

$$H_i=\sum_{j=1}^{4} a_j[x_j^0,x_j^1,x_j^2]f-\sum_{j=1}^{3} b_j[y_j^0,y_j^1,y_j^2]f$$

where $a_1=2(R_i^*+\theta^*h^2\delta p_{i+1});a_2=2T_i^*,a_3=2T_i,a_4=2(R_i+\theta h^2\delta p_i),b_1=6\delta p_i\delta p_{i+1}(\delta p_i+\delta p_{i+1});b_2=2\theta^*h^2\delta p_{i+1},b_3=2\theta h^2\delta p_i$; $x_j^k=x_{i-2+j}-(1-k)h;j=1,2,3,4;y_1^k=t_{i+k};y_2^k=x_{i-1}-(1-k)h$ and $y_3^k=x_{i+2}-(1-k)h$, for $k=0,1,2$.

Clearly, $\sum a_j=\sum b_j$ and, therefore, applying Lemma 4.2 we have

$$|H_i|\le 3p(h^2+6p^2)w(f^{\{2\}},p). \tag{4.7}$$

Now using the equations (3.6), (4.4)-(4.7), we get

$$||e^{\{2\}}||' \leq J(h)w(f^{\{2\}}, p) \tag{4.8}$$

where $J(h) = [1 + y(h)3p(h^2 + 6p^2)]$. Taking $c = t_i, d = t_{i+1}$, $g = e$, in Lemma 4.1 we see that $Lg = 0$ and, therefore $Ue = e$. Thus, it follows from $(4.2)'$ that

$$||e||' \leq (p^2/8)||e^{\{2\}}||'$$

which proves (4.1) for $r = 0$ when we appeal to (4.8). Similarly, we prove (4.1) for $r = 1$ by using $(4.2)''$.

5. Interpolation on a geometric mesh. In order to highlight the importance of nonuniform meshes, we prove the following corollary which shows that the points of interpolation may be chosen anywhere between the successive mesh points, whereas this choice is restricted in the case of uniform meshes (see Dikshit and Powar [**1**]).

COROLLARY 5.1. *Suppose f is a $b - a$ periodic function and $h \leq p'$. Let $(p_i/p_{i+1}) = 1.56$ (or $(1.56)^{-1}$) for all i, then for $0 \leq \theta \leq 1/2$ (or $1/2 \leq \theta \leq 1$), there exists a unique spline interpolant $s(x, h) \in D_1(3, P, h)$ which satisfies the interpolatory condition* (1.3).

PROOF. Writing $(p_i/p_{i+1}) = q = 1.56$, we observe that in the coefficient matrix of (2.4), the excess of the positive value of T_i^* over the sum of the positive values of T_i, R_i and R_i^* is not less than $Q(\theta)$, where

$$\begin{aligned} Q(\theta)/(\theta^* q + \theta)p_{i+1}q^{-2} = [\{(1 - 3\theta^2 + 2\theta^3)q^2 - 3\theta^2\}(1 + q) \\ + 2q\theta^3]p_{i+1}^2 + 2\{(1 - 2\theta)q^2 - \theta q\}h^2, \end{aligned} \tag{5.1}$$

which is, of course, positive for $0 \leq \theta \leq 1/2$. Next we observe that in the case in which $(p_{i+1}/p_i) = q$, the excess of the positive value of T_i over the sum of the positive values of T_i^*, R_i^* and R_i is not less than $Q^*(\theta)$, which is clearly positive for $1/2 \leq \theta \leq 1$. Thus, using the diagonal dominance Theorem, we complete the proof of Corollary 5.1.

REMARK 5.1. The choice of 1.56 in the Corollary is nearly sharp in the sence that the diagonal dominance property fails if we take

$(p_i/p_{i+1}) = 1.55$ (or $(1.55)^{-1}$), say.

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