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A NOTE ON PURE RESOLUTIONS OF POINTS IN GENERIC POSITION IN $\mathbf{P}_k^n$

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Preliminaries. Let $P_1, \dots P_s \in \mathbf{P}_k^n$ denote s distinct points in projective n-space $\mathbf{P}_k^n$. Throughout this paper, k will denote an algebraically closed field whose characteristic is arbitrary (except for a couple of examples near the end). Let $S = k[X_0, \dots, X_n]$ denote the polynomial ring in $n+1$ variables $X_0, \dots, X_n$ over k. If I denotes the ideal of $P_1, \dots, P_s$ in S, then I is a perfect, unmixed, radical ideal of grade n. Let $R = S/I$, the coordinate ring of $P_1, \dots, P_s$. R is a standard graded k-algebra, Cohen-Macaulay of dimension one and has projective dimension n as an S-module. Thus, R has a minimal, free resolution Γ of the form:

$$(1) \qquad \begin{aligned} \Gamma : 0 \to \oplus_{i=1}^{\beta_n} S(-d_i^{(n)}) \xrightarrow{\phi_n} \cdots \to \oplus_{i=1}^{\beta_2} S(-d_i^{(2)}) \xrightarrow{\phi_2} \\ \to \oplus_{i=1}^{\beta_1} S(-d_i^{(1)}) \xrightarrow{\phi_1} S \to R \to 0. \end{aligned}$$

In (1), $\beta_1, \dots, \beta_n$ are the nontrivial betti numbers of R. Each ϕ_1 is a homogeneous, S-module homomorphism of degree zero. Consequently, ϕ_i can be represented by a $\beta_i \times \beta_{i-1}$ matrix $(\alpha_{pq}^{(i)})$ where $\alpha_{pq}^{(i)}$ is a homogeneous form in S of degree $\partial(\alpha_{pq}^{(i)}) = d_p^{(i)} - d_q^{(i-1)}$. Γ being minimal means every $\alpha_{p,q}^{(i)} \in (X_0, \dots, X_n)$. The $d_i^{(j)}$ in (1) are called the twisting numbers of R and, along with $\beta_1, \dots, \beta_n$, are unique.

We say $P_1, \dots, P_s$ have a pure resolution of type $(d_1, \dots, d_n)$ if, in the minimal resolution (1) of R, we have for all $j = 1, \dots, n$ and for all $i = 1, \dots, \beta_j, d_i^{(j)} = d_j$. Thus, $P_1, \dots, P_s$ have a pure resolution of type $(d_1, \dots, d_n)$ with betti numbers $\beta_1, \dots, \beta_n$ if and only if the minimal, free resolution Γ of R has the simple form:

$$(2) \quad \Gamma : 0 \to S(-d_n)^{\beta_n} \to \cdots \to S(-d_2)^{\beta_2} \to S(-d_1)^{\beta_1} \to S \to R \to 0.$$

We note that the minimality of Γ implies $0 < d_1 < d_2 < \cdots < d_n$ in (2).

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If Γ is a pure resolution of type $(e, e+m, \ldots, e+(n-1)m)$, we shall abreviate our notation and say Γ is a pure resolution of type $\langle e; m\rangle$. Thus, when $m = 1$, a pure resolution of type $\langle e; 1\rangle$ is just the usual notion of a linear resolution. Finally, we say Γ is almost linear if Γ is a pure resolution of type $(e, e+m, \ldots, e+(n-2)m, d_n)$.

In this paper, we investigate what points $P_1, \ldots, P_s \in \mathbf{P}_k^n$ have pure resolutions. This problem is almost hopeless unless we put more conditions on the P_i. One such condition which readily comes to mind is to control the Hilbert function $H_R(t)$ of the P_i.

If A is any standard graded k-algebra, we shall let A_t denote the t – th homogeneous piece of A. The Hilbert function, $H_A(t)$, of A is then given by $H_A(t) = \dim_k\{A_t\}$. For example, $H_S(t)$ is equal to the binomial coefficient $\binom{n+t}{n}$ for all $t \geq 0$. The Poincare series, $F_A(z)$, of A is the formal power series $\sum_{t=0}^{\infty} H_A(t)z^t$. Set $\nu(t) = \binom{n+t}{n}$. We say s distinct points $P_1, \ldots, P_s \in \mathbf{P}_k^n$ are in generic s-position if $H_R(t) = \min\{s, \nu(t)\}$ for all $t \geq 0$. We say $P_1, \ldots, P_s \in \mathbf{P}_k^n$ are in uniform position if for every $t = 1, \ldots, s$, and for every subset $P_{i_1}, \ldots, P_{i_t}$ (of $P_1, \ldots, P_s$) consisting of t distinct points, we have $P_{i_1}, \ldots, P_{i_t}$ are in generic t-position. Most sets of s points in $\mathbf{P}_k^n$ are in generic (uniform) position in the sense that the points in generic s-position (uniform position) in $\mathbf{P}_k^n$ form a dense open subset of $\mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n$ (s times).

In this note, we investigate what points in generic position have pure resolutions and what these resolutions look like. We borrow freely from the facts about points in generic position. These facts can be found in [**3**], [**4**] and [**5**] and the author assumes the reader is familiar with these papers.

Pure resolutions of type $\langle \mathbf{e}, \mathbf{m}\rangle$. The first order of business is to dispense with several trivial cases. If $n = 1$, and $P_1, \ldots, P_s$ are s distinct points in $\mathbf{P}_k^1$, then I is a complete intersection, and $P_1, \ldots, P_s$ have pure resolution $0 \to S(-s) \to S \to R \to 0$. Hence, we can assume $n \geq 2$ throughout the rest of this note. Our first theorem tells us we can assume $s \geq n+1$.

THEOREM 1. *Let $P_1, \ldots, P_s \in \mathbf{P}_k^n (n \geq 2)$ be in generic s-position and assume $s \leq n$. Then $P_1, \ldots, P_s$ have a pure resolution if and only if $s = 1$. A single point has a pure resolution of type $\langle 1; 1\rangle$.*

PROOF. Suppose $P_1, \ldots, P_s$ have a pure resolution. Since $s \leq n, d_1 = \min\{j|\nu(j) > s\} = 1$. Thus, $I = I_1 \oplus I_2 \oplus \ldots$ with $I_1 \neq 0$, and

$\dim_k(I_1) = \nu(1) - s$. If $s \geq 2$, then $\dim_k\{I_1\} \leq n - 1$. Since I has height $n, I \neq (I_1)$ by Krull's theorem. But purity implies $I = (I_1)$. Thus, $s = 1$.

The converse as well as the last statement in Theorem 1 are well known.

Henceforth, we assume $s \geq n+1$, and $n \geq 2$. We can easily dispense with the case $n = 2$ because the minimal resolution in (1) is easy to write down. Suppose $P_1, \ldots, P_s \in \mathbf{P}_k^2$ are in generic s-position. Then the minimal free resolution Γ of $P_1, \ldots, P_s$ has the form:

(3)
$$\Gamma : 0 \to S(-e-1)^{\ell'} \oplus S(-e-2)^{\ell} \to S(-e)^r \oplus S(-e-1)^{r'} \to S \to R \to 0.$$

The constants e, r, r', ℓ and ℓ' appearing in (3) are given by the following formulas:

$$(4)\quad \begin{aligned} &(\mathrm{i}) e = \min\{j | \nu(j) > s\}, (\text{Recall for } n = 2, \nu(j) = \binom{j+2}{2})\\ &(\mathrm{ii}) r = \nu(e) - s,\\ &(\mathrm{iii}) r' = \mu - r, \ (\mu \text{ is the minimal number of generators of I})\\ &(\mathrm{iv}) \ell = e + 1 - r,\\ &(\mathrm{v}) \ell' = \mu - \ell - 1. \end{aligned}$$

To see that the resolution in (3) is in fact correct, we note that the first direct sum, $S(-e)^r \oplus S(-e-1)^{r'}$, in Γ follows from [**4**; Cor 3, p.40]. The last betti number, $\beta_2 = \ell + \ell'$, is the Cohen-Macaulay type of $P_1, \ldots, P_s$ and must be $\mu - 1$ when $n = 2$. The fact that the last twisting numbers are $e+2$ and $e+1$, and that ℓ is given by (4,iv) are standard Chern polynomial computations (see [**10**]).

Now Γ is a pure resolution if $r' = 0$, and either ℓ or ℓ' is zero. Some simple arithmetic computations give us the following proposition.

PROPOSITION 1. *Let $P_1, \ldots, P_s \in \mathbf{P}_k^2$ be in generic s-position with $s \geq 3$. If $P_1, \ldots, P_s$ have a pure resolution, then $P_1, \ldots, P_s$ have a pure resolution of type $< e; 1 >$ or $< e; 2 >$. Furthermore,*

(a) *$P_1, \ldots, P_s$ have a pure resolution of type $< e; 1 >$ (i.e., a linear resolution) if and only if $s = \nu(e-1)$ with $e \geq 2$.*

(b) *$P_1, \ldots, P_s$ have a pure resolution of type $< e; 2 >$ if and only if $s = \nu(e-1) + \frac{e}{2}$ and $I = (I_e)$. Here, e is an even integer with $e \geq 2$.*

We note that Proposition 1(b) has two conditions in it which guarantee purity. The next two examples show that neither condition alone implies purity.

EXAMPLE 1. Let $P_1, P_2, P_3, P_4 \in \mathbf{P}_k^2$ be in generic 4-position with three of these points collinear. Then $e = 2$, and $s = \nu(e-1) + e/2$. The Cohen-Macaulay type of these points is known to be 2. Hence $\mu = 3$. Since $\dim_k(I_2) = 2, I \neq (I_2)$. Thus, $r' > 0$, and these points do not have a pure resolution.

EXAMPLE 2. Let $P_1, \ldots, P_7 \in \mathbf{P}_k^2$ be seven points in uniform position in $\mathbf{P}_k^2$. It follows from formulas in [**2**] that $I = (I_3)$. But $7 \neq \nu(e-1)$ or $\nu(e-1) + \frac{e}{2}$ for any integer e. So, Proposition 1 implies $P_1, \ldots, P_7$ cannot have a pure resolution.

Before leaving $\mathbf{P}_k^2$, we remark that there certainly exist sets of points having pure resolutions and not in generic position at all. Complete intersections in general are not in generic position. A concrete example is as follows.

EXAMPLE 3. Suppose $P_1, \ldots, P_9$ are nine distinct points lying on two cubics $f = 0, g = 0$ in $\mathbf{P}_k^2$. These nine points have pure resolution: $0 \to S(-6) \to S(-3)^2 \to S \to R \to 0$. But $9 \neq \nu(e-1)$ or $\nu(e-1) + \frac{e}{2}$ for any integer e. Thus, Proposition 1 implies these nine points are not in generic position.

When $n \geq 3$ and $s \geq n+1$, the resolution in (1) is of course more complicated than (3). In particular, the twisting numbers are harder to compute. We need some of the machinery developed in [**11**]. Set $e = \min\{j | \nu(j) > s\}$. Then $e \geq 2$, and, consequently, $\operatorname{emdim} R = \dim_k\{R_1\} = n+1$. If Γ is pure, then $I = (I_e)$, and $d_1 = e$. We can write the Poincare series of R in two ways:

$$(5) \qquad F_R(z) = \frac{f_R(z)}{1-z} = \frac{g_R(z)}{(1-z)^{n+1}},$$

In (5), $f_R(z)$ and $g_R(z)$ are polynomials in z of degrees $h(R)$ and $g(R)$ respectively.

Let $i(R) = \max\{t | H_R(t) \neq s\} + 1$. $i(R)$ is called the index of regularity of R, and it is easy to see that $i(R) + 1 \geq e$. When $i(R) + 1 = e$, we call R an extremal (Cohen-Macaulay) ring. It follows from [**11**; Thm. C] that $i(R) - 1 = h(R) - 1 = g(R) - n - 1$. If $P_1, \ldots, P_s$ have a pure resolution of type $(d_1, \ldots, d_n)$ and betti numbers $\beta_1, \ldots, \beta_n$, then a simple calculation shows $g_R(z) = 1 + \sum_{i=1}^n (-1)^i \beta_i z^{d_i}$. Thus, $g(R) = d_n$, and $i(R) = d_n - n$ in this case. The index of regularity is easy to compute when $P_1, \ldots, P_s$ are in generic position. The Hilbert polynomial of R is given by

$$H_R(t) = \begin{cases} \nu(t) & \text{if } t = 0, 1, \ldots, e-1 \\ s & \text{if } t \geq e, \end{cases} \tag{6}$$

Therefore

$$i(R) = \begin{cases} e-1 & \text{if } s = \nu(e-1) \\ e & \text{if } \nu(e-1) < s < \nu(e). \end{cases} \tag{7}$$

In particular, R is extremal if and only if $s = \nu(e-1)$. This fact immediately leads to

PROPOSITION 2. *Let $P_1, \ldots, P_s \in \mathbf{P}_k^n$ be in generic s-position with $s \geq n + 1 \geq 3$. Then $P_1, \ldots, P_s$ have a linear resolution if and only if $s = \nu(e-1)$ for some $e \geq 2$.*

PROOF. Suppose $s = \nu(e-1)$ for some $e \geq 2$. Then equation (7) implies R is extremal. It now follows from [**11**; Thm. A] that R has a pure resolution of type $< e; 1 >$.

Conversely suppose $P_1, \ldots, P_s$ have a linear resolution Γ. Then the type of Γ is $(e, e+1, \ldots, e+n-1)$. The twisting numbers of Γ determine the betti numbers (see [**6**;Thm. 1]). In particular, $\beta_1 = \prod_{j=2}^n \left(\frac{e+j-1}{j-1}\right) = \binom{n+e-1}{n-1}$. Since Γ is pure, $I = (I_e)$. Therefore, $\binom{n+e-1}{n-1} = \beta_1 = \dim_k\{I_e\} = \nu(e) - s$. Thus, $s = \nu(e-1)$.

Our next Proposition says pure resolutions of type $< e; m >$ with $m \geq 2$ are impossible when $n \geq 3$.

PROPOSITION 3. *Let $P_1, \ldots, P_s \in \mathbf{P}_k^n$ be in generic s-position with $s \geq n+1$, and $n \geq 3$. $P_1, \ldots, P_s$ have a pure resolution of type $< e; m >$ if and only if $m = 1$ and $s = \nu(e-1)$ for some $e \geq 2$.*

PROOF. Suppose $P_1, \ldots, P_s$ have a pure resolution of type $< e; m >$. We have seen that $e = \min\{j | \nu(j) > s\} \geq 2$. Suppose $s \neq \nu(e-1)$.

Then $\nu(e-1) < s < \nu(e)$, and $i(R) = e$ by equation (7). The type of Γ is $(e, e+m, \ldots, e+(n-1)m)$ and, consequently, $g(R) = e+(n-1)m$. Since $i(R) - 1 = g(R) - (n+1)$, we conclude $m = n/n-1$. Since $n \geq 3$, this last equation is not possible. Hence $s = \nu(e-1)$, and the rest follows from Proposition 2.

We can now summarize the results of this section with

THEOREM 2. *Let $P_1, \ldots, P_s \in \mathbf{P}_k^n$ be in generic s-position with $s \geq n+1 \geq 3$. Then $P_1, \ldots, P_s$ have a pure resolution of type $< e; m >$ if and only if one of the following occurs:*

(i) $s = \nu(e-1)$ *for some* $e \geq 2$, *in which case* $m = 1$, *or*

(ii) $n = 2, s = \nu(e-1)+e/2$ *for some even integer* $e \geq 2$ *and* $I = (I_e)$, *in which case* $m = 2$.

Some results on pure resolutions not of type $< e; m >$. When $n = 2$, a pure resolution must be of type $< e; m >$ by Proposition 1. Hence, we shall assume $s \geq n+1 \geq 4$ in our next theorem. This theorem summarizes about all we can say in general for pure resolutions of points in generic position when $s \neq \nu(e-1)$, for any $e \geq 2$ and $n \geq 3$.

THEOREM 3. *Let $P_1, \ldots, P_s \in \mathbf{P}_k^n$ be in generic s-position. Assume $n \geq 3$ and $\nu(e-1) < s < \nu(e)$ for some $e \geq 2$. Suppose $P_1, \ldots, P_s$ have a pure resolution Γ of type $(d_1, \ldots, d_n)$. Then*

(a) $d_1 = e, d_n = e+n$,

(b) *There exists an integer α between 1 and $n-1$ such that*

$$d_i = \begin{cases} e+i-1, & \text{for } i = 1, \ldots, \alpha, \\ e+i, & \text{for } i = \alpha+1, \ldots, n, \end{cases}$$

(c) *The betti numbers $\beta_1, \ldots, \beta_n$ of Γ are given by the following formulas:*

$$\beta_i = \begin{cases} \begin{pmatrix} e+i-2 \\ i-1 \end{pmatrix} \begin{pmatrix} n+e \\ n-i+1 \end{pmatrix} \begin{pmatrix} \frac{\alpha+i-1}{e+\alpha} \end{pmatrix}, & \text{for } i = 1, \ldots, \alpha, \\ \begin{pmatrix} e+i-1 \\ i \end{pmatrix} \begin{pmatrix} n+e \\ n-i \end{pmatrix} \begin{pmatrix} \frac{i-\alpha}{e+\alpha} \end{pmatrix}, & \text{for } i = \alpha+1, \ldots, n \end{cases},$$

(d) $\nu(e-1) < s < \nu(e) - n$.

PROOF. We have already seen that $\nu(e-1) < s < \nu(e)$ implies $d_1 = e$. From equation (7), $i(R) = e$. Since $g(R) = d_n$, we get $d_n = e+n$. This proves (a). Since $d_1 = e < d_2 < \cdots < d_n = e+n$, (b) is immediate. Part (c) follows from the formula, given in [**6**; Thm. 1], connecting the betti numbers with the twists.

To argue (d), suppose $s = \nu(e) - \lambda$ with $1 \leq \lambda \leq n$. Since $I = (I_e), I_e$ contains a regular sequence of length n. It now follows from [**5**; Cor. 3.15] that $\lambda = \dim_k\{I_e\} = \binom{n+e}{n-1} - \lambda(n-1)$. Thus, $\binom{n+e}{n-1} \leq n^2$ which is impossible when $n \geq 3$, and $e \geq 2$.

Suppose $P_1, \dots, P_s \in \mathbf{P}_k^n$ satisfy the hypotheses of Theorem 3. Then the pure resolution Γ is very close to being linear in the sense that the twisting numbers increase by one from e to $e+\alpha-1$ and then again increase by one from $e+\alpha+1$ to $e+n$. There is a jump of two at α. One can ask what α's between 1 and $n-1$ actually occur in pure resolutions? Since $\beta_0 = 1, \beta_1, \dots, \beta_n$ the expressions in Theorem 3(c) are integers such that $\sum_{i=0}^n (-1)^i \beta_i = 0$, Theorem 3(c) places certain restrictions on α. I suspect that if α is any integer between 1 and $n-1$ such that the expressions in Theorem 3(c) are integers, then there exist points $P_1, \dots, P_s \in \mathbf{P}_k^n$ in generic s-position and having a pure resolution Γ of type $(e, e+1, \dots, e+\alpha-1, e+\alpha+1, \dots, e+n)$. This is certainly the case (if, for example, the characteristic of k is zero) when $n = 3$ as the next two examples show.

EXAMPLE 4. Let k be an algebraically closed field of characteristic zero. We show there exists a nonempty, Zariski open subset $U \subsetneq W = \mathbf{P}_k^3 \times \cdots \times \mathbf{P}_k^3$ (12 times) such that $\langle P_1, \dots, P_{12} \rangle \in U$ implies:

(1) $P_1, \dots, P_{12}$ are in uniform position in $\mathbf{P}_k^3$;

(2) $P_1, \dots, P_{12}$ have the almost linear resolution

$$\Gamma : 0 \to S(-6)^2 \to S(-4)^9 \to S(-3)^8 \to S \to R \to 0.$$

To see this, we first note that there exists a dense open subset U_1 of W such that $\langle P_1, \dots, P_{12} \rangle \in U_1$ implies $P_1, \dots, P_{12}$ are in uniform position in $\mathbf{P}_k^3$, and $I = (I_3)$. The existence of U_1 follows from [**3**; Thm. 4] and [**9**; Cor. 2.2]. (To apply [**9**; Cor. 2.2], we need k to have characteristic zero).

There exists a nonempty, open subset U_2 of W such that $\langle P_1, \dots, P_{12} \rangle \in U_2$ implies $P_1, \dots, P_{12}$ are in generic 12-position in $\mathbf{P}_k^3$ and have Cohen-Macaulay type $t(P_1, \dots, P_{12}) = 2$. This fact follows easily from the algorithm for Cohen-Macaulay type presented in [**1**; pp. 18-20] or by taking a link of the points in Example 5.

Since W is irreducible, $U = U_1 \bigcap U_2$ is a nonempty open subset of W. $\langle P_1, \ldots, P_{12}\rangle \in U$ implies:

(a) $P_1, \ldots, P_{12}$ are in uniform position in $\mathbf{P}_k^3$.

(b) $I = (I_3)$. Consequently, $\beta_1 = \dim_k\{I_3\} = 8$.

(c) $\beta_3 = t(P_1, \ldots, P_{12}) = 2$.

Now let $< P_1, \ldots, P_{12} > \in U$. Then $\beta_2 = 9$. Let $I_3 = \sum_{j=1}^{8} kh_j$. Since $I = (I_3)$, we must have that I_4 is spanned by the 32 vectors $x_0h_1, \ldots, x_0h_8, \ldots, x_3h_1, \ldots, x_3h_8$. Since $\dim_k\{I_4\} = 23$, there are precisely 9 linearly independent vectors $(\ell_{i1}, \ldots, \ell_{i8})$ with $\ell_{ij} \in S_1$, and $\sum_{j=1}^{8} \ell_{ij}h_j = 0$ for $i = 1, \ldots, 9$. Since $\beta_2 = 9$, we conclude that a minimal free resolution of $\langle P_1, \ldots, P_{12}\rangle \in U$ has the form:

$$(8) \qquad 0 \to S(-p) \oplus S(-q) \to S(-4)^9 \to S(-3)^8 \to\to R \to 0.$$

Considering the Chern polynomials from (8), we must have

$$(9) \quad (1 - pt)(1 - qt)(1 - 3t)^8(1 + 24t^3) = (1 - 4t)^9 \text{ in } \mathbf{Z}[t]/(t^4).$$

Thus, $p = q = 6$, and we get Γ as claimed.

In Example 4, $n = 3$, and $\alpha = 2$. In Example 5, $n = 3$, and $\alpha = 1$.

EXAMPLE 5. Let k be an algebraically closed field whose characteristic is not equal to 2 or 3. We show there exists a nonempty open subset $U \subseteq W = \mathbf{P}_k^3 \times \cdots \times \mathbf{P}_k^3$ (15 times) such that $\langle P_1, \ldots, P_{15}\rangle \in U$ implies:

(1) $P_1, \ldots, P_{15} \in \mathbf{P}_k^3$ are in uniform position; and

(2) $P_1, \ldots, P_{15}$ have the pure resolution

$$\Gamma : 0 \to S(-6)^5 \to S(-5)^9 \to S(-3)^5 \to S \to R \to 0.$$

To see this, let us change variables and write $S = k[x, y, z, w]$. From the usual generic considerations, it suffices to exhibit fifteen points $P_1, \ldots, P_{15} \in \mathbf{P}_k^3$ in generic 15-position and having resolution Γ. Consider the ideal $I = (x^3 - w^2x, y^3 - w^2y, z^3 - w^2z, xyz, f) \subseteq S$ where $f = wx^2 + wy^2 + wz^2 - w^2x - w^2y - w^2z - 2w^3 + x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2$.

Clearly, $K = \{x^3 - w^2x, y^3 - w^2y, z^3 - w^2z\}$ is a regular sequence of length three in I. Set $J = (K) : I$. Then J is linked to I. In [**8**], the following facts are proven:

(a) I is a radical ideal, unmixed of grade 3; and

(b) J has a pure resolution Γ' given by

$$\Gamma' : 0 \to S(-6)^2 \to S(-4)^9 \to S(-3)^8 \to S \to S/J \to 0.$$

It now follows from straightforward computations (form the Koszul resolution Γ'' of $S/(K)$ and take the mapping cone of $(\Gamma')^* \to (\Gamma'')^*$) that I has a pure resolution Γ given by

$$\Gamma : 0 \to S(-6)^5 \to S(-5)^9 \to S(-3)^5 \to S \to S/I \to 0.$$

It easily follows from [**7**; Thm. 1.2] that S/I has multiplicity 15. Thus, the zeros of I in $\mathbf{P}_k^3$ are precisely 15 points, necessarily in generic 15-position.

We note that J in Example 5 gives 12 points in $\mathbf{P}_k^3$ with pure resolution as in Example 4. Thus, there are examples of $\alpha = 1$ or 2 in $\mathbf{P}_k^3$ in all characteristics except possibly char$k = 2$ or 3. For $n \geq 4$, it becomes increasingly more difficult to study the behavior of α. The basic problem is finding a suitable open set $U \subseteq W = \mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n$ (s times) such that $\langle P_1, \dots, P_s \rangle \in U$ implies I is generated by its lowest degree forms. This problem seems to be very difficult. Except for [**9**] and some sporadic results in [**4**] and [**5**], little is known concerning U.

In Example 4, we presented 12 points in $\mathbf{P}_k^3$ which have an almost linear resolution. We finish this paper with some theorems which give another collection of points having an almost linear resolution.

THEOREM 4. *Let $P_1, \dots, P_s \in \mathbf{P}_k^n$ be in uniform position with $s \geq n+1 \geq 3$. Then the following are equivalent:*

(a) $s = n+2$;

(b) *R is Gorenstein*;

(c) *$P_1, \cdots, P_s$ have an almost linear resolution Γ of the form*

$$\Gamma : 0 \to S(-n-2)^{\beta_n} \to S(-n)^{\beta_{n-1}} \to S(-n+1)^{\beta_{n-2}} \\ \to \cdots \to S(-2)^{\beta_1} \to S \to R \to 0.$$

PROOF. The equivalence of (a) and (b) is well known [**3**; Thm. 7]. Suppose (c); then $I = (I_2)$. Therefore $2 = \min\{j | v(j) > s\}$.

If $s = n + 1$, then again Proposition 2 implies Γ is linear. Thus, $\nu(1) < s < \nu(2)$. Using [**6**; Thm. 1] we get $\beta_n = \prod_{j=1}^{n-1}(\frac{j+1}{n-j+1}) = 1$. Thus, R is Gorenstein and we have shown (c) implies (b).

Suppose (b). Then, by (a), $s = n + 2$. Therefore, $e = \min\{j|\nu(j) > s\} = 2$, and $i(R) = 2$ by equation (7). In particular, $i(R) + \dim(R) = 2e - 1$ and R is an extremal Gorenstein algebra. Part (c) now follows from [**11**; Thm. B].

A second version of Theorem 4 is worth recording here.

THEOREM 5. *Let $P_1, \cdots, P_s \in \mathbf{P}_k^n$ be in generic s-position with $s \geq n + 1 \geq 3$. Suppose $P_1, \cdots, P_s$ have a pure resolution Γ of type $(d_1, \cdots, d_n)$. Then the following are equivalent:*

(a) $s = n + 2$;

(b) *R is Gorenstein;*

(c) *Γ has the form*

$$\Gamma : 0 \to S(-n-2)^{\beta_n} \to S(-n)^{\beta_{n-1}} \to \cdots \to S(-2)^{\beta_1} \to S \to R \to 0.$$

Furthermore, if (a), (b) or (c) is satisfied, then $P_1, \cdots, P_s$ are in uniform position in $\mathbf{P}_k^n$.

PROOF. We first prove that (b) and (c) are equivalent. Suppose R is Gorenstein. Set $e = d_1$. Then $\nu(e-1) \leq s < \nu(e)$. If $s = \nu(e-1)$, then Proposition 2 implies Γ is linear of type $\langle e; 1\rangle$. Since R is Gorenstein, the twisting numbers in Γ satisfy $d_i = d_n - d_{n-i}$. In particular, $e = d_1 = d_n - d_{n-1} = e + (n-1) - d_{n-1}$. Therefore $d_{n-1} = n - 1$. But then $e = 1$, which is impossible. Thus, $\nu(e-1) < s < \nu(e)$. It now follows from Theorem 3 (or Proposition 1 if $n = 2$) that $d_n = e + n$. Since, $e = d_1 = d_n - d_{n-1} = e + n - d_{n-1}, d_{n-1} = n$. This implies $e = 2$, and (c) follows

If Γ has the form given in (c), then $\beta_n = 1$ from [**6**; Thm. 1]. Thus, R is Gorentsein, and (b) and (c) are equivalent. We also note that (c) implies (a). For $\beta_1 = \prod_{j=2}^{n} |d_j/d_j - d_1| = (\frac{n+2}{n}) \prod_{j=2}^{n-1}(\frac{2+j-1}{j-1}) = \frac{(n+2)(n-1)}{2}$. Therefore, $s = \binom{n+2}{2} - \frac{(n+2)(n-1)}{2} = n + 2$.

Next, we argue (a) implies (c). If $n = 2$, (c) follows immediately from Proposition 1. So, we may assume $n \geq 3$. $d_1 = e = \min\{j|\nu(j) > n + 2\} = 2$, and $d_n = n + 2$ by Theorem 3. $\beta_1 = \dim_k\{I_2\} = \nu(2) - (n+2) = (n+2)(n-1)/2$. Let α be defined as in Theorem 3(b).

Again from [**6**; Thm. 1] we get the equation
(10)
$$\frac{(n+2)(n-1)}{2}=\binom{n+2}{n}\left\{\prod_{j=2}^{\alpha}\binom{j+1}{j-1}\right\}\left\{\prod_{j=\alpha+1}^{n-1}\binom{j+2}{j}\right\}.$$

Equation (10) readily implies $\alpha = n-1$, and thus the proof of (c) is complete.

Finally, we prove the last statement in Theorem 5. If $n = 2$, four points in generic position have either three points collinear or no three points collinear. In the first case, R is not Gorenstein (the Cohen Macaulay type is 2). Thus, no three points of P_1, P_2, P_3, P_4 are collinear. This implies the points are in uniform position in $\mathbf{P}_k^2$. Hence,we may assume $n \geq 3$.

Suppose $P_1, \cdots, P_{n+2} \in \mathbf{P}_k^n$ are not in uniform position. Then these points are not in general position; i.e., $n+1$ of them lie on some hyperplane in $\mathbf{P}_k^n$. We may assume without loss of generality that $P_1, \cdots, P_{n+1} \subseteq V(X_0) \subseteq \mathbf{P}_k^n$. Since $I = I_2 \oplus I_3 \oplus \cdots$ with $I_2 \neq 0, X_0(P_{n+2}) \neq 0$. A straightforward linear algebra argument shows we may assume with no loss of generality that $P_1 = (0:1:0:\cdots:0), \cdots, P_n = (0:\cdots:0:1), P_{n+1} = (0:b_1:\cdots:b_n)$, and $P_{n+2} = (1:0:\cdots:0)$.

Next we identify $V(X_0)$ in $\mathbf{P}_k^n$ with $\mathbf{P}_k^{n-1}$ via $(0:a_1:\cdots:a_n) \rightarrow (a_1:\cdots:a_n)$. Let $T = k[Y_1, \cdots, Y_n]$ denote the coordinate ring of $\mathbf{P}_k^{n-1}$. Then $S/(X_0) \simeq T$, and T can de identified (via $Y_i \rightarrow X_i$) with the subring $k[X_1, \cdots, X_n]$ of S. If J denotes the ideal of $P_1, \cdots P_{n+1}$ in T, then, under this identification, $J \subset I$.

We claim that $P_1, \cdots, P_{n+1}$ are in generic $(n+1)$-position in $\mathbf{P}_k^{n-1}$. To see this, we first note that $n \geq 3$ implies $\min\{j|\binom{n-1+j}{n-1} > n+1\} = 2$. Also, we easily check $J_1 = 0$ and $J_2 \neq 0$. Thus, $P_1, \cdots, P_{n+2}$ fail to be in generic position in $\mathbf{P}_k^{n-1}$ if $\dim_k\{J_2\} \neq \binom{n-1+2}{n-1} - (n+1)$. (See [**3**; Prop. 3]). Suppose then that $P_1, \cdots, P_{n+1}$ are not in generic position in $\mathbf{P}_k^{n-1}$. Then $r = \dim_k\{J_2\} > \binom{n-1+2}{n-1} - (n+1) = \frac{(n+1)(n-2)}{2}$. Let $\{f_1, \cdots, f_r\}$ be a basis of J_2. Then, in S, $I_2 \supset \{f_1, \cdots, f_r, X_0X_1, \cdots, X_0X_n\}$. Thus, $\frac{(n+2)(n-1)}{2} = \dim_k\{I_2\} \geq r + n > \frac{(n+1)(n-2)}{2} + n = \frac{(n+2)(n-1)}{2}$ which is impossible. Hence $P_1, \cdots, P_{n+1} \in \mathbf{P}_k^{n-1}$ are in generic position.

Since Γ is pure, we now conclude $I = (I_2) = (f_1, \cdots, f_r, X_0X_1, \cdots, X_0X_n)$. We finish the proof by arguing that R is not Gorenstein. This contradiction implies $P_1, \cdots, P_{n+2}$ must be in uniform position.

To see that R is not Gorenstein, write $R = k[x_0, \cdots, x_n]$. Since k is infinite, there exists a regular element $\ell \in R_1$. Since $x_0x_i = 0$ for $i = 1, \cdots, n$, we can assume $\ell = x_0 - \alpha_1x_1 - \cdots - \alpha_nx_n$ with $\alpha_i \in k$ and not all zero. Form $\overline{R} = R/(\ell) = k[y_0, \cdots, y_n]$. Then $0 \neq y_0 \in \overline{R}_1$

and $y_0 \in S(\overline{R})$, the socle of $\overline{R}$. The Poincare series, $F_R(z)$, of R is given by $F_R(z) = \frac{(n+2)}{(1-z)} - (n+1) - z$. Since the degree of ℓ is 1, the Poincare series for $\overline{R}$ is given by $F_{\overline{R}}(z) = 1 + nz + z^2$. In particular, $0 \neq \overline{R}_2 \subseteq S(\overline{R})$. Hence $\dim_k\{S(\overline{R})\} > 1$, and R is not Gorenstein.

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